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ALGORITHM FOR THE SIMPLIFICATION OF SOLUTION TO DISCRETE OPTIMIZATION PROBLEMS

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Зазвичай пошук розв'язку в задачах дискретної оптимізації пов'язаний з принциповими обчислювальними труднощами. Відомі методи точного або наближеного розв'язку таких задач вивчаються з урахуванням належності їх до, так званих, задач з класу P та NP (алгоритми поліноміальної та експоненціальної реалізації розв'язку). Сучасні комбінаторні методи для практичного розв'язку задач дискретної оптимізації потребують розробки алгоритмів, які дозволяють отримувати наближений розв'язок з гарантованою оцінкою відхилення від оптимуму. Алгоритми спрощення є ефективним прийомом пошуку розв'язку оптимізаційної задачі. Якщо виконати проектування багатовимірного процесу на двовимірну площину, то такий прийом дозволить наочно відобразити у графічній формі множини розв'язків задачі. В рамках даного дослідження запропоновано спосіб спрощення комбінаторного розв'язку задачі дискретної оптимізації. Він заснований на тому, що виконується декомпозиція системи, яка відображає систему обмежень п'ятивимірної вихідної задачі на двовимірну координатну площину. Такий спосіб дозволяє отримати просту систему графічних розв'язувань складної задачі лінійної дискретної оптимізації. З практичної точки зору запропонований метод дозволяє спростити обчислювальну складність оптимізаційних задач такого класу. Прикладним аспектом запропонованого підходу є використання отриманого наукового результату для забезпечення можливості вдосконалення типових технологічних процесів, що описуються системами лінійних рівнянь з наявністю системами лінійних обмежень. Це складає передумови для подальшого розвитку та удосконалення подібних систем. В даному дослідженні запропоновано методику декомпозиції дискретної оптимізаційної системи шляхом проекції вихідної задачі на двовимірні координатні площини. За такого прийому вихідна задача трансформується в комбінаторне сімейство підсистем, що дозволяє отримати систему графічних розв'язувань складної задачі лінійної дискретної оптимізації

Ключові слова: лінійне оптимізація, дискретна оптимізація, система обмежень, пошук оптимуму, комбінаторний метод, метод Жордана-Гаусса, декомпозиція, графічний розв'язок

1. Introduction

Discrete optimization, which has already been defined as a separate section of the optimization theory, in most cases operates the combinatorial methods of solution [1]. The problems of allocation of raw materials among different products, formation of the nutrition ration, materials cutting, etc. are solved using the methods of linear programming [2]. The typical tasks of the combinatorial methods are to obtain the original reference plan, optimality assessment, improvement of the plan and boundary estimation of objective function. Since most discrete optimization problems belong to the class of *NP* problems, the use of algorithms for a problem simplification without losing control of the solution accuracy is relevant [3]. For a simplification procedure, the known relationship of the systems of linear algebraic equations with the system of linear algebraic inequalities and the classic apparatus of linear algebra are used [4]. Direct computing simplification technique was implemented with the use of the Jordan-Gauss method [5].

2. Literature review and problem statement

In many cases, mathematical models of project management are interpreted as problems of discrete optimization [6]. Solution of problems of discrete optimization is associated with fundamental difficulties. Modern methods of exact and approximated solution of such problems are examined with respect to their belonging to the so-called problems of *P* and *NP* class (algorithms for polynomial and exponential implementation of solution) [7].

Combinatorial methods for accurate and practical solution of discrete optimization problems take one of the important places in obtaining optimal values of such tasks [8]. To realize solution algorithms, it is necessary to obtain the original reference plan, optimality estimates and improvements in case it is not optimal. Modern combinatorial methods for practical solution of discrete optimization problems require development of algorithms that allow obtaining an approximated solution with the guaranteed estimation of deviations from an optimum [9].

Simplification algorithms in discrete optimization problems are an effective technique to search for a solution of an optimization problem [10]. If we perform projecting of a multidimensional process on a two-dimensional plane, such technique will make it possible to clearly observe the permissible set (lattice) of the parameters of a problem [11]. An assessment of values of the objective function of a problem can be performed from the bottom and from the top and dynamically assess the possibility of diversifying the basis optimal variable with guaranteed accuracy [12]. In paper [13], the method of thermoeconomic optimization of power consuming systems with the linear structure on the graphs was developed. Analysis of solutions' stability in the problems of detecting duplicates in electronic documents was presented in research [14]. Complexity of displaying linear relationships in projects was shown with the use of Markov chains in paper [15].

It is possible to resolve the contradictions between requirements regarding the completeness of the model representations of technical and social systems and the methods of obtaining solutions by rational simplification of algorithms for solution of complex systems of equations [16]. The essence of not sufficiently resolved problem regarding the search for solutions in problems of linear programming is the need to create a method of simplification of combinatoric solution of a discrete optimization problem.

In the studies, published by different authors, it is emphasized that one of the unresolved components of the general problem of development of effective models is the stage of obtaining a solution to systems of equations of mathematical description of complex systems [8]. In this case, the methods for solution simplification through introduction of specific constraints [9], transition to iterative search for a solution [11], the use of the decomposition of objects with representation of systems using graphs [13] have become widely used. These solution simplification options are based on the additional transformation of the mathematical description of systems with construction of unique algorithms for obtaining the solution to a problem [15]. The lack of effective in the chosen aspect methods of simplification of combinatorial solution to discrete optimization problems is probably associated with the computational complexity and diversity of displayed systems [16]. The combinatorial methods for solving such problems allow separation of an array of subsets of solutions from a set of admissible values. The algorithm for separation of such subsets is the basis for combinatorial methods, but the usual sorting out solutions can be whatever large and complicated for calculations [3]. Given this, all possible simplifications of the original problem should improve the combinatorial algorithm. Any simplification algorithms decrease the number of combinatorial sorting out from a set of admissible values. This is due to a known fact of the algorithmic complexity of discrete optimization problems.

3. The aim and objectives of the study

The aim of present research is to develop the algorithm of simplification of the solution of multidimensional discrete optimization problems using the standard computational procedures of linear algebra and some techniques of linear optimization.

To accomplish the set goal, the following tasks were to be solved:

– to separate the class of problems, which are subject to simplification;

– to present calculations of the model example.

4. Development of the simplified solution to discrete optimization problems

Assume we have a general linear optimization problem in the form:

$$\mathrm{W}_I = \sum_{j=1}^{n} c_j x_j \to \max,$$

$$\Omega_I : \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i, & i = 1, \ldots, k, \\ \sum_{j=1}^{n} a_{ij} x_j = b_i, & i = k+1, \ldots, m, \end{cases}$$

$$x_j \ge 0, \quad j = 1, \ldots, l.$$

where W_I is the objective function.

It is known that this problem can always be reduced to the canonical form of recording:

$$\mathrm{W}_I = \sum_{j=1}^{n} c_j x_j \to \max,$$

$$\Omega_I : \sum_{j=1}^{n} a_{ij} x_j = b_i, i = 1, \ldots, m,$$

$$x_j \ge 0, \quad j = 1, \ldots, n.$$

We will note that the forms of linear optimization problems are equivalent – they preserve a set of solutions. It is possible to achieve this by using techniques for transformation for a transition from one form of problems to another.

Thus, the equation of the system of constraints to a linear optimization problem is equivalent to the system of two inequalities:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Leftrightarrow \begin{cases} \sum_{j=1}^{n} a_{ij}x_j \le b_i, \\ -\sum_{j=1}^{n} a_{ij}x_j \le -b_i. \end{cases}$$

The variables that are arbitrary by sign can be represented as the difference of two non-negative variables:

$$x_j = u_j - v_j, \quad u_j \ge 0, \quad v_j \ge 0.$$

The transition from constraints-inequalities to constraints–equations is performed by adding a non-negative (balance) variable:

$$\sum_{j=1}^{n} a_{ij}x_j \le b_j \Rightarrow \sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i, \quad x_{n+i} \ge 0, \quad i = 1, \ldots, k.$$

To simplify the transformation of linear optimization problems, the transition from maximization to minimization of the objective function and vice versa is also used:

$$W_I = \sum_{j=1}^{n} c_j x_j \to \max \Leftrightarrow W_I = -\sum_{j=1}^{n} c_j x_j \to \min.$$

Given this, without loss of generality, let us assume that we have a linear discrete optimization problem, given in the canonical form:

$$W_I = CX \to \max,$$

$$\Omega_I : AX = B,$$

$$X \ge 0,$$

where the range of matrix of coefficients of the constraints system is equal to $\operatorname{rang} A = m$.

Then solving the system using the Jordan-Gauss method [5] by arbitrary basis combination of variables, we will obtain the projection of the n-dimensional original problem on $(n-m)$-dimensional space. In case $n-m=2$, we have projecting on a two-dimensional plane.

Let us consider a model example of the solution of a five-dimensional linear optimization problem, which is based on projecting multidimensional space to two-dimensional space.

To solve the linear optimization problem

$$W_I = 13x_1 + 7x_2 + 2x_3 - x_4 + 2x_5 \to \max,$$

$$\Omega_I : \begin{cases} 10x_1 + 10x_2 + x_3 + x_4 + x_5 = 179, \\ 19x_1 + 14x_2 + x_3 + 2x_4 + 2x_5 = 298, \\ 4x_1 + 5x_2 + x_4 = 69, \end{cases} \qquad (1)$$

$$x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 \ge 0, \quad x_4 \ge 0, \quad x_5 \ge 0.$$

Solution. The constraints system (1) consists of three independent equations. Let us proceed from the canonical forms of the problem representation to the standard one. Such transition is performed by solving the system (1) by the Jordan-Gauss method (Table 1). We select the three x_3, x_4, x_5 as basis variables.

Table 1

Calculations by basis x_3, x_4, x_5.

	x_1	x_2	x_3	x_4	x_5	b	$\sum$
	10	10	1	1	1	179	202
	19	14	1	2	2	298	336
	4	5	0	1	0	69	79
W_I	13	7	2	–1	2	151	
	10	10	1	1	1	179	202
	–1	–6	–1	0	0	–60	–68
	4	5	0	1	0	69	79
W_I	–7	–13	0	–3	0	–207	
	9	4	0	1	1	119	134
	1	6	1	0	0	60	68
	4	5	0	1	0	69	79
W_I	–7	–13	0	–3	0	–207	
	5	–1	0	0	1	50	55
	1	6	1	0	0	60	68
	4	5	0	1	0	69	79
W_I	5	2	0	0	0	0	

From the last chain of Table 1, we have the resolved system

$$\begin{cases} 4x_1 + 5x_2 + x_3 = 69, \\ 5x_1 - x_2 + x_4 = 50, \\ x_1 + 6x_2 + x_5 = 60. \end{cases} \qquad (2)$$

4. 1. Projection onto Ox_1x_2

Rejecting the basis variables, we ensure a transition to two-dimensional inequalities. Projection of the five-dimensional original problem onto coordinate plane Ox_1x_2 takes the form:

$$W_I = 4x_1 + 5x_2 \to \max,$$

$$\Omega_I^{Ox_1x_2} : \begin{cases} 4x_1 + 5x_2 \le 69, \\ 5x_1 - x_2 \le 50, \\ x_1 + 6x_2 \le 60, \end{cases}$$

$$x_1 \ge 0, \quad x_2 \ge 0.$$

Graphical solution will be shown in Fig. 1.

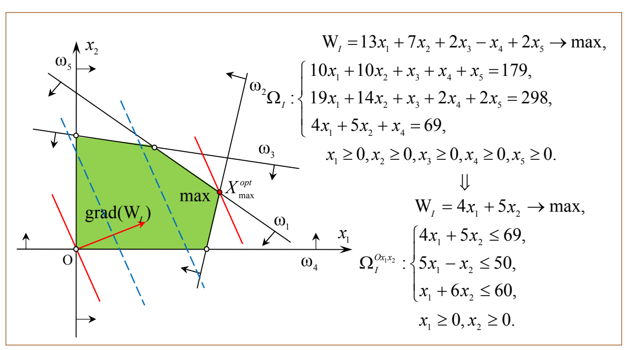

Fig. 1. Projection onto Ox_1x_2

The coordinates of the extreme vertex are the solution to the system

$$X_{\max}^{opt}:\omega_1\times\omega_2\Leftrightarrow\begin{cases}4x_1+5x_2=69,\\5x_1-x_2=50,\end{cases}\Leftrightarrow\begin{cases}x_1=11,\\x_2=5.\end{cases}$$

Other coordinates will be obtained from the resolved system (2). Thus, the optimal solution to the original problem is equal to

$$X_{\max}^{opt}=[\ 11,\ 5,\ 19,\ 0,\ 0\ ].$$

The largest value of objective function will be $W_I^{\max}=65$.

Note. Selection of three variables out of five is possible with the use of ten methods $C_5^3=10$. Let us consider all possible reductions of the problem.

4. 2. Projection onto Ox_1x_5

We select x_2, x_3, x_4 as basis variables. We solve the original system of constraints relative to variables x_2, x_3, x_4 by the Jordan-Gauss method (Table 2), but we use system (2) that is equivalent to (1).

Table 2

Calculations by basis x_2, x_3, x_4.

	x_1	x_2	x_3	x_4	x_5	b	Σ
	5	–1	0	0	1	50	55
	1	6	1	0	0	60	68
	4	5	0	1	0	69	79
W_I	5	2	0	0	0	0	
	–5	1	0	0	–1	–50	–55
	31	0	1	0	6	360	398
	29	0	0	1	5	319	354
W_I	15	0	0	0	2	100	

Table 2 gives the resolved system with basis variables x_2, x_3, x_4,

$$\begin{cases}-5x_1-x_5+x_2=-50,\\31x_1+6x_5+x_3=360,\\29x_1+5x_5+x_4=319.\end{cases}\qquad(3)$$

Neglecting non-negative basis variables x_2, x_3, x_4, we carry out a transition to the constraints-inequalities. The projection of the five-dimensional polyhedral of the original problem (1) is onto coordinate plane Ox_1x_5, otherwise the equivalent transition from the canonical form of the LO problem to the standard one takes the form of:

$$W_I=15x_1+2x_5-100\rightarrow\max,$$

$$\Omega_I^{Ox_1x_5}:\begin{cases}-5x_1-x_5\le-50,\\31x_1+6x_5\le360,\\29x_1+5x_5\le319,\end{cases}$$

$$0\le x_1,\quad 0\le x_5.$$

The graphical solution to the problem is shown in Fig. 2 where

$$\omega_1:5x_1+x_5=50,\ \ \omega_2:31x_1+6x_5=360,$$

$$\omega_3:29x_1+5x_5=319,\ \ \omega_4:x_2=0,\ \ \omega_5:x_1=0.$$

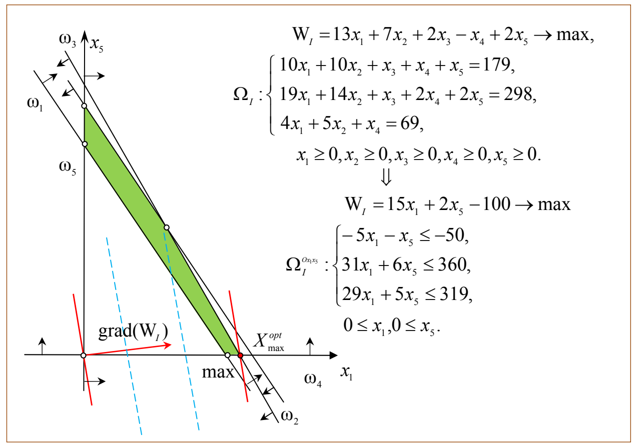

Fig. 2. Projection onto Ox_1x_5

The optimal vertex has coordinates $x_1=11$, $x_5=0$. Coordinates of x_2, x_3, x_4 will be calculated from system (3). Finally, $X_{\max}^{\text{opt}}=[\ 11, 5, 19, 0, 0\ ]$. The obtained optimal solution coincides with the previous ones, which proves correctness of the performed calculations.

4. 3. Projection onto Ox_2x_5

We select x_1, x_3, x_4 as basis variables.

We solve the original system of constraints with respect to variables x_1, x_3, x_4, using the table of the solution of the previous calculation (Table 2).

Table 3

Calculations by basis x_1, x_3, x_4

	x_1	x_2	x_3	x_4	x_5	b	Σ
	–5	1	0	0	–1	–50	–55
	31	0	1	0	6	360	398
	29	0	0	1	5	319	354
W_I	15	0	0	0	2	100	
	1	–1/5	0	0	1/5	10	11
	0	31/5	1	0	–1/5	50	57
	0	29/5	0	1	–4/5	29	35
W_I	0	3	0	0	–1	–50	

We have the resolved system with basis variables x_1, x_3, x_4

$$\begin{cases}-\dfrac{1}{5}x_2+\dfrac{1}{5}x_5+x_1=10,\\ \dfrac{31}{5}x_2-\dfrac{1}{5}x_5+x_3=50,\\ \dfrac{29}{5}x_2-\dfrac{4}{5}x_5+x_4=29.\end{cases}\qquad(4)$$

Neglecting the non-negative basis variables, we ensure a transition to constraints-inequalities. Projection of a five-dimensional polyhedral of the original LO problem onto coordinate plane Ox_2x_5 takes the form:

$$W_I=3x_2-x_5+50\rightarrow\max,$$

$$\Omega_I^{Ox_2x_5}:\begin{cases}-\frac{1}{5}x_2+\frac{1}{5}x_5\le 10,\\ \frac{31}{5}x_2-\frac{1}{5}x_5\le 50,\\ \frac{29}{5}x_2-\frac{4}{5}x_5\le 29,\end{cases}$$

$0\le x_2, 0\le x_5.$

Graphical solution to the problem is shown in Fig. 3 where

$$\omega_1:-\frac{1}{5}x_2+\frac{1}{5}x_5=10,\quad \omega_2:\frac{31}{5}x_2-\frac{1}{5}x_5=50,$$

$$\omega_3:\frac{29}{5}x_2-\frac{4}{5}x_5=29,\quad \omega_4:x_5=0,\quad \omega_5:x_2=0.$$

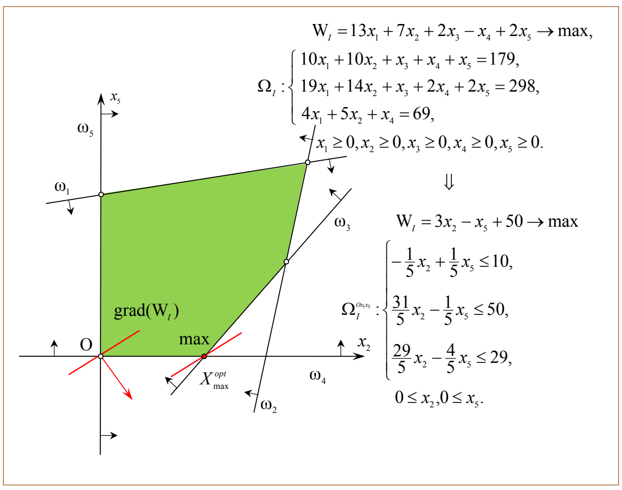

Fig. 3. Projection onto Ox_2x_5

The optimal vertex has coordinates $x_2=5,\ \ x_5=0$. Coordinates x_1, x_3, x_4 will be calculated from system (4). Finally, $X_{\max}^{\text{opt}}=[\ 11, 5, 19, 0, 0\ ]$.

4. 4. Projection onto Ox_4x_5

We select x_1, x_2, x_3 as basis variables. We solve the original constraints system with respect to variables x_1, x_2, x_3 (Table 4), using the previous table of the solution (Table 3).

Table 4

Calculations by basis x_1, x_2, x_3

	x_1	x_2	x_3	x_4	x_5	b	Σ
	–5	1	0	0	–1	–50	–55
	31	0	1	0	6	360	398
	29	0	0	1	5	319	354
W_I	15	0	0	0	2	100	
	0	1	0	5/29	–4/29	5	175/29
	0	0	1	–31/29	19/29	19	568/29
	1	0	0	1/29	5/29	11	354/29
W_I	0	0	0	–15/29	–17/29	–65	

We have the resolved system with basis variables x_1, x_2, x_3

$$\begin{cases}\frac{5}{29}x_4-\frac{4}{29}x_5+x_2=5,\\ -\frac{31}{29}x_4+\frac{19}{29}x_5+x_3=19,\\ \frac{1}{29}x_4+\frac{5}{29}x_5+x_1=11.\end{cases}\qquad (5)$$

Projection of the five-dimensional polyhedral of the original LO problem onto coordinate plane Ox_4x_5 takes the form:

$$W_I=-\frac{15}{29}x_4-\frac{17}{29}x_5+65\to\max,$$

$$\Omega_I^{Ox_4x_5}:\begin{cases}5x_4-4x_5\le 145,\\ -31x_4+19x_5\le 551,\\ x_4+5x_5\le 319,\end{cases}$$

$0\le x_4,\ \ 0\le x_5.$

The graphical solution to the problem is shown in Fig. 4 where

$$\omega_1:\frac{5}{29}x_4-\frac{5}{29}x_5=5,$$

$$\omega_2:-\frac{31}{29}x_4+\frac{19}{29}x_5+19,$$

$$\omega_3:\frac{1}{29}x_4+\frac{5}{29}x_5=11,$$

$$\omega_4:x_5=0,\quad \omega_5:x_4=0.$$

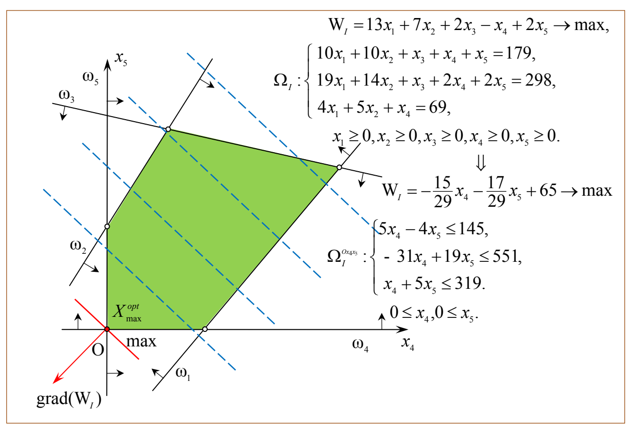

Fig. 4. Projection onto Ox_4x_5

The optimal vertex has coordinates $x_4=0,\ x_5=0$. Coordinates x_1, x_2, x_3 will be calculated from system (5). Finally, $X_{\max}^{\text{opt}}=[\ 11, 5, 19, 0, 0\ ]$. The obtained optimal solution coincides with the previous calculations.

4. 5. Projection onto $Ox_2\ x_3$

We select x_1, x_4, x_5 as basis variables.

We solve the original constraints system with respect to variables x_1, x_4, x_5 from Table 5, using the previous table of the solution (Table 1).

Table 5

Calculations by basis x_1, x_4, x_5.

	x_1	x_2	x_3	x_4	x_5	b	$\sum$
	5	–1	0	0	1	50	55
	1	6	1	0	0	60	68
	4	5	0	1	0	69	79
W_I	5	2	0	0	0	0	
	0	–31	–5	0	1	–250	–285
	1	6	1	0	0	60	68
	0	–19	–4	1	0	–171	–193
W_I	0	–28	–5	0	0	–300	

We have the resolved system with basis variables x_1, x_4, x_5. From the last chain, we obtain the solved system

$$\begin{cases} -31x_2 - 5x_3 + x_5 = -250, \\ 6x_2 + x_3 + x_1 = 60, \\ -19x_2 - 4x_3 + x_4 = -171. \end{cases} \quad (6)$$

Rejecting the basis variables, we provide a transition to two-dimensional inequalities. Projection of the five-dimensional original problem onto coordinate plane $Ox_2\,x_3$ takes the form:

$$W_I = -28x_2 - 5x_3 + 300 \to \max,$$

$$\Omega_I^{Ox_2x_3}: \begin{cases} -31x_2 - 5x_3 \le -250, \\ 6x_2 + x_3 \le 60, \\ -19x_2 - 4x_3 \le -171, \end{cases}$$

$$x_2 \ge 0, \; x_3 \ge 0.$$

The graphical solution is shown in Fig. 5 where

$$\omega_1: 31x_2 + 5x_3 = 250, \quad \omega_2: 6x_2 + x_3 = 60,$$

$$\omega_3: 19x_2 + 4x_3 = 171, \quad \omega_4: x_3 = 0, \quad \omega_5: x_2 = 0.$$

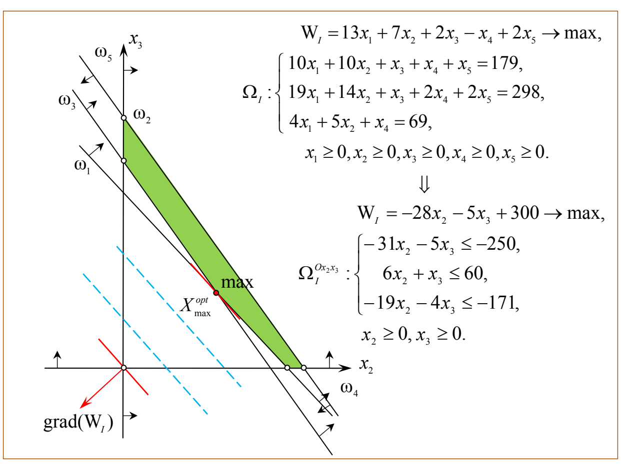

Fig. 5. Projection onto Ox_2x_3

The coordinates of the extreme vertex are the solution to the system

$$X_{\max}^{opt}: \omega_1 \times \omega_3 \Leftrightarrow \begin{cases} 31x_2 + 5x_3 = 250 \\ 19x_2 + 4x_3 = 171 \end{cases} \Leftrightarrow \begin{cases} x_2 = 5, \\ x_3 = 19. \end{cases}$$

Other coordinates will be obtained from the solved systems (6). Thus, the optimal solution to the original problem is equal to

$$X_{\max}^{opt} = [\, 11,\ 5,\ 19,\ 0,\ 0 \,].$$

The largest value of the objective function will be $W_I^{\max} = 65$.

4. 6. Projection onto $Ox_2\,x_4$

We select the combination of x_1, x_3, x_5 as basis variables. We solve the original constraints system with respect to variables x_1, x_3, x_5 from Table 6, using the previous table of solution (Table 5).

Table 6

Calculations by basis x_1, x_3, x_5.

	x_1	x_2	x_3	x_4	x_5	b	$\sum$
	0	–31	–5	0	1	–250	–285
	1	6	1	0	0	60	68
	0	–19	–4	1	0	–171	–193
W_I	0	–28	–5	0	0	–300	
	0	–29/4	0	–5/4	1	–145/4	–175/4
	1	5/4	0	1/4	0	69/4	79/4
	0	19/4	1	–1/4	0	171/4	193/4
W_I	0	–17/4	0	–5/4	0	–345/4	

We have the resolved system with basis variables x_1, x_3, x_5

$$\begin{cases} -\dfrac{29}{4}x_2 - \dfrac{5}{4}x_4 + x_5 = -\dfrac{145}{4}, \\ \dfrac{5}{4}x_2 + \dfrac{1}{4}x_4 + x_1 = \dfrac{69}{4}\dfrac{69}{4}, \\ \dfrac{19}{4}x_2 - \dfrac{1}{4}x_4 + x_3 = \dfrac{171}{4}. \end{cases} \quad (7)$$

We proceed to constraints-inequalities. This transition ensures projecting of the original five-dimensional problem (1) onto coordinate plane $Ox_2\,x_4$ and takes the form:

$$W_I = -\frac{17}{4}x_2 - \frac{5}{4}x_3 + \frac{345}{4} \to \max,$$

$$\Omega_I^{Ox_2x_4}: \begin{cases} -29x_2 - 5x_4 \le -145, \\ 5x_2 + x_4 \le 69, \\ 19x_2 - x_4 \le 171, \end{cases}$$

$$x_2 \ge 0, \quad x_4 \ge 0.$$

The graphical solution is shown in Fig. 6 where

$$\omega_1: 29x_2 + 5x_4 = 145,$$

$$\omega_2: 5x_2 + x_4 = 69,$$

$$\omega_3: 19x_2 - x_4 = 171,$$

$$\omega_4: x_4 = 0, \quad \omega_5: x_2 = 0.$$

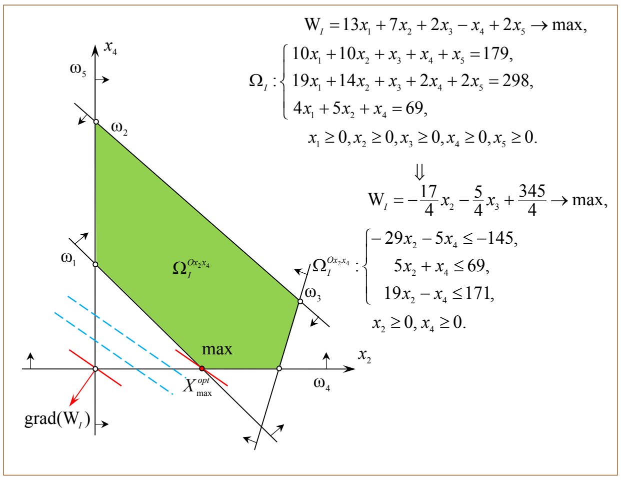

Fig. 6. Projection onto Ox_2x_4

Coordinates of extreme vertex are the solution to the system

$$X_{\max}^{opt} : \omega_1 \times \omega_4 \Leftrightarrow \begin{cases} 29x_2 + 5x_4 = 145 \\ x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_2 = 5, \\ x_4 = 0. \end{cases}$$

Other coordinates will be obtained from the solved system (7). The optimal solution to the original problem is equal to:

$$X_{\max}^{opt} = [11,\ 5,\ 19,\ 0,\ 0].$$

The largest value of the objective function will be $W_I^{\max} = 65$.

4. 7. Projection onto $Ox_1\ x_3$

Let x_2, x_4, x_5 be the basis variables. We solve the original constraints system regarding variables x_2, x_4, x_5 (Table 7), but, to reduce the amount of calculations, we use the previous table of solution (Table 5).

Table 7

Calculations by basis x_2, x_4, x_5

	x_1	x_2	x_3	x_4	x_5	b	Σ
	0	−31	−5	0	1	−250	−285
	1	6	1	0	0	60	68
	0	−19	−4	1	0	−171	−193
W_I	0	−28	−5	0	0	−300	
	31/6	0	1/6	0	1	60	199/3
	1/6	1	1/6	0	0	10	34/3
	19/6	0	−5/6	1	0	19	67/3
W_I	14/3	0	−1/3	0	0	−20	

We have the resolved system with basis variables x_2, x_4, x_5. It takes the algebraic form of:

$$\begin{cases} \frac{31}{6}x_1 + \frac{1}{6}x_3 + x_5 = 60, \\ \frac{1}{6}x_1 + \frac{1}{6}x_3 + x_2 = 10, \\ \frac{19}{6}x_2 - \frac{5}{6}x_4 + x_4 = 19. \end{cases} \qquad (8)$$

We proceed to constraints–inequalities. The projection of the original five-dimensional problem (1) onto coordinate plane $Ox_1\ x_3$ takes the form:

$$W_I = \frac{14}{3}x_1 - \frac{1}{3}x_3 + 20 \to \max,$$

$$\Omega_I^{Ox_1x_3} : \begin{cases} 31x_1 + x_3 \leq 360, \\ x_1 + x_3 \leq 60, \\ 19x_1 - 5x_3 \leq 114, \end{cases}$$

$$x_1 \geq 0, \quad x_3 \geq 0.$$

The graphical solution is shown in Fig. 7 where

$$\omega_1 : 31x_1 + x_3 = 360,$$

$$\omega_2 : x_1 + x_3 = 60,$$

$$\omega_3 : 19x_1 - 5x_3 = 114,$$

$$\omega_4 : x_3 = 0, \quad \omega_5 : x_1 = 0.$$

Coordinates of the extreme vertex are the solution to the system

$$X_{\max}^{opt} : \omega_1 \times \omega_3 \Leftrightarrow \begin{cases} 31x_1 + x_3 = 360 \\ 19x_1 - 5x_3 = 114 \end{cases} \Leftrightarrow \begin{cases} x_1 = 11, \\ x_3 = 19. \end{cases}$$

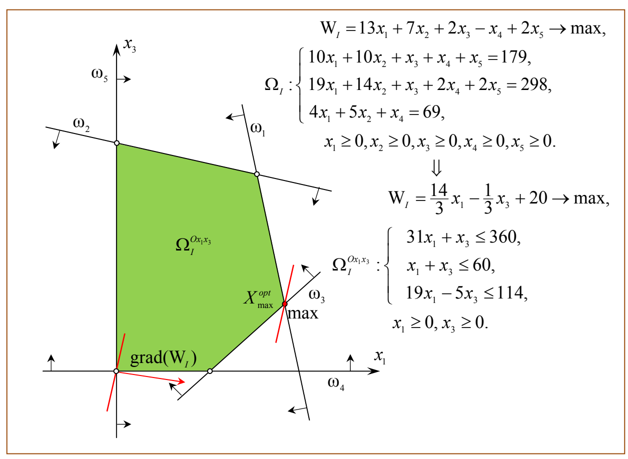

Fig. 7. Projection onto Ox_1x_3

Other coordinates will be obtained from the solved systems (8). Thus, the optimal solution to the original problem is equal to

$$X_{\max}^{opt} = [11,\ 5,\ 19,\ 0,\ 0].$$

The largest value of the objective function will be

$$W_I^{\max} = 65.$$

4. 8. Projection onto $Ox_3\ x_4$

Let x_1, x_2, x_5 be the basis variables. We solve the original constraints system with respect to variables x_1, x_2, x_5 (Table 8). To reduce the amount of calculations, we use the previous table of solution (Table 7).

Table 8

Calculations by basis x_1, x_2, x_5

	x_1	x_2	x_3	x_4	x_5	b	$\sum$
	31/6	0	1/6	0	1	60	199/3
	1/6	1	1/6	0	0	10	34/3
	19/6	0	–5/6	1	0	19	67/3
W_I	14/3	0	–1/3	0	0	–20	
	0	0	29/19	–31/19	1	29	568/19
	0	1	4/19	–1/19	0	9	193/19
	1	0	–5/19	6/19	0	6	134/19
W_I	0	0	17/19	–28/19	0	–48	

We have the resolved system with basis variables x_1, x_2, x_5. It takes the algebraic form of:

$$\begin{cases} \frac{29}{19}x_3 - \frac{31}{19}x_4 + x_5 = 29, \\ \frac{4}{19}x_3 - \frac{1}{19}x_4 + x_2 = 9, \\ -\frac{5}{19}x_3 + \frac{6}{19}x_4 + x_1 = 6. \end{cases} \qquad (9)$$

We proceed to constraints–inequalities. Projection of the five-dimensional problem (1) onto coordinate plane Ox_3 x_4 takes the form:

$$W_I = \frac{17}{19}x_3 - \frac{28}{19}x_4 + 48 \to \max,$$

$$\Omega_I^{Ox_3x_4} : \begin{cases} 29x_3 - 31x_4 \le 551, \\ 4x_3 - x_4 \le 171, \\ -5x_3 + 6x_4 \le 114, \end{cases}$$

$$x_3 \ge 0, \quad x_4 \ge 0.$$

The graphical solution is shown in Fig. 8 where

$$\omega_1 : 29x_3 - 31x_4 = 551, \quad \omega_2 : 4x_3 - x_4 = 171,$$

$$\omega_3 : \text{-}5x_3 + 6x_4 = 114, \quad \omega_4 : x_3 = 0, \quad \omega_5 : x_4 = 0.$$

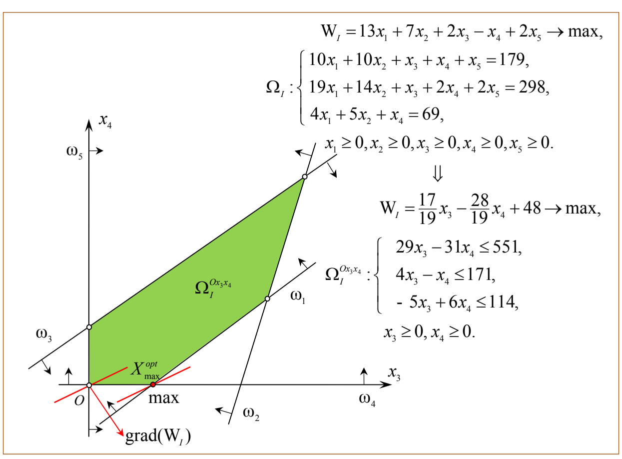

Fig. 8. Projection onto Ox_3x_4

Coordinates of the optimal vertex are the solution to the system

$$X_{\max}^{opt} : \omega_1 \times \omega_4 \Leftrightarrow \begin{cases} 29x_3 - 31x_4 = 551 \\ x_4 = 0 \end{cases} \Leftrightarrow \begin{cases} x_3 = 19, \\ x_4 = 0. \end{cases}$$

Other coordinates will be obtained from the solved system (9). The optimal solution to the original problem is equal to

$$X_{\max}^{opt} = [\ 11, 5, 19, 0, 0\ ].$$

The largest value of the objective function will be $W_I^{\max} = 65$.

4. 9. Projection onto $Ox_1\ x_4$

Let x_1, x_2, x_5 be the basis variables. We solve the original system of constraints regarding variables x_1, x_2, x_5 (Table 9), but, to reduce the amount of calculation, we will use the previous table of solution (Table 8).

Table 9

Calculations by basis x_1, x_2, x_5

	x_1	x_2	x_3	x_4	x_5	b	$\sum$
	0	0	29/19	–31/19	1	29	568/19
	0	1	4/19	–1/19	0	9	193/19
	1	0	–5/19	6/19	0	6	134/19
W_I	0	0	17/19	–28/19	0	–48	
	29/5	0	0	1/5	1	319/5	354/5
	4/5	1	0	1/5	0	69/5	79/5
	–19/5	0	1	–6/5	0	–114/5	–134/5
W_I	17/5	0	0	–2/5	0	–138/5	

We have the resolved system with basis variables x_1, x_2, x_5. It takes the algebraic form:

$$\begin{cases} \frac{29}{5}x_1 + \frac{1}{5}x_4 + x_5 = \frac{319}{5}, \\ \frac{4}{5}x_1 + \frac{1}{5}x_4 + x_2 = \frac{69}{5}, \\ -\frac{19}{5}x_1 - \frac{6}{5}x_4 + x_3 = -\frac{114}{5}. \end{cases} \qquad (10)$$

We proceed to constraints–inequalities. The projection of the five-dimensional problem (1) onto coordinate plane Ox_1x_4 takes the form:

$$W_I = \frac{17}{5}x_1 - \frac{2}{5}x_4 + \frac{138}{5} \to \max,$$

$$\Omega_I^{Ox_1x_4} : \begin{cases} 29x_1 + x_4 \le 319, \\ 4x_1 + x_4 \le 69, \\ -19x_1 - 6x_4 \le -114, \end{cases}$$

$$x_1 \ge 0, \quad x_4 \ge 0.$$

The graphical solution is shown in Fig. 9 where

$$\omega_1 : \ 29x_1 + x_4 = 319,$$

$$\omega_2 : 4x_1 + x_4 = 69,$$

$$\omega_3 : 19x_1 + 6x_4 = 114, \quad \omega_4 : x_1 = 0, \quad \omega_5 : x_4 = 0.$$

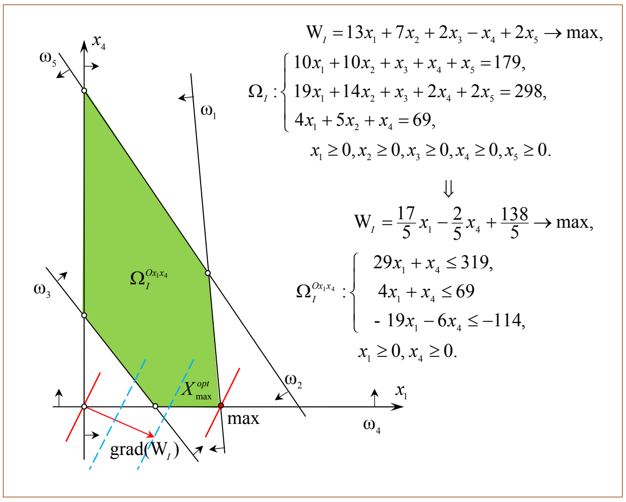

Fig. 9. Projection onto $Ox_1\,x_4$

Coordinates of the optimal vertex are the solution to the system

$$X_{\max}^{opt}: \omega_1 \times \omega_4 \Leftrightarrow \begin{cases} 29x_1 + x_4 = 319 \\ x_4 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} x_1 = 11, \\ x_4 = 0. \end{bmatrix}$$

Other coordinates will be obtained from the solved system (10). The optimal solution to the original problem is equal to

$$X_{\max}^{opt} = [11,\ 5,\ 19,\ 0,\ 0].$$

The largest value of the objective function will be $W_I^{\max} = 65$.

4. 10. Projection onto $Ox_3\,x_5$

We accept x_1, x_2, x_4 as basis variables. We solve the original system of constraints with respect to variables x_1, x_2, x_4 (Table 10), but, to reduce the amount of calculation, we use the table of solution (Table 8).

Table 10

Calculations by basis x_1, x_2, x_4

	x_1	x_2	x_3	x_4	x_5	b	Σ
	0	0	29/19	–31/19	1	29	568/19
	0	1	4/19	–1/19	0	9	193/19
	1	0	–5/19	6/19	0	6	134/19
W_I	0	0	17/19	–28/19	0	–48	
	0	0	–29/31	1	–19/31	–551/31	–568/31
	0	1	5/31	0	–1/31	250/31	285/31
	1	0	1/31	0	6/31	360/31	398/31
W_I	0	0	–15/31	0	–28/31	–2300/31	

We have the resolved system with basis variables x_1, x_2, x_4. It takes the algebraic form:

$$\begin{cases} -\dfrac{29}{31}x_3 - \dfrac{19}{31}x_5 + x_4 = -\dfrac{551}{31}, \\ \dfrac{5}{31}x_3 - \dfrac{1}{31}x_5 + x_2 = \dfrac{250}{31}, \\ \dfrac{1}{31}x_3 + \dfrac{6}{31}x_5 + x_1 = \dfrac{360}{31}. \end{cases} \quad (11)$$

We proceed to constraints–inequalities. The projection of the original five-dimensional problem (1) onto coordinate plane $Ox_3\,x_5$ takes the form:

$$W_I = -\frac{15}{31}x_3 - \frac{28}{31}x_5 + \frac{2300}{31} \to \max,$$

$$\Omega_I^{Ox_3x_5}: \begin{cases} 29x_3 + 19x_4 \le 551, \\ 5x_3 - x_5 \le 250, \\ x_3 + 6x_5 \le 360, \end{cases}$$

$$x_3 \ge 0, \quad x_5 \ge 0.$$

The graphical solution is shown in Fig. 10 where

$$\omega_1:\ 29x_3 + 19x_4 = 551, \quad \omega_2: 5x_3 - x_5 = 250,$$

$$\omega_3: x_3 + 6x_5 = 360, \quad \omega_4: x_3 = 0, \quad \omega_5: x_5 = 0.$$

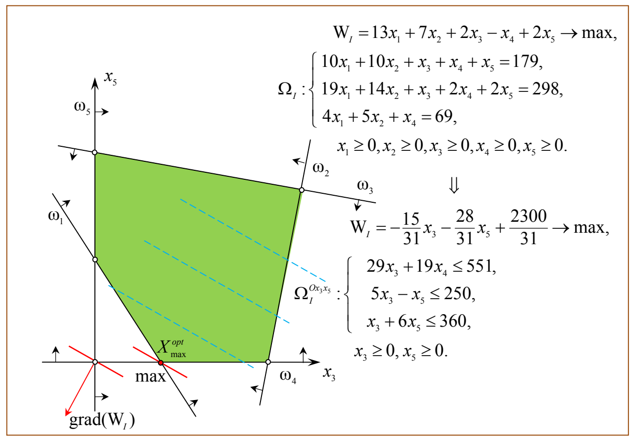

Fig. 10. Projection onto $Ox_3\,x_5$

Coordinates of the optimal vertex are the solution to the system

$$X_{\max}^{opt}: \omega_1 \times \omega_4 \Leftrightarrow \begin{cases} 29x_3 + 19x_4 = 551 \\ x_4 = 0 \end{cases} \Leftrightarrow \begin{bmatrix} x_3 = 19, \\ x_4 = 0. \end{bmatrix}$$

Other coordinates will be obtained from the solved system (10). The optimal solution of the original problem is equal to

$$X_{\max}^{opt} = [11,\ 5,\ 19,\ 0,\ 0].$$

The largest value of the objective function will be $W_I^{\max} = 65$.

8. Discussion of results of studying the optimization calculations

The proposed approach to simplification of the combinatorial solution to a discrete optimization problem has significant benefits unlike the known methods of determining the optimal solution, such as the simplex method or the method of artificial basis [3]. In fact, the performed decomposition of the system reduces the dimensionality of the system of equations to be solved. The projection of the multi-dimensional system of an original problem onto a two-dimensional coordinate plane makes it possible to obtain a simple system of graphic solution to a complex problem of linear discrete optimization. From the practical point of view, the proposed method makes it possible to simplify the calculation complexity of optimization problems of such a class.

It was shown that the optimal solution to the original problem by all 10 projections is equal to:

$$X_{\max}^{opt} = [11,\ 5,\ 19,\ 0,\ 0].$$

The largest value of the objective function: $W_I^{\max} = 65.$

The obtained scientific result allows us to conclude that in the general case there is no need to search for a solution by all projections. It will suffice to determine the solution only for one projection.

The applied value of the proposed approach is the use of obtained results to provide a possibility to improve complex systems that are described by systems of linear equations with existence of the systems of linear constraints. Multivaluedness of combinatorial projections determines the possibility to change the set of parameters of the problem. Projecting a multi-dimensional optimization process on a two-dimensional plane was proposed.

This simplification method can be applied only to the prepared classes of problems. The rank of *m* matrix of coefficients of the system of constraints for a linear discrete optimization must problem must satisfy the condition of n–m=2, where n is the dimensionality of the problem. Generalization of such projecting on a three-dimensional space is expedient.

9. Conclusions

1. It is shown that the solution to a linear optimization problem is possible by simplification with the use of decomposition of a system due to construction of projections of a multi-dimensional system of the original problem onto two-dimensional coordinate planes.

2. It was proved by the example of solving a typical model problem that the proposed approach allows obtaining a simple system of graphic solutions of a complex problem of linear discrete optimization. The obtained result makes it possible to conclude that in the general case, there is no need to search for solutions by all projections. It is enough to determine the solution only by one projection.

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