Досліджується проблема існування обмежених розв'язків на всій дійсній осі лінійних неоднорідних систем диференційних рівнянь, які зазнають імпульсні збурення у фіксовані моменти часу. Знайдено достатні умови гіперболічності розв'язків однорідної багатовимірної системи диференціальних рівнянь з імпульсним впливом. Отримані умови застосовано для дослідження обмежених розв'язків неоднорідної імпульсної системи. Сформульовано достатні умови існування єдиного обмеженого розв'язку до неоднорідної системи для випадку слабкої регулярності відповідної однорідної системи. Перевагою такого підходу є те, що знайдені умови можуть бути ефективно перевірені для конкретних класів імпульсно-збурених систем, оскільки сформульовані у термінах коефіцієнтів вихідних задач. Отримані умови дозволяють застосувати класичні методи диференціальних рівнянь для одержання тверджень про розв'язність та неперервну залежність розв'язків від параметрів імпульсної системи.

Теорія систем з імпульсним впливом має широке коло застосувань. Численні еволюційні процеси у фізиці, техніці, автоматичному регулюванні, біології, економіці на протязі свого розвитку можуть піддаватись короткочасним впливам, тривалістю яких можна знехтувати. Наприклад, процеси із скачкоподібними змінами спостерігаються у механіці (рух пружини при ударному впливі, робота годинникового механізму, зміна швидкості ракети при відокремленні ступенів), в радіотехніці (генерація імпульсів), в біології (робота серця, поділ клітин, передача сигналів нейронами), в теорії контролю (робота промислових роботів). Тому якісне дослідження імпульсних систем у даній роботі є актуальною задачею в сучасній теорії математичного моделювання

Ключові слова: диференціальні рівняння, імпульсна система, обмежений роз'вязок, функція Гріна-Самойленка, регулярні роз'вязки

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FINDING OF BOUNDED SOLUTIONS TO LINEAR IMPULSIVE SYSTEMS

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1. Introduction

During the mathematical description of evolution of real processes with short-term perturbations, it is convenient to neglect the durations of these perturbations and consider that these perturbations have an "instantaneous" character. Such idealization leads to the necessity of studying dynamical systems with discontinuous trajectories or, as they are also called, differential equations with impulsive actions.

The growing interest in the systems with discontinuous trajectories in recent years is primarily due to numerous applications. Nowadays, various impulsive systems of automatic regulation, impulsive computing systems occupy a very significant place in modern mathematical modeling. Therefore, the study of the qualitative behavior of impulsive systems is a relevant task both for the theory of differential equations and for its modern applications.

Studying various problems of natural science, one often has to deal with evolutionary processes, which are described by ordinary differential equations and undergo short-term perturbations. During the mathematical modeling of such processes, it is often convenient to neglect the duration of such perturbations and consider them as an impulse (push, shock).

Such idealization leads to the need to study the systems of differential equations, the solutions of which change abruptly. However, the idealization of the replacement of short-term perturbations with "instantaneous" ones is not the only reason for the emergence of differential equations with discontinuous trajectories. Often the discontinuities of certain dependences in the studied system are their essential characteristics.

An example of such behavior is the movement of a steel ball falling freely from a certain height onto a horizontal steel surface. We can see that in the mathematical model of this process the velocity of the ball "instantly" changes its sign at the moment of hitting the steel plate.

The results of the paper can be successfully used for the study of vibrational processes in a variety of mechanical and electromechanical systems with discontinuous characteristics and in the study of multi-frequency vibrational processes of discontinuous systems.

2. Literature review and problem statement

The most fruitful and effective studies of impulsive systems have been conducted in the last decade. Sufficient conditions for the asymptotic stability of zero solution to nonlinear systems on the basis of the Lyapunov's direct method are obtained in [1]. In [2], the review of the most modern methods of stability analysis of solutions to impulse differential equations and their application to the problems of impulse control is carried out. In [3], the invariance and stability of global attractors in multi-valued impulsive dynamical systems are investigated.

In [4], sufficient conditions for the existence of asymptotically stable invariant toroidal manifold of linear extensions of dynamical systems on torus in the case when the system matrix commutes with its integral are obtained. The proposed approach is applied to the study of stability of invariant sets of a class of discontinuous dynamical systems. Exponential stability of the trivial torus for a certain class of nonlinear extensions of dynamical systems on torus is proved in [5]. The obtained results are applied to the study of stability of toroidal manifolds of impulsive dynamical systems. In [6], the problem of constructing approximate adaptive control, including the case of impulse control, for a certain infinite-dimensional problem with the cost functional of the Nemytsky type is considered. The method of averaging for obtaining approximate adaptive control is justified. The concept of impulsive non-autonomous dynamical system is introduced in [7]. The existence and properties of the impulsive attracting set are investigated for such systems. The obtained results are applied to the study of stability of the two-dimensional impulsively perturbed Navier-Stokes system. In [8], the recursive properties of almost periodic motions of impulsive dynamical systems are studied. The obtained results are applied to the study of qualitative behavior of discrete-time systems. In [9], the stability properties with respect to the external (controlling) perturbations for systems of the differential equations with impulsive influences at the fixed moments of time are considered. Necessary and sufficient stability conditions for the classes of impulsive systems possessing the function of Lyapunov type are obtained. In [10], non-autonomous systems of the reaction-diffusion type with impulsive effects at fixed moments of time are considered. The corresponding non-autonomous dynamical system is constructed, for which the existence of a uniform attractor is proved. In [11], a non-autonomous evolution inclusion with impulse effects at fixed time moments is considered. The corresponding non-autonomous multi-valued dynamical system is constructed, for which a compact global attractor in the phase space is proved to exist. In [12], the existence of global attractors in discontinuous infinite-dimensional dynamical systems, which can have trajectories with an infinite number of impulsive perturbations, is proved. The obtained results are applied to the study of asymptotic behavior of weakly nonlinear impulsively perturbed parabolic equations. In [13], the model of neuron firing is considered, which is defined by the system of differential equations with impulsive influence. The system describes the dynamics of the potential on the membrane, when some current is applied to the input, the voltage on the membrane increases with time to some critical value.

In all the above papers, the bases of the qualitative theory of differential equations with impulses are presented. In fact, the foundations of the qualitative theory of impulsive systems are established, based on the qualitative theory of ordinary differential equations, methods of asymptotic integration of such equations, and theory of difference equations and generalized functions. Nevertheless, the question of the existence of solutions to hyperbolic linear impulsive systems is not considered. At the same time, some classes of impulsive systems, such as multi-dimensional nonhomogeneous impulsive systems, are not fully investigated. The questions of the existence and structural arrangement of integral sets of systems of differential equations subject to impulsive perturbations at the fixed moments of time, and at the moments when a phase point hits predefined subsets of the phase space remain unsolved.

3. The aim and objectives of the study

The aim of the present study is to find bounded solutions to multidimensional nonhomogeneous systems of differential equations with impulsive perturbations at fixed moments of time. This will allow using impulsive systems of the specified type for modeling the dynamics of evolutionary processes which parameters are exposed to sharp changes at the predefined moments of time as a result of almost instantaneous external influences.

In order to achieve this aim, the following objectives are set:

– to find sufficient conditions of hyperbolicity of solutions to the homogeneous multidimensional system of differential equations with impulsive influences;

– to use the obtained conditions for the theoretical study of bounded solutions to the nonhomogeneous impulsive system;

– to test the possibility of obtaining solutions on the example of the "integrate-and-fire" neuron model.

4. Finding bounded solutions of multidimensional nonhomogeneous systems of differential equations with impulsive influences at fixed moments of time

4. 1. Sufficient conditions for the hyperbolicity of a homogeneous system

We consider a linear system of ordinary differential equations with impulses at fixed moments of time

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P(t)x + f(t), \quad t \neq \tau_i,$$
$$\Delta x\big|_{t=\tau_i} \equiv x(t_i+0) - x(t_i-0) = B_i x + a_i, \tag{1}$$

where $x \in \mathbb{R}^n$, $P(t)$ – continuous (piecewise continuous) in the interval $I=(\alpha,\beta)$ $n\times n$ – dimensional matrix and bounded on $\mathbb{R}$; B_i – constant matrices; a_i – constant vectors, $f(t)$ – vector-function. The sequence of the moments of impulsive jumps $\{\tau_i\}$ are such that $\tau_i \to -\infty$ when $i \to -\infty$ and $\tau_i \to +\infty$ when $i \to +\infty$. This assumption rules out the possibility of the existence of finite accumulation points in the impulsive sequence and prevents the emergence of so-called Zeno solutions. Additionally, we assume that uniformly with respect to $\tau \in \mathbb{R}$ there exists a finite limit

$$\lim_{T\to\infty} \frac{i(t,t+T)}{T}. \tag{2}$$

Limit (2) is widely used in stability analysis of impulsive systems [1] and may be interpreted as an inverse average time between moments of jumps [9].

In sequel, we are interested in the study of the existence of solutions to linear nonhomogeneous system (1) that are bounded on the entire real axis, based on the properties of the corresponding homogeneous system, which has the following form

$$\frac{\mathrm{d}x}{\mathrm{d}t}=P(t)x+f(t),\quad t\neq\tau_i,\quad \Delta x\big|_{t=\tau_i}=B_i x. \tag{3}$$

The relationship between the properties of solutions of systems (1) and (3) will be defined in Theorems 1 and 2.

We introduce the necessary definitions.

Definition 1. Homogeneous system (3) is called weakly regular on $\mathbb{R}$ if the corresponding nonhomogeneous system (1) has at least one bounded on $\mathbb{R}$ solution for any bounded on $\mathbb{R}$ vector-function $f(t)$.

Definition 2. Homogeneous system (3) is called regular on $\mathbb{R}$ if the corresponding nonhomogeneous system (1) has a unique bounded on $\mathbb{R}$ solution for any bounded on $\mathbb{R}$ vector-function $f(t)$ and any bounded sequence $\{a_i\}$.

Definition 3. Homogeneous system (3) is called hyperbolic (or exponential dichotomous) on $\mathbb{R}$ if the phase space $\mathbb{R}^n$ can be decomposed into a direct sum of subspaces $\mathbb{R}^n=\mathbb{R}^k\oplus\mathbb{R}^{n-k}$ so that any solution $x(t,x_0)$ to (3) with $x_0\in\mathbb{R}^k$ satisfying the estimate

$$\|x(t,x_0)\|\le Ke^{-\gamma_1(t-\tau)}\|x(\tau,x_0)\|,\quad t\ge\tau \tag{4}$$

and any solution $x(t,x_0)$ to (3), with $x_0\in\mathbb{R}^{n-k}$ satisfying the estimate

$$\|x(t,x_0)\|\le K_1e^{-\gamma_1(t-\tau)}\|x(\tau,x_0)\|,\quad t\le\tau, \tag{5}$$

for any $\tau\in\mathbb{R}$ and some positive K, γ, K_1, γ_1, which do not depend on x_0 and τ.

The following lemma holds true.

Lemma 1. For system (3) to be a hyperbolic on the entire real axis it is necessary and sufficient that there exist a projecting map $P:\mathbb{R}^n\to\mathbb{R}^n$ and positive numbers K and γ, such that

$$\|X(t)PX^{-1}(\tau)\|\le Ke^{-\gamma(t-\tau)},\quad t\ge\tau,$$

$$\|X(t)(E-P)X^{-1}(\tau)\|\le Ke^{-\gamma(t-\tau)},\quad t\le\tau. \tag{6}$$

Definition 4. A set of initial values x_0 that lead to bounded on the entire axis $\mathbb{R}$ (or semi-axis $\mathbb{R}_+$, $\mathbb{R}_-$, respectively) solutions $x(t,x_0)$ to (3) will be denoted by $P(P_+,P_-)$ in the assumption that system (3) is defined on the entire axis $\mathbb{R}$ (on semi-axis $\mathbb{R}_+$, $\mathbb{R}_-$, respectively).

4. 2. Bounded solutions of a nonhomogeneous impulsive system

Theorem 1. Let system (3) be weakly regular on $\mathbb{R}$ and the projection map P project $\mathbb{R}^n$ on A. Then, for any bounded on $\mathbb{R}$ function $f(t)$ and sequence $\{a_i\}$ the system has a unique bounded on the entire axis solution $x=\varphi(t)$ that satisfies the condition $P\varphi(0)=0$. Moreover, there exists a constant c that does not depend on $f(t)$ and $\{a_i\}$ such that

$$\sup_{t\in R}\|\varphi(t)\|\le c\max\left\{\sup_{t\in R}\|f(t)\|,\sup_{i\in Z}\|a_i\|\right\}. \tag{7}$$

Proof. Let $f(t)$ be arbitrary continuous (piecewise continuous with discontinuities of the first kind at $t=\tau_i$) bounded on $\mathbb{R}$ function, $\{a_i\}$ be a bounded sequence. Due to the weak regularity of (3) the nonhomogeneous system (1) has a bounded solution $x=\varphi(t)$.

Let us denote by $\psi(t)$ a solution to the homogeneous system (3) satisfying the condition $\psi(0)=P\varphi(0)$. Obviously, the difference $z(t)=\varphi(t)-\psi(t)$ is a solution to the nonhomogeneous system (1) and $Pz(0)=0$. The solution $z(t)$ is unique, since assuming the existence of another solution $z_1(t)$ the nonhomogeneous system (1) satisfies the condition $Pz_1(0)=0$. Then, their difference should be bounded on the entire real axis solution to the homogeneous system (3) satisfying the condition $P[z(t)-z(0)]=0$, which is possible for the zero solution only.

Let us denote by M the set of tuples $(f(t),\{a_i\})$, where $f(t)$ is continuous (piecewise continuous with discontinuities of the first kind at $t=\tau_i$) bounded on the entire axis vector-function, $\{a_i\}$ is bounded sequence from $\mathbb{R}^n$. By introducing the norm

$$\|f(t),\{a_i\}\|_M=\max\left\{\sup_{t\in R}\|f(t)\|,\sup_{i\in Z}\|a_i\|\right\},$$

we turn M into a normed space. On the normed space M, let us define the operator F

$$F:(f(t),\{a_i\})\to\varphi(t),$$

that maps every element $(f(t),\{a_i\})\in M$ to the unique bounded solution of (1) satisfying $P\varphi(0)=0$. Operator F is invertible.

Let $M=FM$ be its image. By introducing the norm

$$\|\varphi(t)\|_M=\sup_{t\in R}\|\varphi(t)\|+\|F^{-1}\varphi(t)\|_M,$$

operator F becomes continuous. Let us check that space M is a complete metric space. Indeed, if $\{\varphi_m\}\subset M$, $m\in\mathbb{N}$ is a fundamental sequence, then it converges to some piecewise bounded function. Due to the continuity of the operator F^{-1} the sequence

$$\{F^{-1}\varphi_m(t)\}=\left\{(f_m(t),\{a_i^{(m)}\}\right\},\quad m\in\mathbb{N}$$

is fundamental in M and converges to some element $(f(t),\{a_i\})\in M$.

On the continuity intervals $(\tau_i,\tau_{i+1}), i\in\mathbb{Z}$ the functions $\varphi_m(t)$ are solutions to the corresponding differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=P(t)x+f_n(t),$$

and, hence, for $t\neq\tau_i$ function $\varphi(t)$ satisfies the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t}=P(t)x+f(t).$$

At $t=\tau_i$ functions $\varphi_m(t)$ undergo the discontinuity of the first kind with jumps

$$\Delta\varphi_m\big|_{t=\tau_i}=B_i\,\varphi_m(\tau_i)+a_i^{(m)},$$

and, hence, the limit function $\varphi(t)$ satisfies the relation

$$\Delta\varphi\big|_{t=\tau_i}=B_i\,\varphi(\tau_i)+a_i.$$

Since every function $\varphi_m(t)$ satisfies the condition $P\varphi_m(0)=0$ it holds that $P\varphi(0)=0$ and, hence, $\varphi(t)\in M$ which proves the completeness of the space M.

Since operator F^{-1} is continuous, due to Banach theorem on inverse operator, the operator F is also continuous and there exists a positive constant k such that if $F:(f(t),\{a_i\})\to\varphi(t)$, then

$$\sup_{t\in R}\|\varphi(t)\|\le\|\varphi(t)\|_M\le k\max\left\{\sup_{t\in R}\|f(t)\|,\sup_{i\in Z}\|a_i\|\right\},$$

which completes the proof.

In sequel, we develop conditions for weak regularity of (3) on semi-axis $t\ge 0$ and on the entire axis.

Theorem 2. Let in system (3) matrix $P(t)$ be continuous (piecewise continuous with discontinuities of the first kind at $t=\tau_i$) and bounded for all $t\ge 0$, matrices $E+B_i$, $i=1,2,\ldots$ be non-degenerate, and $(E+B_i)$ and $(E+B_i)^{-1}$ be bounded. Additionally, we assume that there exists a finite limit (2). Then for the weak regularity of system (3) on semi-axis $t\ge 0$ it is necessary and sufficient that system (3) is hyperbolic on semi-axis $t\ge 0$.

Proof. Assuming the hyperbolicity of (3) we may assume (without loss of generality) that the matrices $P(t)$ and B_i have the following block-diagonal structure

$$P(t)=\begin{pmatrix}P_1(t)&0\\0&P_2(t)\end{pmatrix},\quad B_i=\begin{pmatrix}B_i^1&0\\0&B_i^2\end{pmatrix},$$

and if $X_1(t,\tau)$ and $X_1(t,\tau)$ – the matriciants of the corresponding linear systems

$$\frac{\mathrm{d}X_1}{\mathrm{d}t}=P_1(t)X_1,\quad t\ne\tau_i;\ \Delta X_1\big|_{t=\tau_i}=B_i^1X_1$$

and

$$\frac{\mathrm{d}X_2}{\mathrm{d}t}=P_2(t)X_2,\quad t\ne\tau_i;\quad \Delta X_2\big|_{t=\tau_i}=B_i^2X_2,$$

then it holds that

$$\|X_1(t,\tau)\|\le Ke^{-\gamma(t-\tau)},\quad t\ge\tau\ge 0,$$

$$\|X_2(t,\tau)\|\le Ke^{-\gamma(t-\tau)},\quad 0\le t\le\tau. \tag{8}$$

Let us denote by $G(t,\tau)$ the Green-Samoilenko function

$$G(t,\tau)=\begin{cases}\operatorname{diag}(X_1(t,\tau),0), & t>\tau\ge 0,\\ \operatorname{diag}(0,X_2(t,\tau)), & 0\le t\le\tau.\end{cases}$$

Due to the inequality (1) the Green-Samoilenko function $G(t,\tau)$ satisfies the estimate

$$\|G(t,\tau)\|\le Ke^{-\gamma|t-\tau|},\quad t\ge 0,\quad \tau\ge 0. \tag{9}$$

Utilizing the Green-Samoilenko function $G(t,\tau)$, let us define the function

$$x(t)=\int_0^\infty G(t,\tau)f(\tau)\mathrm{d}\tau+\sum_{i=0}^\infty G(t,\tau_i+0)a_i. \tag{10}$$

The right-hand side of (10) is well-defined since the conditions of the existence of the limit (2) and the inequality (9) guarantee the uniform convergence of the integral and the sum from (10).

Indeed, from (10) we have

$$\|x(t)\|\le\int_0^\infty Ke^{-\gamma|t-\tau|}\|f(\tau)\|\mathrm{d}\tau+$$

$$+\sum_{i=0}^\infty Ke^{-\gamma|t-\tau_i|}\|a_i\|\le\frac{K}{\gamma}\sup_{i\ge 0}\|f(t)\|+K\sup_{i\ge 0}\|a_i\|\sum_{i=0}^\infty e^{-\gamma|t-\tau_i|}.$$

Since, from (2), any interval of time of the length l contains no more than q elements of the sequence $\{\tau_i\}$, for $j\ge i$ we have

$$j-i+1\le q\left(\left[\frac{\tau_j-\tau_i}{i}\right]+1\right)\le q\left(\frac{\tau_j-\tau_i}{i}+1\right).$$

Hence,

$$\tau_j-\tau_i\le l\left(\frac{1}{q}-1\right)+\frac{i}{q}(j-1).$$

Consequently,

$$\sum_{i=0}^\infty e^{-\gamma|t-\tau_i|}=\sum_{0<\tau_i\le t}e^{-\gamma(t-\tau_i)}+\sum_{\tau_i>t}e^{-\gamma(\tau_i-t)}\le\sum_{0\le\tau_i\le\tau_j}e^{-\gamma(\tau_j-\tau_i)}+$$

$$+\sum_{\tau_i>\tau_j}e^{-\gamma(\tau_i-\tau_{j+1}}\le e^{\gamma l\left(1-\frac{1}{q}\right)}\sum_{m=0}^\infty e^{-\frac{\gamma lm}{q}}=\frac{e^{\gamma l\left(1-\frac{1}{q}\right)}}{1-e^{-\frac{\gamma l}{q}}}.$$

Summarizing, we have shown that

$$\|x(t)\|\le c\left(\sup_{t\ge 0}\|f(t)\|+\sup_{i>0}\|a_i\|\right),$$

where

$$C=\max\left(\frac{K}{l},\frac{Ke^{\gamma l\left(1-\frac{1}{q}\right)}}{1-e^{-\frac{\gamma l}{q}}}\right).$$

Let us show that function $x(t)$ defined in (10) is the solution to (1). For this purpose, we represent $x(t)$ in the following form

$$x(t)=\int_0^t G(t,\tau)f(\tau)\mathrm{d}\tau+\int_t^\infty G(t,\tau)f(\tau)\mathrm{d}\tau+$$

$$+\sum_{0<\tau_i<t}G(t,\tau_i+0)a_i+\sum_{\tau_i>t}G(t,\tau_i+0)a_i. \tag{11}$$

Differentiating $x(t)$ at $t\ne\tau_i$ we get

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\int_0^t\frac{\mathrm{d}G(t,\tau)}{\mathrm{d}t}f(\tau)\mathrm{d}\tau+\sum_{\tau_i<t}\frac{\mathrm{d}G(t,\tau_i+0)}{\mathrm{d}t}a_i+$$

$$+\int_t^\infty\frac{\mathrm{d}G(t,\tau)}{\mathrm{d}t}f(\tau)\mathrm{d}\tau+\sum_{\tau_i>t}\frac{\mathrm{d}G(t,\tau)}{\mathrm{d}t}a_i+$$

$$+\left(G(t,t-0)-G(t,t+0)\right)f(t)=P(t)x+f(t),$$

or

$$\frac{\mathrm{d}G(t,\tau)}{\mathrm{d}t}=P(t)G(t,\tau),\quad t\ne\tau,\quad t\ne\tau_i,$$

and for $t=\tau,\ \ t\ne\tau_i$

$$G(t,t-0)-G(t,t+0)=E.$$

From (12) we also conclude that for $t=\tau$,

$$x(\tau_i+0)-x(\tau_i)=\int_0^\infty\begin{pmatrix}G(\tau_i+0,\tau)-\\-G(\tau_i,\tau)\end{pmatrix}f(\tau)\mathrm{d}\tau+$$
$$+\sum_{k=1}^\infty\begin{bmatrix}G(\tau_i+0,\,\tau_k+0)-\\-G(\tau_i,\,\tau_k+0)\end{bmatrix}a_k+a_i=B_ix(\tau_i)+a_i,$$

meaning that the function $x(t)$ is bounded for all $t\ge 0$ solution to (1).

Let us prove the necessity of the conditions of theorem 2.

Assume that $x_0\in A_+$. Let us show that the solution $x(t,x_0)$ to (1), which comes out when $t=0$ from the point x_0 satisfies the estimate (5).

For any $s\ge 0$ and $\sigma\ge 0$ we define function $\xi(t)$ given by the formula

$$\xi(t)=\begin{cases}1, & \text{if}\quad 0\le t<s+\sigma,\\ 1-t+s+\sigma, & \text{if}\quad s+\sigma\le t\le s+\sigma+1,\\ 0, & \text{if}\quad t>s+\sigma+1,\end{cases}$$

and utilizing this function we construct another function

$$y(t)=x(t)\int_0^t\frac{\xi(\tau)}{\|x(\tau)\|}\mathrm{d}\tau.$$

By a direct check, we verify that this function is the bounded for $t\ge 0$ solution to (1) when

$$f(t)=\xi(t)\cdot\frac{x(t)}{\|x(t)\|},$$

$$a_i=0.$$

Based on the statement of theorem 1, there exists a constant $c_1>0$ such that

$$\sup_{t\ge 0}\|y(t)\|\le c\cdot\sup_{t\ge 0}\|f(t)\|=c_1.$$

Therefore, when $0\le\theta\le\sigma$, we have

$$\|y(s+\theta)\|=\|x(s+\theta)\|\int_0^{s+\theta}\frac{\mathrm{d}\tau}{\|x(t)\|}\le c_1. \qquad (12)$$

Assuming that

$$\varphi(t)=\int_0^t\frac{\mathrm{d}\tau}{\|x(\tau)\|}$$

we rewrite the estimate (12) in the form

$$\frac{\varphi'(s+\theta)}{\varphi(s+\theta)}=\left(\|x(s+\theta)\|\cdot\int_0^{s+\theta}\frac{\mathrm{d}\tau}{\|x(t)\|}\right)^{-1}\ge\frac{1}{c_1}.$$

If $\sigma>1$ then integrating the latter inequality with respect to θ on the interval $1\le\theta\le\sigma$ we obtain

$$\varphi(s+\sigma)\ge(s+1)e^{\frac{\sigma-1}{c_1}}. \qquad (13)$$

By the theorem, there exists such a positive number b that $\|E+B_i\|\le b$ for any $i=1,2,\ldots$ and the interval $[t,t+l]$ holds no more than q elements of the sequence $\{\tau_i\}$. Hence, for any $s\in[\tau,\tau+l]$, $\tau\ge 0$

$$\|x(s)\|=\|x(s,\tau)x(\tau)\|\le e^{al}\times b^q\cdot\|x(\tau)\|=e^{a_1}\|x(\tau)\|, \qquad (14)$$

where

$$a=\sup_{t\ge 0}\|A(t)\|,\quad a_1=al+q\ln b$$

is a constant which does not depend on τ. Without limiting generality of reasoning, we may always consider $l\ge 1$. Then,

$$\varphi(s+1)=\int_0^{s+1}\frac{\mathrm{d}\tau}{\|x(\tau)\|}\ge\int_s^{s+1}\frac{\mathrm{d}\tau}{\|x(\tau)\|}\ge\|x(s)\|^{-1}e^{-a_1}, \qquad (15)$$

and consequently, when $\sigma>1$ we get

$$\|x(s+\sigma)\|\le\frac{c_1}{\varphi(s+\sigma)}\le\frac{\mu}{\varphi(s+1)}e^{-\frac{\sigma-1}{c_1}}\le c_1e^{\left(a_1+\frac{1}{c_1}\right)}\times$$
$$\times e^{-\frac{\sigma}{c_1}}\cdot\|x[s]\|\le e^{a_2}e^{-\frac{\sigma}{c_1}}\|x(s)\|, \qquad (16)$$

where

$$a_2=a_1+\frac{1}{c_1}+\ln c_1.$$

If $0\le\sigma\le 1$ then based on (14) we have

$$\|x(s+\sigma)\|\le e^{a_1}\|x(s)\|\le e^{\left(a_1+\frac{1}{c_1}\right)}\times e^{-\frac{\sigma}{c_1}}\cdot\|x(s)\|. \qquad (17)$$

Inequalities (16) and (17) state the existence of an estimate for the solution $x(t)$ of the form (5), that is

$$\|x(t,x_0)\|\le K_1e^{-\gamma(t-\tau)}\|x(\tau,x_0)\|,\ t\ge\tau\ge 0,$$

where

$$K_1=e^{a_1+\frac{1}{c_1}}\max\{1,c_1\}.$$

Assume that $x_0\in R^n\setminus A_+$. Let us show that the solution

$$x(t,x_0),\ x(0,x_0)=x_0\in R^n\setminus A_+$$

satisfies an estimate of the form (6).

For the solution $x(t,x_0)=x(t)$ and defined above function $\xi(t)$ consider the following function

$$\varphi(t)=x(t)\int_t^\infty\frac{\xi(\tau)}{\|x(\tau)\|}\mathrm{d}\tau.$$

It is easy to verify that this function is the bounded, when $t\ge 0$, solution to the system of equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=A(t)x+f(t),\quad t\ne\tau_i;\quad \Delta x\big|_{t=\tau_i}=B_ix,$$

that is, equations (1) when $a_i=0,\ i=1,\ 2,\ \ldots$.

Due to theorem 1, the following estimate is true

$$\|\varphi(t)\| = \|x(t)\| \int_t^{\infty} \frac{\xi(\tau)}{\|x(\tau)\|} \mathrm{d}\tau \le c_1.$$

Let us rewrite the latter relation as follows:

$$\|x(t)\| \cdot \gamma(t) \le c_1,$$

where

$$\gamma(t) = \int_t^{\infty} \frac{\mathrm{d}\tau}{\|x(\tau)\|}$$

or

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} \le -\frac{1}{c_1}\gamma(t).$$

Hence, we get that

$$\gamma(s) \le \gamma(t) e^{\frac{t-s}{c_1}}. \tag{18}$$

If

$$\tau_i < t \le \tau_{i+1} \le \tau_j < \tau \le \tau_{j+1},$$

then

$$\|x(\tau)\| \le \|X(\tau,t)x(t)\| \le e^{a(\tau-t)} \times$$
$$\times b^{j-i}\|x(t)\| \le e^{a(\tau-t)}\left[\frac{q}{l}(\tau-t)+q\right]\ln b\|x(t)\|.$$

Since the estimate is the same when $t \in [\tau_i, \tau_{i+1}]$, we have

$$\|x(t)\|\gamma(t) = \|x(t)\| \int_t^{\infty} \frac{\mathrm{d}\tau}{\|x(\tau)\|} \ge e^{-a} b^{-q}\left(a + \frac{q}{l}\ln c\right) = c_2.$$

Taking into consideration (18), it follows that

$$\|x(s)\| \ge \frac{c_2}{\gamma(s)} \ge \frac{c_1}{\gamma(t)} e^{-\frac{t-s}{c_1}} \ge \frac{c_2}{c_1} e^{-\frac{t-s}{c_1}} \|x(t)\|,$$

which means that

$$\|x(t,x_0)\| \le K_1 e^{\gamma_1(t-\tau)} \|x(\tau,x_0)\|,$$

where

$$K_1 = \frac{c_2}{c_1}, \quad \gamma_1 = \frac{1}{c_1}.$$

This completes the proof of theorem 2.

4. 3. Model of the "integrate-and-fire" neuron

The results obtained in the paper can be used in the study of various problems of natural science, mathematical models of which are reduced to the study of the qualitative behavior of solutions of impulse differential equations. In particular, the results of the work can be applied to the study of the existence and finding bounded solutions to the nonhomogeneous impulsive system that describes the behavior of the neuronal system and is based on the "integrate-and-fire" model proposed by Louis Lapicque. The model is described by the following system:

$$\begin{cases} c_i \dot{v}_i = f_i(v_1, v_2, \ldots, v_n), \\ \left[v_i(t+0) - v_i(t)\right]\Big|_{v_i(t)=E_{T_i}} = E_{K_i} - v_i(t), \end{cases} \quad i = \overline{1,n},$$

where $v_i(t)$ – the membrane potential on the neuron i, $i = \overline{1,n}$, E_{T_i} – the threshold value of the i-th neuron potential, E_{K_i} – the value of the potential corresponding to the resting state of the i-th neuron, c_i – the parameter of the i-th neuron.

The impulse occurs when the membrane potential on the neuron $v(t)$ reaches the threshold value E_T, after which the static currents are activated immediately and the potential value drops down to E_K, as shown in Fig. 1.

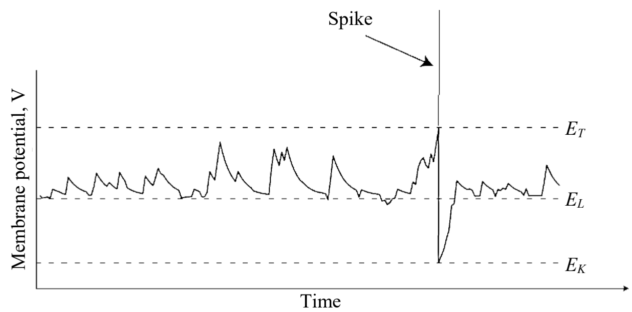

Fig. 1. The process of impulse exposure in the "integrate-and-fire" neuron model for $n=1$

5. Discussion on the results of finding bounded solutions to linear impulsive systems

The problem of studying the bounded solutions to the differential equations with impulsive influences is poorly studied in the general case of multidimensional spaces. Until recently, the main interest of researchers was concentrated on one-dimensional and two-dimensional systems, where it is possible to obtain a general solution to the impulsive system in an explicit form. This approach is not applicable for the multidimensional systems. Therefore, a fundamentally different approach based on the analysis of the qualitative properties of the corresponding homogeneous systems, such as regularity and hyperbolicity, is developed (Lemma 1). On the one hand, these properties can be effectively tested for wide classes of impulsive systems, and on the other hand, they provide an opportunity to prove a number of qualitative properties for nonhomogeneous impulsively perturbed systems. Thus, the conditions of the existence of bounded solutions to linear differential equations can be extended to the classes of linear impulsive systems (Theorem 1). New effective conditions are also derived (Theorem 2), which guarantee the existence of bounded solutions to linear systems of differential equations with impulsive jumps at fixed moments of time.

6. Conclusions

1. The conditions guaranteeing the hyperbolicity of the systems of differential equations with impulsive jumps

are established. An important aspect from the viewpoint of applications is the fact that these conditions are formulated in terms of coefficients of the initial problem. This essentially expands the class of evolutionary systems with short-term perturbations to which these conditions can be applied.

2. The obtained hyperbolicity conditions allow for the investigation of the existence of bounded solutions to non-homogeneous multidimensional systems of differential equations with impulsive perturbations. This makes it possible to study such important qualitative characteristics of solutions as stability, periodicity and quasi-periodicity in the future.

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