UDC 519.718
DOI: 10.15587/1729-4061.2021.239149

IMPLEMENTATION OF THE METHOD OF FIGURATIVE TRANSFORMATIONS TO MINIMIZING SYMMETRIC BOOLEAN FUNCTIONS

Mykhailo Solomko
Corresponding author
PhD, Associate Professor
Department of Computer Engineering*
E-mail: doctrinas@ukr.net

Petro Tadeyev
Doctor of Pedagogical Sciences, PhD, Professor
Department of Higher Mathematics*

Liudmyla Zubyk
PhD, Associate Professor
Department of Software Systems and Technologies
Taras Shevchenko National University of Kyiv
Volodymyrska str., 60, Kyiv, Ukraine, 01033

Stepaniia Babych
PhD, Associate Professor
Department of Information and Communication Technologies and Methods of Teaching Informatics**

Yuliia Mala
PhD
Department of Computer Science and Software Engineering
University of Customs and Finance
Volodymyra Vernadskoho str., 2/4, Dnipro, Ukraine, 49000

Oksana Voitovych
Doctor of Pedagogical Sciences, Associate Professor
Department of Ecology, Geography and Tourism**

*National University of Water and Environmental Engineering
Soborna str., 11, Rivne, Ukraine, 33028
**Rivne State University of Humanities
Stepana Bandery str., 12, Rivne, Ukraine, 33028

This paper reports a study that has established the possibility of improving the effectiveness of the method of figurative transformations in order to minimize symmetrical Boolean functions in the main and polynomial bases. Prospective reserves in the analytical method were identified, such as simplification of polynomial function conjuncterms using the created equivalent transformations based on the method of inserting the same conjuncterms followed by the operation of super-gluing the variables.

The method of figurative transformations was extended to the process of minimizing the symmetrical Boolean functions with the help of algebra in terms of rules for simplifying the functions of the main and polynomial bases and developed equivalent transformations of conjuncterms. It was established that the simplification of symmetric Boolean functions by the method of figurative transformations is based on a flowchart with repetition, which is the actual truth table of the assigned function. This is a sufficient resource to minimize symmetrical Boolean functions that makes it possible to do without auxiliary objects, such as Karnaugh maps, cubes, etc.

The perfect normal form of symmetrical functions can be represented by binary matrices that would represent the terms of symmetrical Boolean functions and the OR or XOR operation for them.

The experimental study has confirmed that the method of figurative transformations that employs the 2-(n, b)-design, and 2-(n, x/b)-design combinatorial systems improves the efficiency of minimizing symmetrical Boolean functions. Compared to analogs, this makes it possible to enhance the productivity of minimizing symmetrical Boolean functions by 100–200 %.

There are grounds to assert the possibility of improving the effectiveness of minimizing symmetrical Boolean functions in the main and polynomial bases by the method of figurative transformations. This is ensured, in particular, by using the developed equivalent transformations of polynomial function conjuncterms based on the method of inserting similar conjuncterms followed by the operation of super-gluing the variables

Keywords: minimization of symmetric Boolean functions by the method of figurative transformations, singular function, main basis

Received date 20.05.2021
Accepted date 09.08.2021
Published date 30.08.2021

How to Cite: *Solomko, M., Tadeev, P., Zubyk, L., Babich, S., Mala, J., Voitovich, O. (2021). Implementation of the method of figurative transformations to minimizing symmetric Boolean functions. Eastern-European Journal of Enterprise Technologies, 4 (4 (112)), 23–39. doi: https://doi.org/10.15587/1729-4061.2021.239149*

1. Introduction

Given the practical expediency, the special classes of Boolean functions are typically preferred, in particular symmetrical functions (SFs) [1–3]. Due to their extensive functionality, symmetrical functions simulate a considerable number of computational components, in particular, n-input 1-bit binary code adders or (non)parity schemes, comparators, error detection devices, defective code decoders, etc. On the other hand, to synthesize symmetrical functions, special partial approaches that are not inherent in the general case can be used, which mainly produce better results in the implementation of digital components [4–6].

The importance of such functions was first recognized in work [1], in which the basic concepts, definitions, and properties of SFs were introduced. In practice, the problems of SF decomposition and the problems of synthesizing the optimal logical schemes based on them [7–14] are of interest.

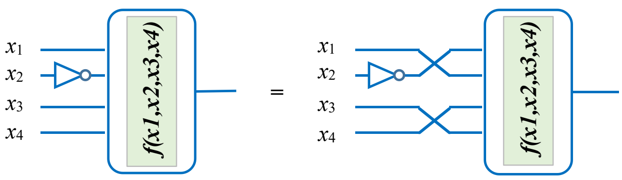
The Boolean function $f(x_1, x_2, ..., x_n)$ is symmetric with respect to the variables $x_1, x_2, ..., x_n$ if, for any substitution:

$$\begin{pmatrix} 1 & 2 ... k \\ x_1 & x_2 ... k \end{pmatrix}$$

the following equality holds:

$$f(x_1,...,x_k,x_{k+1},...,x_n)=f(x_{j_1},...,x_{j_k},x_{j_{k+1}},...,x_{j_n}).$$

Typically, symmetry refers to the permutations of object parameters that leave it unchanged. They give an idea of the structure of an object that can be used to facilitate calculations on it. Permutations can also serve as a guideline for maintaining this structure when an object is transformed in a certain way. Thus, the symmetry for Boolean functions is the permutation of variables with a possible addition, which leaves the values of functions unchanged (Fig. 1).



$$f(x_1,\overline{x_2},x_3,x_4)=f(x_1,\overline{x_2},x_3,x_4)$$

Fig. 1. Illustration of the Boolean function symmetry

This property of symmetrical functions makes it possible to optimize logical synthesis in the design of digital circuits.

The simplest examples of symmetrical functions are functions submitted by the disjunction and conjunction of non-inverted variables:

$$f(A,B)=A+B=B+A;$$

$$f(A,B)=AB=BA.$$

The peculiarity of minimizing Boolean functions by the method of figurative transformations is the use of binary matrices with a list of rules for the equivalent transformation of conjuncterms or max terms of the predefined functions. The result of simplifying the terms in a binary matrix is some universal function, metadata that can explain other data, for example, to derive a minimum Boolean function for another basis. That defines the eidos (hermeneutics) of logical operations on binary structures, as well as instances of classes of functions of logical bases on binary matrices. The eidos of logical operations on 2-dimensional binary structures, as a clear presence of the abstract, makes it possible to focus on what the object does, not how it does it.

The evolution of the visual-matrix form of the analytical method is the result of the introduction of new logical operations for simplifying logical functions. In particular, these are the operations of super-gluing the variables, incomplete super-gluing the variables, doubling the constituents followed by the operation of simple gluing the variables, inserting similar conjuncterms followed by the operation of super-gluing the variables.

Those objects (logical operations) still make it possible in practice to increase the hardware capabilities of minimizing symmetrical Boolean functions by analytical method, to increase the control function, which warrants the optimal result, and to practically bypass, to some extent, without the use of automation of the process of minimizing symmetrical functions in the main and polynomial bases. The interpretation of the result of using those logical operations is that there are no symmetric logical functions (except for minimal ones), which cannot be simplified.

Thus, a relevant aspect of theoretical research into the minimization of symmetrical Boolean functions by the method of figurative transformations is to identify opportunities for improvement and expansion of the apparatus of synthesis of arithmetic components based on symmetrical functions for their application in digital technologies. Specifically, still relevant are theoretical studies on minimizing symmetrical Boolean functions, aimed at improving such factors as:

– the visual-matrix methods for minimizing symmetrical Boolean functions of the main and polynomial bases;

– the cost of technology to minimize symmetrical Boolean functions;

– ensuring the reliability of the result from minimizing symmetrical Boolean functions.

2. Literature review and problem statement

The method of synthesis of symmetric logical functions generated by schemes in the worst case with a depth of $O(\log^2 n)$ was proposed in work [15]. It was noted that the reported method for synthesizing symmetrical functions is the first, which is aimed at reducing the depth of logical schemes generated for symmetrical Boolean functions. The experimental results demonstrated that the approach in question reduces the depth of the final implementation of the function to 25.93 % compared to other methods of synthesis of symmetrical functions.

The matrix method of parallel decomposition to minimize symmetrical Boolean functions in orthogonal form is presented in work [16]. The results obtained when using that method, compared to the use of Zhegalkin polynomials, demonstrate the improved indicators of the complexity of the implementation of schemes of digital devices. Due to the polarization of inputs of Boolean functions, the method can be used as one of the components of the complete matrix method of parallel decomposition to obtain a complex minimum form of Boolean functions, which has better implementation indicators compared to classical forms of representation of Boolean functions. The peculiarity of the method is the use of already prepared extended matrices and tables of a complete list of conjunctive sets, which can significantly reduce the time of minimization of the predefined function.

Two parallel algorithms for solving the problem of finding exact ESOP expressions for an arbitrary Boolean function are proposed in work [17]. Since this minimization problem is very complex, the solution is only available for the seven variables of the assigned function. The processing time of some symmetric functions of the seven variables is about a week. With the help of the proposed algorithm, which is hybrid (OpenMP, MPI), for a cluster of three nodes and with four cores, it is possible to achieve more than nine times the acceleration of the calculation of the task.

A special metric that motivates Boolean functions is multiplicative complexity (MC): the minimum number of AND gateways, which is sufficient to implement a Boolean function based on {XOR, AND, NOT}. Paper [18] examines MC of the symmetrical Boolean functions, the output of which is invariant in the reordering of input variables. Based on the Hemming weights method, new methods are introduced that allow the synthesis of circuits with fewer logical elements of AND, compared to the upper limit. Work [18] presents the generation of schemes for all such functions up to 25 variables. As a special focus, the authors report specific upper limits for MC of the elementary symmetric functions $\sum_k^n$ and counting functions $\sum_k^n$ up to $n=25$ input variables. In addition, the upper limits of the maximum MC in the class of n-variable symmetric Boolean functions for each n to 132 are demonstrated.

Classic two-variable symmetry plays an important role in many EDA applications, ranging from logical synthesis to formal verification. Paper [19] proposes a complete circuit-based method that uses structural analysis, integrated modeling, and logical matching for the rapid and scalable detection of classic symmetry of fully-set Boolean functions. Experimental results demonstrate that the proposed method works for Boolean functions with a large number of variables for which BDD cannot be constructed.

The algorithm for minimizing the Reed-Muller functions at fixed polarity (FPRM) for polynomial time for fully symmetrical Boolean functions based on ordered functional decision-making diagrams (OFDD) is presented in paper [20]. The generalization of the minimization algorithm for partially symmetrical functions was investigated. The minimization algorithm is implemented as a Sympathy program. The advantages of the proposed algorithm are illustrated by examples and experimental results of minimizing symmetric functions in the FPRM class.

The new PSDKRO implementation method is considered in paper [21]. The Pseudo Kronecker (PSDKRO) expressions are the AND/EXOR class of logical functions. The paper proves that the exact minimization of PSDKRO for fully symmetrical functions can be done over a polynomial time. Experimental results of simplification of symmetrical functions are presented to compare the effectiveness of the approach in question with other methods of AND/EXOR minimization.

Abbreviated ordered decision-making binary diagrams (ROBDDs) are data structures that are often used to present and manipulate logical functions. Because ROBDD size is extremely sensitive to ordering variables in a chart, many heuristics have been designed to get the optimal order of variables. For a class of partially symmetric Boolean functions, paper [22] demonstrates a new general method of improving the quality of heuristics of ordering ROBDD variables based on the exchange of variables. To demonstrate the effectiveness of the approach in question, statistical and control results of ordering variables on ROBDD are presented.

Symmetrical and partially symmetrical functions are studied from an algebraic point of view. Work [23] considers tests for the detection of such properties. A more general approach is presented, which includes the concept of ρ-symmetrical Boolean functions. The canonical form for ρ-symmetrical functions is derived, which leads to synthesis procedures that improve Shannon's results.

Paper [24] proposes a method for identifying types of symmetry based on the specified classification (asymmetry, simple symmetry, anti-symmetry, poly symmetry, pseudo symmetry) in Boolean functions n variables using so-called decomposition clones formed by q-separation of specified minterms. The theorem on the basis of which it is possible to find and identify different types of symmetry of Boolean functions by methods easier than known ones is formulated. The advantages of the described authentication algorithm are illustrated using an example.

The reviewed literary sources [15–24] mainly report methods of minimizing symmetrical Boolean functions in the Boolean and Reed-Muller bases. There are methods for simplifying symmetrical Boolean functions that use theoretical objects of contiguous theory as Hemming scales, ordered binary decision-making diagrams (ROBDDs), etc. The algorithms for the implementation of the considered methods are evaluated by polynomial complexity. A mandatory technological point for the implementation of those algorithms and methods is the need for automated calculations. In the complex search for the optimal function, compensation may be an approximate synthesis – the tendency of logical synthesis, when some results of the logical specification change within the permissible non-optimality of the digital circuit to be designed.

A method of figurative transformations based on binary combinatorial systems with repeated 2-(n, b)-design, 2-$(n, x/b)$-design belongs to the visual-matrix form of the analytical method [25] by qualification and does not exclude the manual technique for minimizing symmetric Boolean functions.

Thus, the algorithms and methods, the software tools designed for them, covering the general procedure for minimizing symmetrical Boolean functions [15–24] and the method of figurative transformations take different approaches (principles of minimization). Therefore, they imply different prospects regarding the possibility of algorithmic minimization of symmetrical Boolean functions.

The prospect of the method of figurative transformations, as a descendant of the analytical method, regarding the proper minimization of symmetric logical functions in the Reed-Muller basis is the creation of the necessary algebra in terms of the rules for the equivalent transformation of polynomial functions [26]. As well as new developed equivalent transformations of conjuncterms in the polynomial normal form (PNF) based on the method of inserting similar conjuncterms with the following operation of super-gluing the variables (chapter 5. 1). Thus, the classical analytical method still has the prospect of increasing its hardware capabilities in relation to the minimization of symmetrical Boolean functions. And this is the reason to believe that the software and technological base, which is represented by algorithms and methods with theoretical objects of adjacent theories [15–24], is insufficient to conduct theoretical research on the optimal minimization of symmetrical Boolean functions, in particular in the Reed-Muller basis.

This predetermines the need for research involving equivalent figurative transformations in order to minimize symmetrical Boolean functions. In particular, with a procedure that uses equivalent transformations of PNF conjuncterms based on the method of inserting similar conjuncterms of polynomial functions with the following operation of super-gluing the variables (chapter 5. 1.).

In the practical aspect, a method of figurative transformations will ensure the expansion of possibilities of technology design of digital components based on symmetric Boolean functions in the main $\{\vee, \wedge, \neg\}$ and polynomial $\{\wedge, \oplus, 1\}$ bases.

3. The aim and objectives of the study

The purpose of this work is to extend the method of figurative transformations to minimize symmetrical Boolean functions in the class of perfect disjunctive normal forms (PDNFs), perfect conjunctive normal forms (PCNFs), and perfect polynomial normal forms (PPNFs). This will make it possible to simplify, increase the productivity of minimization of symmetrical Boolean functions in the main and polynomial bases by using the algebraic apparatus of the specified bases.

To accomplish the aim, the following tasks have been set:

– to establish the equivalent transformations of a normal polynomial form by inserting similar conjuncterms with the following operation of super-gluing the variables;

– to analyze the results of simplification of symmetric Boolean functions in the main basis {I, OR, NOT} by the method of figurative transformations and examples of minimizing symmetric functions in the Boole basis in order to compare the cost of implementing the minimum symmetric function and the number of procedural steps;

– to analyze the results of simplification of symmetric Boolean functions in the Reed-Muller basis by the method of figurative transformations and examples of minimizing symmetric functions in a polynomial basis in order to compare the cost of implementing a minimum symmetric function and the number of procedural steps;

– to conduct a comparative analysis of the results of simplification of Boolean functions with partial symmetry by the method of figurative transformations and decomposition methods in order to compare the cost of implementing the minimum symmetric function and the number of procedural steps;

– to optimize the logical structure of the symmetrical 4-input binary code adder.

4. The study materials and methods

The peculiarity of minimizing symmetrical Boolean functions is that not all such functions are simplified in a perfect disjunctive normal form (PDNF) or in a perfect conjunctive normal form (PCNF) of the main basis $\{\vee,\wedge,\neg\}$. Using an element basis of only one functionally complete system of switch functions, in a general case, does not provide conditions for obtaining the optimal combination scheme. As the practice of designing logical circuits by combining element bases that belong to several functionally complete systems (for example, {I, OR, NOT} systems) makes it possible to build optimal combination schemes (in terms of hardware complexity and performance), including the use of symmetrical Boolean functions.

Example 1. The Boolean symmetrical function (SF) is a function equal to 1 on C_n^a sets of variables that have exactly a unities on all these sets [27]. The number a is termed the SF index. The fundamental SF is a function with one index, the peculiarity of which is the impossibility of using the operation of gluing the variables for the Boole basis $\{\vee,\wedge,\neg\}$. Thus, the fundamental SF of 3 variables with the index $a=1$ in the Boole basis is written in the following form (1) [27]:

$$H_3(1)=x_1\overline{x_2}\,\overline{x_3}+\overline{x_1}x_2\overline{x_3}+\overline{x_1}\,\overline{x_2}x_3. \qquad (1)$$

Symmetrical function (1) represents a 3-level logic whose cost of implementation is $k_\theta/k_l/k_{in}=3/9/6$, where k_θ, k_l, k_{in} is the number of conjuncterms, literals, and inverters, respectively [28].

However, symmetric function (1) can be simplified on a polynomial basis $\{\wedge,\oplus,1\}$ with the transition to a mixed basis. Since function (1) is singular [26], let us dig into the Reed-Muller algebra. We obtain:

$$H_3(1)_{\min}=\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix}=\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&1&1\\0&0&0\\0&1&1\end{vmatrix}=\begin{vmatrix}&0&0\\0&&\\0&1&1\end{vmatrix}=$$

$$=\overline{x_2}\,\overline{x_3}\oplus\overline{x_1}\oplus\overline{x_1}x_2x_3=\overline{x_2}\,\overline{x_3}\oplus\overline{x_1}\,\overline{x_2x_3}=$$

$$=\overline{x_2}\,\overline{x_3}\oplus\overline{x_1}\left(\overline{x_2}+\overline{x_3}\right)=\overline{x_2}\,\overline{x_3}\oplus\overline{x_1}\,\overline{x_2x_3}=$$

$$=\left(x_1+x_2x_3\right)\oplus\left(x_2+x_3\right). \qquad (2)$$

The minimum symmetric function (MSF) of the function $H_3(1)$ (1) in a mixed base is:

$$H_3(1)_{\min}=\left(x_1+x_2x_3\right)\oplus\left(x_2+x_3\right). \qquad (3)$$

MSF (3) represents a 3-level logic whose cost of implementation is:

$$k_\theta/k_l/k_{in}=2/5/0,$$

which is four literals less compared to (1) [27].

To simplify function (2), the procedure of inserting similar conjuncterms was applied to the polynomial normal form (PNF) with the following operation of super-gluing the variables [26].

When meditating on MSF (3), we envision that the permutation of the variables x_2 and x_3 would not change the value of the function. Thus, function (3) has partial symmetry for variables with indices 2 and 3.

The verification of the derived MSF (3) is given in Table 1.

Table 1

MSF (3) verification – $\left(x_1+x_2x_3\right)\oplus\left(x_2+x_3\right)$

No.	x_1	x_2	x_3	$H_3(1)$ (4)	$\left(x_1+x_2x_3\right)\oplus\left(x_2+x_3\right)$	$H_3(1)_{\min}$
1	0	0	1	1	$\left(0_1+0_21_3\right)\oplus\left(0_2+1_3\right)$	1
2	0	1	0	1	$\left(0_1+1_20_3\right)\oplus\left(1_2+0_3\right)$	1
4	1	0	0	1	$\left(1_1+0_20_3\right)\oplus\left(0_2+0_3\right)$	1

Table 1 demonstrates that MSF (3) $\left(x_1+x_2x_3\right)\oplus\left(x_2+x_3\right)$ satisfies the assigned logical function $H_3(1)$ (1).

Example 2. It is required to simplify the Boolean function with partial symmetry given in algebraic form (4):

$$y=\overline{x_1}\,\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}\,\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2\overline{x_3}\,\overline{x_4}+\overline{x_1}x_2x_3x_4+$$

$$+x_1\overline{x_2}\,\overline{x_3}\,\overline{x_4}+x_1\overline{x_2}x_3x_4+x_1x_2\overline{x_3}x_4+x_1x_2x_3\overline{x_4}. \qquad (4)$$

Permutation of variables x_3 and x_4 does not change the value of function (4).

Solution.

Simplification of function (4) implies the PPNF representation with the transition to the Reed-Muller basis:

$$y_{\min}=\left|\begin{array}{cccc}0&0&0&1\\0&0&1&0\\0&1&0&0\\0&1&1&1\\1&0&0&0\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{array}\right|=\left|\begin{array}{cccc}0&0&&1\\0&0&1&\\0&1&0&\\0&1&&1\\1&0&0&\\1&0&&1\\1&1&&1\\1&1&1&\end{array}\right|=$$

$$=\overline{x_1}\,\overline{x_2}\left(x_3\oplus x_4\right)+\overline{x_1}x_2\left(\overline{x_3\oplus x_4}\right)+$$

$$+x_1\overline{x_2}\left(\overline{x_3\oplus x_4}\right)+x_1x_2\left(x_3\oplus x_4\right)=$$

$$=\left(x_3\oplus x_4\right)\left(\overline{x_1}\,\overline{x_2}+x_1x_2\right)+\left(\overline{x_3\oplus x_4}\right)\left(\overline{x_1}x_2+x_1\overline{x_2}\right)=$$

$$=\left(x_3\oplus x_4\right)\left(\overline{x_1\oplus x_2}\right)+\left(\overline{x_3\oplus x_4}\right)\left(x_1\oplus x_2\right)=$$

$$=\left(x_1\oplus x_2\right)\oplus\left(x_3\oplus x_4\right)=x_1\oplus x_2\oplus x_3\oplus x_4.$$

The minimum symmetric function (MSF) of function (4) in the Reed-Muller basis is (5):

$$y_{\min}=x_1\oplus x_2\oplus x_3\oplus x_4. \quad (5)$$

Function (5) has complete symmetry – permutation of any pair of variables does not change the value of the function.

5. Results of minimizing the symmetric Boolean functions by the method of figurative transformations

The equivalent figurative transformations when minimizing symmetric Boolean functions produce the following result:

– they make it possible to set the equivalent transformations of a normal polynomial form by inserting similar conjuncterms with the following operation of super-gluing the variables;

– they provide analysis of the results of simplification of symmetrical Boolean functions in the main basis {I, OR, NOT} and examples of minimization of symmetrical functions in the Boole basis in order to compare the cost of implementing the minimum symmetrical function and the number of procedural steps;

– they provide analysis of the results of simplification of symmetrical Boolean functions in the Reed-Muller basis and examples of minimizing symmetrical functions in a polynomial basis in order to compare the cost of implementing a minimum symmetric function and the number of procedural steps;

– they provide a comparative analysis of the results of simplification of Boolean functions with partial symmetry and examples of reducing the complexity of the implementation of Boolean functions with partial symmetry;

– they provide a comparative analysis of the results of simplification of Boolean functions with partial symmetry by the method of figurative transformations and decomposition methods;

– they optimize the logical structure of the symmetrical 4-input binary code adder.

5. 1. Deriving equivalent transformations of a polynomial function by the method of inserting similar conjuncterms

The procedure for inserting similar conjuncterms with the following operation of super-gluing the variables [26] makes it possible to derive equivalent transformations of the normal polynomial form of Boolean functions.

Some equivalent transformations of a normal polynomial form are derived as follows:

$$\begin{array}{|ccc|}\hline x_1&x_2&x_3\\\hline 0&0&0\\1&1&1\\\hline\end{array}=\left|\begin{array}{ccc}0&0&0\\0&0&1\\0&0&1\\0&1&0\\0&1&0\\0&1&1\\0&1&1\end{array}\right|\left|\begin{array}{ccc}1&0&0\\1&0&0\\1&0&1\\1&0&1\\1&1&0\\1&1&0\\1&1&1\end{array}\right|=1\oplus\left|\begin{array}{ccc}0&0&1\\0&1&0\\0&1&1\\1&0&0\\1&0&1\\1&1&0\end{array}\right|=$$

$$=1\oplus\left|\begin{array}{ccc}0&0&1\\0&1&\\1&0&\\1&1&0\end{array}\right|=1\oplus\left|\begin{array}{ccc}0&&1\\0&1&\\1&0&\\1&&0\end{array}\right|=1\oplus\left(x_1\oplus x_2\right)+\left(x_1\oplus x_3\right).$$

Since the fourth matrix is singular, it is necessary to proceed to the algebra of the main basis $\{\vee,\wedge,\neg\}$ in it and perform the operation of semi-gluing the variables. Thus:

$$\overline{x_1}\,\overline{x_2}\,\overline{x_3}\oplus x_1x_2x_3=1\oplus\left(x_1\oplus x_2\right)+\left(x_1\oplus x_3\right)=$$

$$=\overline{\left(x_1\oplus x_2\right)+\left(x_1\oplus x_3\right)}=\left(\overline{x_1\oplus x_2}\right)\left(\overline{x_1\oplus x_3}\right)=$$

$$=\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)=\left(x_1\oplus\overline{x_2}\right)\left(x_1\oplus\overline{x_3}\right).$$

$$\begin{array}{|cccc|}\hline x_1&x_2&x_3&x_4\\\hline 0&0&0&0\\1&1&1&1\\\hline\end{array}=\left|\begin{array}{cccc}0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&1&0\\0&0&1&0\\0&0&1&1\\0&0&1&1\\0&1&0&0\\0&1&0&0\\0&1&0&1\\0&1&0&1\\0&1&1&0\\0&1&1&0\\0&1&1&1\\0&1&1&1\end{array}\right|\left|\begin{array}{cccc}1&0&0&0\\1&0&0&0\\1&0&0&1\\1&0&0&1\\1&0&1&0\\1&0&1&0\\1&0&1&1\\1&0&1&1\\1&1&0&0\\1&1&0&0\\1&1&0&1\\1&1&0&1\\1&1&1&0\\1&1&1&0\\1&1&1&1\end{array}\right|=$$

$$=1\oplus\left|\begin{array}{cccc}0&0&0&1\\0&0&1&0\\0&0&1&1\\0&1&0&0\\0&1&0&1\\0&1&1&0\\0&1&1&1\\1&0&0&0\\1&0&0&1\\1&0&1&0\\1&0&1&1\\1&1&0&0\\1&1&0&1\\1&1&1&0\end{array}\right|=1\oplus\left|\begin{array}{cccc}0&0&0&1\\0&0&1&\\0&1&&\\1&0&&\\1&1&0&\\1&1&1&0\end{array}\right|=1\oplus\left|\begin{array}{cccc}0&0&&1\\0&0&1&\\0&1&&\\1&0&&\\1&1&0&\\1&1&&0\end{array}\right|=$$

$$=1\oplus\left|\begin{array}{cccc}0&&&1\\0&&1&\\0&1&&\\1&0&&\\1&&0&\\1&&&0\end{array}\right|=1\oplus\left(x_1\oplus x_2\right)+\left(x_1\oplus x_3\right)+\left(x_1\oplus x_4\right).$$

Since the fourth matrix is singular, we select in it the algebra of the main basis $\{\vee,\wedge,\neg\}$ and perform the operation of semi-gluing the variables. In the fifth matrix, the operation of semi-gluing the variables is also carried out. Thus:

$$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}\oplus x_1x_2x_3x_4=$$
$$=1\oplus(x_1\oplus x_2)+(x_1\oplus x_3)+(x_1\oplus x_4)=$$
$$=\overline{(x_1\oplus x_2)+(x_1\oplus x_3)+(x_1\oplus x_4)}=$$
$$=\left(\overline{x_1\oplus x_2}\right)\left(\overline{x_1\oplus x_3}\right)\left(\overline{x_1\oplus x_4}\right)=\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right).$$

The next equivalent transformation is derived by induction.

$$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}\ \overline{x_5}\oplus x_1x_2x_3x_4x_5=$$
$$=1\oplus(x_1\oplus x_2)+(x_1\oplus x_3)+(x_1\oplus x_4)+(x_1\oplus x_5)=$$
$$=(\overline{(x_1\oplus x_2)+(x_1\oplus x_3)+(x_1\oplus x_4)+(x_1\oplus x_5)}=$$
$$=\left(\overline{x_1\oplus x_2}\right)\left(\overline{x_1\oplus x_3}\right)\left(\overline{x_1\oplus x_4}\right)\left(\overline{x_1\oplus x_5}\right)=$$
$$=\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)\left(\overline{x_1}\oplus x_5\right).$$

Some examples of the equivalent transformations of PNF and DNF, based on the results of the procedure for inserting similar conjuncterms with the following operation of super-gluing the variables, are given in Tables 2, 3, respectively.

Algebraic expressions in the left column in Table 2 are singular [26], so the latter can be represented in the main basis $\{\vee,\wedge,\neg\}$ (Table 3).

When meditating on Tables 2, 3, we envision that the logical operation of the $\overline{x_1}\ \overline{x_2}+x_1x_2$ equivalence is a partial case on a set of similar equivalent transformations of Boolean expressions.

Table 2

Table of some equivalent transformations of PNF conjuncterms

$\overline{x_1}\ \overline{x_2}\oplus x_1x_2$	$\left(\overline{x_1}\oplus x_2\right)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\oplus x_1x_2x_3$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)$
$x_1\overline{x_2}x_3\oplus\overline{x_1}x_2\overline{x_3}$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_2}\oplus x_3\right)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}\oplus x_1x_2x_3x_4$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)$
$x_1\overline{x_2}x_3\overline{x_4}\oplus\overline{x_1}x_2\overline{x_3}x_4$	$(x_1\oplus x_2)(x_1\oplus x_4)(x_2\oplus x_3)$
$\overline{x_1}x_2x_3x_4\oplus x_1\overline{x_2}\ \overline{x_3}\ \overline{x_4}$	$(x_1\oplus x_2)(x_1\oplus x_3)(x_1\oplus x_4)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}\ \overline{x_5}\oplus x_1x_2x_3x_4x_5$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)\left(\overline{x_1}\oplus x_5\right)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}\ \overline{x_5}\ \overline{x_6}\oplus x_1x_2x_3x_4x_5x_6$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)\left(\overline{x_1}\oplus x_5\right)\left(\overline{x_1}\oplus x_6\right)$

Table 3

Table of some equivalent transformations of DNF minterms

$\overline{x_1}\ \overline{x_2}+x_1x_2$	$\left(\overline{x_1}\oplus x_2\right)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}+x_1x_2x_3$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)$
$x_1\overline{x_2}x_3+\overline{x_1}x_2\overline{x_3}$	$(x_1\oplus x_2)(x_2\oplus x_3)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}+x_1x_2x_3x_4$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)$
$x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2\overline{x_3}x_4$	$(x_1\oplus x_2)(x_1\oplus x_4)(x_2\oplus x_3)$
$\overline{x_1}x_2x_3x_4+x_1\overline{x_2}\ \overline{x_3}\ \overline{x_4}$	$(x_1\oplus x_2)(x_1\oplus x_3)(x_1\oplus x_4)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}\ \overline{x_5}+x_1x_2x_3x_4x_5$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)\left(\overline{x_1}\oplus x_5\right)$
$\overline{x_1}\ \overline{x_2}\ \overline{x_3}\ \overline{x_4}\ \overline{x_5}\ \overline{x_6}+x_1x_2x_3x_4x_5x_6$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)\left(\overline{x_1}\oplus x_5\right)\left(\overline{x_1}\oplus x_6\right)$

Example 3. It is required to simplify the Boolean function set in canonical form (6) [29]:

$$f(x_1,x_2,x_3x_4)=(0,1,6,8,11,14,15),\qquad(6)$$

by applying the method of inserting similar conjuncterms with the following operation of super-gluing the variables.

Solution.

When simplifying function (6), work [29] reported the following result:

$$Y^{\oplus}=\{(--1-),(-0--),(10-1),(0111)\}^{\oplus}.\qquad(7)$$

We shall apply to the conjuncterms – $\{(10-1),(0111)\}^{\oplus}==x_1\overline{x_2}x_4\oplus\overline{x_1}x_2x_3x_4$ of function (7) a method of inserting similar conjuncterms with the following operation of super-gluing the variables:

$$\left|\begin{array}{cccc}1&0&&1\\0&1&1&1\end{array}\right|=\left|\begin{array}{cccc}1&0&0&1\\1&0&1&1\\0&1&1&1\end{array}\right|=$$

$$=\left|\begin{array}{cccc}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&1\\0&0&1&0\\0&0&1&0\\0&0&1&1\\0&0&1&1\\0&1&0&0\\0&1&0&0\\0&1&0&1\\0&1&0&1\\0&1&1&0\\0&1&1&0\\0&1&1&1\end{array}\right\|\begin{array}{cccc}1&0&0&0\\1&0&0&0\\1&0&0&1\\1&0&1&0\\1&0&1&0\\1&0&1&1\\1&1&0&0\\1&1&0&0\\1&1&0&1\\1&1&0&1\\1&1&1&0\\1&1&1&0\\1&1&1&1\\1&1&1&1\end{array}\right|=1\oplus\left|\begin{array}{cccc}0&0&0&0\\0&0&0&1\\0&0&1&0\\0&0&1&1\\0&1&0&0\\0&1&0&1\\0&1&1&0\\1&0&0&0\\1&0&1&0\\1&1&0&0\\1&1&0&1\\1&1&1&0\\1&1&1&1\end{array}\right|=$$

$$=1\oplus\left|\begin{array}{cccc}0&0&&\\0&1&0&0\\0&1&0&1\\0&1&1&0\\1&0&0&0\\1&0&1&0\\1&1&&\end{array}\right|=1\oplus\left|\begin{array}{cccc}0&0&&\\0&1&0&\\0&1&1&0\\1&0&&0\\1&1&&\end{array}\right|=$$

$$=1\oplus\left|\begin{array}{cccc}0&0&&\\0&1&0&\\0&1&&0\\1&0&&0\\1&1&&\end{array}\right|=1\oplus\left|\begin{array}{cccc}0&0&&\\&1&0&\\&1&&0\\1&&&0\\1&1&&\end{array}\right|=$$

$$=1\oplus\left|\begin{array}{cccc}0&0&&\\&1&0&\\&1&&0\\1&&&0\\&0&&0\\1&1&&\end{array}\right|=1\oplus\left|\begin{array}{cccc}0&0&&\\&1&0&\\&&&0\\1&&&0\\1&1&&\end{array}\right|=$$

$$=1\oplus\left|\begin{array}{cccc}0&0&&\\&1&0&\\&&&0\\1&1&&\end{array}\right|=1\oplus\left(\left(\overline{x_1\oplus x_2}\right)+x_2\overline{x_3}+\overline{x_4}\right)=$$

$$=\overline{\overline{x_1\oplus x_2}+x_2\overline{x_3}+\overline{x_4}}=(x_1\oplus x_2)\left(\overline{x_2\overline{x_3}}\right)x_4=$$

$$=(x_1\oplus x_2)\left(\overline{x_2}+x_3\right)x_4.$$

Since the sixth matrix is singular, in it we choose the algebra of the main basis $\{\vee,\wedge,\neg\}$ and perform the operation of semi-gluing the variables. The result of semi-gluing the variables is written to the seventh matrix. In the seventh matrix, the operation of semi-gluing the variables is also carried out. The result is written to the eighth matrix. In matrices 8, 9, the operation of generalized gluing of variables was carried out. In matrix 10, the operation of absorption of variables was carried out. The result of the absorption of variables is recorded to matrix 11.

Post-simplification $x_1\overline{x_2}x_4\oplus\overline{x_1}x_2x_3x_4$ conjunctures take the following form (8):

$$(x_1\oplus x_2)(\overline{x_2}+x_3)x_4. \tag{8}$$

Thus, expression (8) includes two literals less compared to the expression of conjuncterms $x_1\overline{x_2}x_4\oplus\overline{x_1}x_2x_3x_4$. The verification of the result in (8) is given in Table 4.

Table 4

Verification of expression $(x_1\oplus x_2)(\overline{x_2}+x_3)x_4$ (8)

No.	x_1	x_2	x_3	x_4	$x_1\overline{x_2}\,\overline{x_3}x_4\oplus x_1\overline{x_2}x_3x_4\oplus\overline{x_1}x_2x_3x_4$	$(x_1\oplus x_2)(\overline{x_2}+x_3)x_4$	$(x_1\oplus x_2)(\overline{x_2}+x_3)x_4$
9	1	0	0	1	1	$(1_1\oplus 0_2)(\overline{0_2}+0_3)1_4$	1
11	1	0	1	1	1	$(1_1\oplus 0_2)(\overline{0_2}+1_3)1_4$	1
7	0	1	1	1	1	$(0_1\oplus 1_2)(\overline{1_2}+1_3)1_4$	1

When meditating on Table 4, we envision that the simplified expression (8) – $(x_1\oplus x_2)(\overline{x_2}+x_3)x_4$ satisfies the assigned conjuncterms $x_1\overline{x_2}x_4\oplus\overline{x_1}x_2x_3x_4$ of function (7). Replacing the conjuncterms $x_1\overline{x_2}x_4\oplus\overline{x_1}x_2x_3x_4$ of function (7) with expression (8) produces the minimum function in the mixed basis:

$$Y=x_3\oplus\overline{x_2}\oplus(x_1\oplus x_2)(\overline{x_2}+x_3)x_4,$$

which includes two literals less compared to (7) [29].

5. 2. Minimizing the symmetrical Boolean functions in the main basis by the method of figurative transformations

In a general case, SF can have several indexes. For example, the following SF:

$$H_3(1,2)=x_1\overline{x_2}\,\overline{x_3}+\overline{x_1}x_2\overline{x_3}+\overline{x_1}\,\overline{x_2}x_3+$$
$$+x_1x_2\overline{x_3}+\overline{x_1}x_2x_3+x_1\overline{x_2}x_3, \tag{9}$$

has two indexes: $a=1$ and $a=2$. Here, the variable sets $x_1\overline{x_2}\,\overline{x_3}$; $\overline{x_1}x_2\overline{x_3}$; $\overline{x_1}\,\overline{x_2}x_3$ have one 1, and the variable sets $x_1x_2\overline{x_3}$; $\overline{x_1}x_2x_3$; $x_1\overline{x_2}x_3$ have two 1's. Since the SF indexes (9) differ by unity, the operation of gluing the variables in the main basis $\{\vee,\wedge,\neg\}$ is possible [27].

Example 4. Use the method of figurative transformations to simplify the symmetric function $H_3(1,2)$ (9) [27]:

Solution.

Represent the equivalent binary matrix of the function $H_3(1,2)$ in lexicographic order and simplify $H_3(1,2)$ in the PPNF representation with the transition to the mixed basis.

$$H_3(1,2)_{\min}=\begin{vmatrix}0&0&1\\0&1&0\\0&1&1\\1&0&0\\1&0&1\\1&1&0\end{vmatrix}=\begin{vmatrix}0&0&1\\0&1&\\1&0&\\1&1&0\end{vmatrix}=$$
$$=\begin{vmatrix}0&&1\\0&1&\\1&0&\\1&&0\end{vmatrix}=x_1\oplus x_2+x_1\oplus x_3. \tag{10}$$

In the first matrix (10), simple gluing of variables was carried out, the result of the gluing is recorded in the second matrix. In the second matrix (10), the semi-gluing of variables was carried out, the result of semi-gluing the variables is written to the third matrix.

The MFS of function $H_3(1,2)$ (10) in the mixed basis is (11):

$$H_3(1,2)_{\min}=$$
$$=x_1\oplus x_2+x_1\oplus x_3. \tag{11}$$

MSF (11) presents a 2-level logic whose cost of implementation is:

$$k_\theta/k_l/k_{in}=2/4/0,$$

which is two literals less compared to [27].

Considering MSF (11), it is clear that the permutation of the variables x_2 and x_3 would not change the value of the function. Thus, the minimum function (11) in the mixed basis has a partial symmetry of the variables x_2 and x_3.

Example 5. Use the method of figurative transformations to simplify the symmetric function $H_4(2,3)$ set by a Karnaugh map (Fig. 2) [27].

We shall represent the equivalent binary matrix of function $H_4(2,3)$ (Fig. 2) in lexicographic order and simplify $H_4(2,3)$ (Fig. 2) in the PPNF representation with a transition to the mixed basis:

$$H_4(2,3)_{\min}=$$
$$=\begin{vmatrix}0&0&1&1\\0&1&0&1\\0&1&1&0\\0&1&1&1\\1&0&0&1\\1&0&1&0\\1&0&1&1\\1&1&0&0\\1&1&0&1\\1&1&1&0\end{vmatrix}=\begin{vmatrix}0&0&1&1\\0&1&0&1\\0&1&1&\\1&&0&1\\1&0&1&\\1&1&&0\end{vmatrix}=\begin{vmatrix}0&&1&1\\0&1&&1\\0&1&1&\\1&&0&1\\1&0&1&\\1&1&&0\end{vmatrix}=$$
$$=\begin{vmatrix}0&&1&1\\0&1&&1\\0&1&1&\\1&&0&1\\1&0&1&\\1&1&&0\end{vmatrix}=\begin{vmatrix}&&1&1\\&1&&1\\&1&1&\\1&&&1\\1&&1&\\1&1&&\end{vmatrix}=$$
$$=x_4(x_1\oplus x_3)+x_3(x_1\oplus x_2)+x_2(x_1\oplus x_4). \tag{12}$$

x_1x_2	x_3x_4			
	00	01	11	10
00	0	0	1	0
01	0	1	1	1
11	1	1	0	1
10	0	1	1	1

Fig. 2. Karnaugh map for the symmetric function $H_4(2,3)$

In the first matrix (12), the simple gluing of variables was carried out, the result of the gluing is recorded in the second matrix. In the second matrix (12), the semi-gluing of variables was carried out, the result of semi-gluing the variables is written to the third matrix. In the fourth matrix, we execute a transition to the mixed basis.

The MSF of function $H_4(2,3)$ (Fig. 2) [27] in the mixed basis is:

$$H_4(2,3)_{\min} = x_4(x_1 \oplus x_3) + x_3(x_1 \oplus x_2) + x_2(x_1 \oplus x_4). \quad (13)$$

MSF (13) represents a 3-level logic whose cost of implementation is:

$$k_\theta / k_l / k_{in} = 3/9/0,$$

which is three literals less compared to [27].

The permutation of variables x_1 and x_2 in MSF (13) yields the following function (14):

$$H_4(2,3)_{\min} = x_4(x_2 \oplus x_3) + x_3(x_1 \oplus x_2) + x_1(x_2 \oplus x_4). \quad (14)$$

The permutation of variables x_3 and x_4 in MSF (13) produces the following function (15):

$$H_4(2,3)_{\min} = x_3(x_1 \oplus x_4) + x_4(x_1 \oplus x_2) + x_2(x_1 \oplus x_3). \quad (15)$$

Both functions (14) and (15) do not change the value of function (13). This means that MSF (13) is symmetrical with respect to the variables x_1, x_2, and x_3, x_4. A similar symmetry of MSF (13) is expected with respect to the pairs of variables x_1, x_3; x_1, x_4; x_2, x_3, and x_2, x_4.

Example 6. Use the method of figurative transformations to simplify the Boolean function $f(x_1, x_2, x_3)$ with partial symmetry given in algebraic form [30]:

$$f(x_1,x_2,x_3) = \overline{x}_1\overline{x}_2\overline{x}_3 + \overline{x}_1\overline{x}_2x_3 + \overline{x}_1x_2x_3 + x_1\overline{x}_2\overline{x}_3 + x_1x_2\overline{x}_3 + x_1x_2x_3. \quad (16)$$

Solution.

Function (16) is partially symmetrical. The permutation of variables x_1 and x_3 does not change the value of the function.

We shall represent the equivalent binary matrix of function $f(x_1, x_2, x_3)$ (16) in lexicographic order and simplify $f(x_1, x_2, x_3)$ (16) in the PDNF representation with a transition to the mixed basis:

$$f(x_1,x_2,x_3)_{\min} = \left|\begin{array}{ccc} 0&0&0\\ 0&0&1\\ 0&1&1\\ 1&0&0\\ 1&1&0\\ 1&1&1 \end{array}\right| = \left|\begin{array}{ccc} &0&0\\ 0&0&1\\ &1&1\\ 1&1&0 \end{array}\right| = \left|\begin{array}{ccc} &0&0\\ 0&0& \\ &1&1\\ 1&1& \end{array}\right| = (\overline{x}_1 \oplus x_2) + (\overline{x}_3 \oplus x_4).$$

The MSF of function $f(x_1, x_2, x_3)$ (16) [30] in the mixed basis takes the following form (17):

$$f(x_1,x_2,x_3)_{\min} = (\overline{x}_1 \oplus x_2) + (\overline{x}_3 \oplus x_4). \quad (17)$$

MSFS (17) represents a 3-level logic whose cost of implementation is:

$$k_\theta / k_l / k_{in} = 2/4/2,$$

which is two literals and one term less compared to [30].

Function (17) remains partially symmetrical. The permutation of variables with indexes (1,3) does not change the value of the function.

5. 3. Minimizing the symmetric Boolean functions in the polynomial basis by the method of figurative transformations

Example 7. Use the method of figurative transformations to simplify the Boolean function with partial simple symmetry given in the following algebraic form (18) [31]:

$$f(x_1,x_2,x_3,x_4) = (x_1 \oplus x_2 \oplus \overline{x}_3\overline{x}_4 \oplus x_2x_3\overline{x}_4) + x_1x_2. \quad (18)$$

Solution.

$$f(x_1,x_2,x_3,x_4) = (x_1 \oplus x_2 \oplus \overline{x}_3\overline{x}_4 \oplus x_2x_3\overline{x}_4) + x_1x_2 =$$

$$= \left(\begin{array}{l} x_1\overline{x}_2 \oplus \overline{x}_1x_2 \oplus x_2\overline{x}_3\overline{x}_4 \oplus \overline{x}_2\overline{x}_3\overline{x}_4 \oplus \\ \oplus x_1x_2x_3\overline{x}_4 \oplus \overline{x}_1x_2x_3\overline{x}_4 \end{array}\right) + x_1x_2x_3 + x_1x_2\overline{x}_3 =$$

$$= \left(\begin{array}{l} x_1\overline{x}_2x_3 \oplus x_1\overline{x}_2\overline{x}_3 \oplus \overline{x}_1x_2x_3 \oplus \overline{x}_1x_2\overline{x}_3 \oplus \\ \oplus x_1x_2\overline{x}_3\overline{x}_4 \oplus \overline{x}_1x_2\overline{x}_3\overline{x}_4 \oplus x_1\overline{x}_2\overline{x}_3\overline{x}_4 \oplus \\ \oplus \overline{x}_1\overline{x}_2\overline{x}_3\overline{x}_4 \oplus x_1x_2x_3\overline{x}_4 \oplus \overline{x}_1x_2x_3\overline{x}_4 \end{array}\right) +$$

$$+ x_1x_2x_3x_4 + x_1x_2x_3\overline{x}_4 + x_1x_2\overline{x}_3x_4 + x_1x_2\overline{x}_3\overline{x}_4 =$$

$$= \left(\begin{array}{l} x_1\overline{x}_2x_3x_4 \oplus x_1\overline{x}_2x_3\overline{x}_4 \oplus x_1\overline{x}_2\overline{x}_3x_4 \oplus \cancel{x_1\overline{x}_2\overline{x}_3\overline{x}_4} \oplus \\ \oplus \overline{x}_1x_2x_3x_4 \oplus \cancel{\overline{x}_1x_2x_3\overline{x}_4} \oplus \overline{x}_1x_2\overline{x}_3x_4 \oplus \cancel{\overline{x}_1x_2\overline{x}_3\overline{x}_4} \oplus \\ \oplus x_1x_2\overline{x}_3\overline{x}_4 \oplus \cancel{\overline{x}_1x_2\overline{x}_3\overline{x}_4} \oplus \cancel{x_1\overline{x}_2\overline{x}_3\overline{x}_4} \oplus \\ \oplus \overline{x}_1\overline{x}_2\overline{x}_3\overline{x}_4 \oplus x_1x_2x_3\overline{x}_4 \oplus \cancel{\overline{x}_1x_2x_3\overline{x}_4} \end{array}\right) +$$

$$+ x_1x_2x_3x_4 + x_1x_2x_3\overline{x}_4 + x_1x_2\overline{x}_3x_4 + x_1x_2\overline{x}_3\overline{x}_4 =$$

$$= \left(\begin{array}{l} x_1\overline{x}_2x_3x_4 \oplus x_1\overline{x}_2x_3\overline{x}_4 \oplus x_1\overline{x}_2\overline{x}_3x_4 \oplus \overline{x}_1x_2x_3x_4 \oplus \\ \oplus \overline{x}_1x_2\overline{x}_3x_4 \oplus x_1x_2\overline{x}_3\overline{x}_4 \oplus \overline{x}_1\overline{x}_2\overline{x}_3\overline{x}_4 \oplus x_1x_2x_3\overline{x}_4 \end{array}\right) +$$

$$+ x_1x_2x_3x_4 + x_1x_2x_3\overline{x}_4 + x_1x_2\overline{x}_3x_4 + x_1x_2\overline{x}_3\overline{x}_4 =$$

$$= x_1\overline{x}_2x_3x_4 + x_1\overline{x}_2x_3\overline{x}_4 + x_1\overline{x}_2\overline{x}_3x_4 + \overline{x}_1x_2x_3x_4 +$$

$$+ \overline{x}_1x_2\overline{x}_3x_4 + x_1x_2\overline{x}_3\overline{x}_4 + \overline{x}_1\overline{x}_2\overline{x}_3\overline{x}_4 + x_1x_2x_3\overline{x}_4 +$$

$$+ x_1x_2x_3x_4 + x_1x_2x_3\overline{x}_4 + x_1x_2\overline{x}_3x_4 + x_1x_2\overline{x}_3\overline{x}_4 =$$

$$= x_1\overline{x}_2x_3x_4 + x_1\overline{x}_2x_3\overline{x}_4 + x_1\overline{x}_2\overline{x}_3x_4 + \overline{x}_1x_2x_3x_4 +$$

$$+ \overline{x}_1x_2\overline{x}_3x_4 + x_1x_2\overline{x}_3\overline{x}_4 + \overline{x}_1\overline{x}_2\overline{x}_3\overline{x}_4 +$$

$$+ x_1x_2x_3\overline{x}_4 + x_1x_2x_3x_4 + x_1x_2\overline{x}_3x_4 =$$

$$= \left|\begin{array}{cccc} 0&0&0&0\\ 0&1&0&1\\ 0&1&1&1\\ 1&0&0&1\\ 1&0&1&0\\ 1&0&1&1\\ 1&1&0&0\\ 1&1&0&1\\ 1&1&1&0\\ 1&1&1&1 \end{array}\right| = \left|\begin{array}{cccc} 0&0&0&0\\ 0&1&0&1\\ 0&1&1&1\\ 1&0&0&0\\ 1&0&0&0\\ 1&0&0&1\\ 1&0&1&0\\ 1&0&1&1\\ 1&1&0&0\\ 1&1&0&1\\ 1&1&1&0\\ 1&1&1&1 \end{array}\right| = \left|\begin{array}{cccc} 0&0&0&0\\ 0&1&0&1\\ 0&1&1&1\\ 1&0&0&0\\ 1&&& \end{array}\right| = \left|\begin{array}{cccc} &0&0&0\\ 0&1&&1\\ 1&&& \end{array}\right| =$$

$$= (x_1 + x_2x_4) \oplus \overline{x}_2\overline{x}_3\overline{x}_4. \quad (19)$$

In the first matrix (19), the procedure for inserting similar PNF conjuncterms with the following operation of super-gluing the variables [26] was applied. The result of the procedure is written to the second matrix (19).

The MSF of function (18) [31] in the mixed basis takes the following form (20):

$$f\left(x_1,x_2,x_3,x_4\right)_{\min}=\left(x_1+x_2x_4\right)\oplus\overline{x_2}\;\overline{x_3}\;\overline{x_4}. \qquad (20)$$

MSF (20) represents a 3-level logic whose cost of implementation is:

$$k_\theta/k_l/k_{in}=2/6/3,$$

which is three literals less compared to (18).

Considering MSF (20), it is clear that the permutation of variables x_2 and x_4; $\overline{x_2}$ and $\overline{x_4}$ would not change the value of the function.

Therefore, function (20) has a partial simple symmetry with respect to the variables x_2 and x_4 and $\overline{x_2}$ and $\overline{x_4}$.

Example 8. It is required to simplify a partially symmetrical Boolean function given in the following canonical form (21) [32]:

$$f=\left(0,3,5,6,7,8,9,10,12,15\right), \qquad (21)$$

by applying the method of inserting similar conjuncterms with the following operation of super-gluing the variables.

Solution.

In the polynomial format, by applying a Karnaugh map, the authors of work [32] simplified function (21) to the following form:

$$f=x_1\oplus x_2x_3\oplus x_2x_4\oplus x_3x_4\oplus x_1x_2x_3x_4\oplus\overline{x}_1\overline{x}_2\overline{x}_3\overline{x}_4,$$

whose cost of implementation is $k_\theta/k_l=6/15$.

In paper [29], the same function (21) was simplified to the following form:

$$f=x_1\oplus x_2\oplus\overline{x}_2x_3x_4\oplus\overline{x}_1\overline{x}_3\overline{x}_4\oplus x_1x_2\overline{x}_3\oplus x_1x_2x_4,$$

whose cost of implementation is $k_\theta/k_l=6/14$.

Function (21) in article [26] was simplified to the following form:

$$Y_{\text{PNF}}=x_1\oplus x_2\oplus\overline{x_3}\oplus x_4\oplus\overline{x_1}x_2x_3x_4\oplus x_1\overline{x_2}\;\overline{x_3}\;\overline{x_4}. \qquad (22)$$

whose cost of implementation is $k_\theta/k_l=6/12$.

We shall apply to the conjuncterms $\overline{x_1}x_2x_3x_4\oplus x_1\overline{x_2}\;\overline{x_3}\;\overline{x_4}$ of function (22) the method of inserting similar conjuncterms with the following operation of super-gluing the variables; we obtain:

$$\overline{x_1}x_2x_3x_4\oplus x_1\overline{x_2}\;\overline{x_3}\;\overline{x_4}=\left(x_1\oplus x_2\right)\left(x_1\oplus x_3\right)\left(x_1\oplus x_4\right).$$

The function (22) minimization can be continued:

$$Y_{\text{PNF}}=x_1\oplus x_2\oplus\overline{x_3}\oplus x_4\oplus\overline{x_1}x_2x_3x_4\oplus x_1\overline{x_2}\;\overline{x_3}\;\overline{x_4}=$$

$$=x_1\oplus x_2\oplus\overline{x_3}\oplus x_4\oplus\left(x_1\oplus x_2\right)\left(x_1\oplus x_3\right)\left(x_1\oplus x_4\right)=$$

$$=\overline{x_3}\oplus x_4\oplus\left(x_1\oplus x_2\right)\overline{\left(x_1\oplus x_3\right)\left(x_1\oplus x_4\right)}=$$

$$=\left(\overline{x_3}\oplus x_4\right)\oplus\left(x_1\oplus x_2\right)\left(\left(\overline{x_1\oplus x_3}\right)+\left(\overline{x_1\oplus x_4}\right)\right).$$

The MSF of function (21) will take the following form (23):

$$Y_{\text{PNF}}=\left(\overline{x_3}\oplus x_4\right)\oplus\left(x_1\oplus x_2\right)\left(\left(\overline{x_1\oplus x_3}\right)+\left(\overline{x_1\oplus x_4}\right)\right). \qquad (23)$$

The cost of implementing the MSF of (23) is:

$$k_\theta/k_l/k_{in}=2/8/3,$$

which is seven literals less compared to [32].

The result of the direct simplification of the conjuncterms $\overline{x}_1x_2x_3x_4\oplus x_1\overline{x_2}\;\overline{x_3}\;\overline{x_4}$ of the function

$$Y_{\text{PNF}}=x_1\oplus x_2\oplus\overline{x_3}\oplus x_4\oplus\overline{x_1}x_2x_3x_4\oplus x_1\overline{x_2}\;\overline{x_3}\;\overline{x_4}$$

by the method of inserting similar conjuncterms with the following operation of super-gluing the variables is the expression $\left(x_1\oplus x_2\right)\left(x_1\oplus x_3\right)\left(x_1\oplus x_4\right)$. Next, the polynomial absorption of the variables $x_1\oplus x_2$ and $\left(x_1\oplus x_2\right)\left(x_1\oplus x_3\right)\left(x_1\oplus x_4\right)$ is carried out [26], the result of which is the logical expression – $\left(x_1\oplus x_2\right)\overline{\left(x_1\oplus x_3\right)\left(x_1\oplus x_4\right)}$.

5. 4. Comparing the method of figurative transformations with the methods of decomposition in the simplification of Boolean functions

It is believed that Boolean functions that depend on a large number of variables are not minimized optimally [33]. In this regard, there is a problem with the decomposition of Boolean functions. The solution is to break the source function into a minimum number of Boolean functions, each of which depends on a smaller number of variables compared to the original function. There is a decomposition dividing the variables of the source function [34] and one that does not separate the variables [35].

Example 9. Use the method of figurative transformations to simplify the Boolean function given in the disjunctive normal form [34]:

$$F=\overline{a}bd+bcd+a\overline{c}\;\overline{d}+a\overline{b}\;\overline{c}. \qquad (24)$$

Solution.

Function (24) is partially symmetrical. The permutation of variables with indexes (2,4) does not change the value of function (24):

$$F_{\min}=\begin{vmatrix}0&1&&1\\&1&1&1\\1&&0&0\\1&0&0&\end{vmatrix}=\begin{vmatrix}0&1&1&1\\0&1&0&1\\&1&1&1\\1&&0&0\\1&0&0&0\\1&0&0&1\end{vmatrix}=$$

$$=\begin{vmatrix}0&1&0&1\\&1&1&1\\1&&0&0\\1&0&0&1\end{vmatrix}=\begin{vmatrix}&1&0&1\\&1&1&1\\1&&0&0\\1&&0&1\end{vmatrix}=$$

$$=\begin{vmatrix}&1&&1\\1&&0&\end{vmatrix}=a\overline{c}\oplus bd.$$

The minimum function remains partially symmetrical. The permutation of variables with indexes (2,4) does not change the value of the minimum function.

The results of function (24) simplification by the method of figurative transformations and the method of decomposition coincide but the method of figurative transformations in a given example is simpler.

Example 10. Use the method of figurative transformations to simplify the Boolean function given in the disjunctive normal form [35].

$$f(x_1,x_2,x_3)=\overline{x_1}\,\overline{x_2}\,\overline{x_3}+x_1x_2x_3. \quad (25)$$

Solution.

Function (25) is symmetrical. The permutation of variables with any pair of indexes does not change the value of function (25).

According to the equivalent transformations of PNF conjuncterms of Boolean functions (chapter 5. 1.) (Table 2), we find the minimum function:

$$f(x_1,x_2,x_3)=\overline{x_1}\,\overline{x_2}\,\overline{x_3}+x_1x_2x_3=\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right). \quad (26)$$

The minimum function remains partially symmetrical. The permutation of variables with indexes (2,3) does not change the value of the minimum function.

The results of the simplification of function (25) by the method of figurative transformations and by the method of decomposition, which does not separate the variables, coincide but the method of figurative transformations in a given example is simpler. The logical structure of minimum function (26) coincides with the logical structure of a two-block decomposition, which does not separate the variables [35].

5. 5. Synthesizing a symmetrical 4-input adder of binary operands

The symmetrical 4-input adder of binary codes is set by the truth table (Table 5).

Table 5

Truth table of 4-input symmetric adder

Input					Output		
No.	a_1	a_2	a_3	a_4	S_2	S_1	S_0
0	0	0	0	0	0	0	0
1	0	0	0	1	0	0	1
2	0	0	1	0	0	0	1
3	0	0	1	1	0	1	0
4	0	1	0	0	0	0	1
5	0	1	0	1	0	1	0
6	0	1	1	0	0	1	0
7	0	1	1	1	0	1	1
8	1	0	0	0	0	0	1
9	1	0	0	1	0	1	0
10	1	0	1	0	0	1	0
11	1	0	1	1	0	1	1
12	1	1	0	0	0	1	0
13	1	1	0	1	0	1	1
14	1	1	1	0	0	1	1
15	1	1	1	1	1	0	0

According to Table 5, adding any single operand a_1, a_2, a_3, a_4 gives the same value of the amount – 001; adding any two pairs of operands a_3a_4, a_2a_4, a_2a_3, a_1a_4, a_1a_3, a_1a_2 gives the same value of the sum – 010; adding any trio of operands $a_2a_3a_4$, $a_1a_3a_4$, $a_1a_2a_4$, $a_1a_2a_3$ gives the same value of the sum – 011; adding four operands gives the value of the sum – 100. Thus, any permutation π of variable values does not change the value of the function, and, therefore the 4-input adder, given in Table 5, is symmetrical.

The system of equations of a 4-input symmetric adder in PDNF is as follows:

$$\begin{cases} S_0=\overline{x_1}\,\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}\,\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2\overline{x_3}\,\overline{x_4}+ \\ +\overline{x_1}x_2x_3x_4+x_1\overline{x_2}\,\overline{x_3}\,\overline{x_4}+x_1\overline{x_2}x_3x_4+ \\ +x_1x_2\overline{x_3}x_4+x_1x_2x_3\overline{x_4}; \\ S_1=\overline{x_1}\,\overline{x_2}x_3x_4+\overline{x_1}x_2\overline{x_3}x_4+ \\ +\overline{x_1}x_2x_3\overline{x_4}+\overline{x_1}x_2x_3x_4+ \\ +x_1\overline{x_2}\,\overline{x_3}x_4+x_1\overline{x_2}x_3\overline{x_4}+ \\ +x_1\overline{x_2}x_3x_4+x_1x_2\overline{x_3}\,\overline{x_4}+ \\ +x_1x_2\overline{x_3}x_4+x_1x_2x_3\overline{x_4}; \\ S_2=x_1x_2x_3x_4. \end{cases} \quad (27)$$

There are two approaches to minimizing the systems of Boolean functions depending on n variables [36, 37]. We shall minimize the system of functions (27) separately for each function.

Minimizing the function S_0.

The S_0 function is singular [26]; to minimize S_0, we select Reed-Muller's algebra. We use identities (28) and (29) [26]:

$$x_1\overline{x_2}\oplus\overline{x_1}x_2=x_1\oplus x_2=\overline{x_2}\oplus\overline{x_1}. \quad (28)$$

$$xy\oplus\overline{x}\,\overline{y}=\overline{x\oplus y}=\overline{\overline{x}\oplus\overline{y}}=x\oplus\overline{y}=\overline{x}\oplus y. \quad (29)$$

$$S_0=\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}=\begin{vmatrix} 0 & 0 & & 1 \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & \\ 0 & 1 & & 1 \\ 1 & 0 & 0 & \\ 1 & 0 & & 1 \\ 1 & 1 & & 1 \\ 1 & 1 & 1 & \end{vmatrix}=$$

$$=\overline{x_1}\,\overline{x_2}\left(x_3\oplus x_4\right)\oplus x_1x_2\left(x_3\oplus x_4\right)\oplus$$

$$\oplus\overline{x_1}x_2\left(\overline{x_3}\oplus x_4\right)\oplus x_1\overline{x_2}\left(\overline{x_3}\oplus x_4\right)=$$

$$=\left(x_3\oplus x_4\right)\left(\overline{x_1}\,\overline{x_2}\oplus x_1x_2\right)\oplus$$

$$\oplus\left(\overline{x_3}\oplus x_4\right)\left(\overline{x_1}x_2\oplus x_1\overline{x_2}\right)=$$

$$=\left(x_3\oplus x_4\right)\left(\overline{x_1\oplus x_2}\right)\oplus\left(\overline{x_3\oplus x_4}\right)\left(x_1\oplus x_2\right)=$$

$$=\left(x_1\oplus x_2\right)\oplus\left(x_3\oplus x_4\right)=$$

$$=x_1\oplus x_2\oplus x_3\oplus x_4.$$

Minimizing the function S_1:

$$S_1=\begin{vmatrix}0&0&1&1\\0&1&0&1\\0&1&1&0\\0&1&1&1\\1&0&0&1\\1&0&1&0\\1&0&1&1\\1&1&0&0\\1&1&0&1\\1&1&1&0\end{vmatrix}=\begin{vmatrix}0&&1&1\\0&1&&1\\&1&1&0\\&1&1&1\\1&0&&1\\1&0&1&\\1&&1&1\\1&1&&0\\1&1&&1\\1&1&1&\end{vmatrix}=\begin{vmatrix}&&1&1\\&1&&1\\&1&1&\\1&&&1\\1&1&&\\1&&1&\end{vmatrix}=$$

$$x_1x_2\oplus x_3x_4\oplus x_1x_3\oplus x_1x_4\oplus x_2x_3\oplus x_2x_4=$$
$$=x_1x_2\oplus x_3x_4\oplus x_1(x_3\oplus x_4)\oplus x_2(x_3\oplus x_4)=$$
$$=x_1x_2\oplus x_3x_4\oplus(x_3\oplus x_4)(x_1\oplus x_2).$$

$$S_1=x_1x_2\oplus x_3x_4\oplus(x_3\oplus x_4)(x_1\oplus x_2).$$

$$S_1=x_1x_2\oplus x_3x_4\oplus(x_3\oplus x_4)(x_1\oplus x_2)=$$
$$=(x_1x_2\oplus x_3x_4)\overline{(x_3\oplus x_4)(x_1\oplus x_2)}+$$
$$+\overline{x_1x_2\oplus x_3x_4}(x_3\oplus x_4)(x_1\oplus x_2).$$

$$(x_1x_2\oplus x_3x_4)\overline{(x_3\oplus x_4)(x_1\oplus x_2)}=$$
$$=(x_1x_2\oplus x_3x_4)\left(\overline{x_3\oplus x_4}+\overline{x_1\oplus x_2}\right)=$$
$$=(x_1x_2\oplus x_3x_4)\left(x_3x_4\oplus\overline{x_3}\,\overline{x_4}+x_1x_2\oplus\overline{x_1}\,\overline{x_2}\right)=$$
$$=\left(x_1x_2\overline{x_3x_4}+\overline{x_1x_2}x_3x_4\right)\begin{pmatrix}x_3x_4\overline{\overline{x_3}}\,\overline{\overline{x_4}}+\overline{x_3x_4}\,\overline{x_3}\,\overline{x_4}+\\+x_1x_2\overline{\overline{x_1}}\,\overline{\overline{x_2}}+\overline{x_1x_2}\,\overline{x_1}\,\overline{x_2}\end{pmatrix}=$$
$$=\left(x_1x_2\left(\overline{x_3}+\overline{x_4}\right)+\left(\overline{x_1}+\overline{x_2}\right)x_3x_4\right)\times$$
$$\times\begin{pmatrix}x_3x_4(x_3+x_4)+\left(\overline{x_3}+\overline{x_4}\right)\overline{x_3}\,\overline{x_4}+\\+x_1x_2(x_1+x_2)+\left(\overline{x_1}+\overline{x_2}\right)\overline{x_1}\,\overline{x_2}\end{pmatrix}=$$
$$=\left(x_1x_2\overline{x_3}+x_1x_2\overline{x_4}+\overline{x_1}x_3x_4+\overline{x_2}x_3x_4\right)\times$$
$$\times\begin{pmatrix}x_3x_4+x_3x_4+\overline{x_3}\,\overline{x_4}+\overline{x_3}\,\overline{x_4}+\\+x_1x_2+x_1x_2+\overline{x_1}\,\overline{x_2}+\overline{x_1}\,\overline{x_2}\end{pmatrix}=$$
$$=\left(x_1x_2\overline{x_3}+x_1x_2\overline{x_4}+\overline{x_1}x_3x_4+\overline{x_2}x_3x_4\right)\times$$
$$\times\left(x_3x_4+\overline{x_3}\,\overline{x_4}+x_1x_2+\overline{x_1}\,\overline{x_2}\right)=$$
$$=x_1x_2\overline{x_3}\,\overline{x_4}+x_1x_2\overline{x_3}+x_1x_2\overline{x_3}\,\overline{x_4}+x_1x_2\overline{x_4}+$$
$$+\overline{x_1}x_3x_4+\overline{x_1}\,\overline{x_2}x_3x_4+\overline{x_2}x_3x_4+\overline{x_1}\,\overline{x_2}x_3x_4=$$
$$=\begin{vmatrix}1&1&0&0\\1&1&0&\\1&1&0&0\\1&1&&0\\0&&1&1\\0&0&1&1\\&0&1&1\\0&0&1&1\end{vmatrix}=\begin{vmatrix}1&1&0&\\1&1&0&0\\1&1&&0\\0&&1&1\\0&0&1&1\\&0&1&1\end{vmatrix}=\begin{vmatrix}1&1&0&\\1&1&&0\\0&&1&1\\&0&1&1\end{vmatrix}=$$
$$=x_1x_2\left(\overline{x_3}+\overline{x_4}\right)+x_3x_4\left(\overline{x_1}+\overline{x_2}\right)=$$
$$=x_1x_2\overline{x_3x_4}+x_3x_4\overline{x_1x_2}=x_1x_2\oplus x_3x_4.$$

$$\overline{x_1x_2\oplus x_3x_4}(x_3\oplus x_4)(x_1\oplus x_2)=$$
$$=\left(x_1x_2x_3x_4\oplus\overline{x_1x_2}\,\overline{x_3x_4}\right)\left(x_3\overline{x_4}+\overline{x_3}x_4\right)\left(x_1\overline{x_2}+\overline{x_1}x_2\right)=$$
$$=\left(x_1x_2x_3x_4\overline{\overline{x_1x_2}\,\overline{x_3x_4}}+\overline{x_1x_2x_3x_4}\,\overline{x_1x_2}\,\overline{x_3x_4}\right)\times$$
$$\times\left(x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\right)=$$
$$=\begin{pmatrix}x_1x_2x_3x_4\left(\overline{x_1x_2}+\overline{x_3x_4}\right)+\\+\left(\overline{x_1}+\overline{x_2}+\overline{x_3}+\overline{x_4}\right)\left(\overline{x_1}+\overline{x_2}\right)\left(\overline{x_3}+\overline{x_4}\right)\end{pmatrix}\times$$
$$\times\left(x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\right)=$$
$$=\begin{pmatrix}x_1x_2x_3x_4\left(\overline{x_1}+\overline{x_2}+\overline{x_3}+\overline{x_4}\right)+\\+\left(\overline{x_1}+\overline{x_2}+\overline{x_3}+\overline{x_4}\right)\left(\overline{x_1}\,\overline{x_3}+\overline{x_1}\,\overline{x_4}+\overline{x_2}\,\overline{x_3}+\overline{x_2}\,\overline{x_4}\right)\end{pmatrix}\times$$
$$\times\left(x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\right)=$$
$$=\left(\overline{x_1}+\overline{x_2}+\overline{x_3}+\overline{x_4}\right)\left(\overline{x_1}\,\overline{x_3}+\overline{x_1}\,\overline{x_4}+\overline{x_2}\,\overline{x_3}+\overline{x_2}\,\overline{x_4}\right)\times$$
$$\times\left(x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\right)=$$
$$=\begin{pmatrix}\overline{x_1}\,\overline{x_3}+\overline{x_1}\,\overline{x_4}+\overline{x_2}\,\overline{x_3}+\overline{x_2}\,\overline{x_4}+\overline{x_1}\,\overline{x_2}\,\overline{x_3}+\overline{x_1}\,\overline{x_2}\,\overline{x_4}+\\+\overline{x_2}\,\overline{x_3}+\overline{x_2}\,\overline{x_4}+\overline{x_1}\,\overline{x_3}+\overline{x_1}\,\overline{x_3}\,\overline{x_4}+\overline{x_2}\,\overline{x_3}+\overline{x_2}\,\overline{x_3}\,\overline{x_4}\\+\overline{x_1}\,\overline{x_3}\,\overline{x_4}+\overline{x_1}\,\overline{x_4}+\overline{x_2}\,\overline{x_3}\,\overline{x_4}+\overline{x_2}\,\overline{x_4}\end{pmatrix}\times$$
$$\times\left(x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\right)=$$
$$=\begin{vmatrix}0&&0&\\0&&&0\\&0&0&\\&0&&0\\0&0&0&\\0&0&&0\\&0&0&\\&0&&0\\0&&0&\\0&&0&0\\&0&0&\\&0&0&0\\0&&0&0\\0&&&0\\&0&0&0\\&0&&0\end{vmatrix}\begin{pmatrix}x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+\\+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\end{pmatrix}=$$
$$=\begin{vmatrix}0&&&0\\0&&0&\\&0&0&\\&0&&0\end{vmatrix}\begin{pmatrix}x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+\\+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\end{pmatrix}=$$
$$=\begin{pmatrix}\overline{x_1}\,\overline{x_4}+\overline{x_1}\,\overline{x_3}+\\+\overline{x_2}\,\overline{x_3}+\overline{x_2}\,\overline{x_4}\end{pmatrix}\begin{pmatrix}x_1\overline{x_2}x_3\overline{x_4}+\overline{x_1}x_2x_3\overline{x_4}+\\+x_1\overline{x_2}\,\overline{x_3}x_4+\overline{x_1}x_2\overline{x_3}x_4\end{pmatrix}=$$
$$=\overline{x_1}x_2x_3\overline{x_4}+\overline{x_1}x_2\overline{x_3}x_4+x_1\overline{x_2}\,\overline{x_3}x_4+x_1\overline{x_2}x_3\overline{x_4}=$$
$$=\begin{vmatrix}0&1&1&0\\0&1&0&1\\1&0&0&1\\1&0&1&0\end{vmatrix}=\overline{x_1}x_2(x_3\oplus x_4)+x_1\overline{x_2}(x_3\oplus x_4)=$$
$$=(x_3\oplus x_4)\left(\overline{x_1}x_2+x_1\overline{x_2}\right)=(x_1\oplus x_2)(x_3\oplus x_4).$$

Thus,

$$S_1 = (x_1x_2 \oplus x_3x_4) + (x_1 \oplus x_2)(x_3 \oplus x_4).$$

The system of equations (27) takes the following form:

$$\begin{cases} S_0 = x_1 \oplus x_2 \oplus x_3 \oplus x_4; \\ S_1 = x_1x_2 \oplus x_3x_4 + (x_1 \oplus x_2)(x_3 \oplus x_4); \\ S_2 = x_1x_2x_3x_4. \end{cases} \quad (30)$$

Based on the system of equations (30), we build a scheme of a 4-input symmetrical adder of binary codes (Fig. 3).

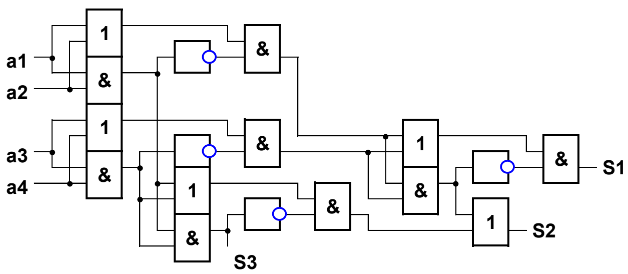


Fig. 3. Symmetrical 4-input binary code adder

The scheme of a 4-input symmetrical adder in Fig. 3 is optimal; it complies with the verification truth table (Table 5); it has less complexity compared to the schemes of open patents back in the USSR [38–45] about symmetrical 4-input adders.

6. Discussion of results of minimizing the symmetric Boolean functions by the method of figurative transformations

The mathematical apparatus of minimizing the Boolean functions by the method of figurative transformations was considered in works [25, 26, 37, 46–49, 51], and others. The technology of the method of figurative transformations is given in Table 6.

A new component in the technology for minimizing Boolean functions by the method of figurative transformations (MFT) are equivalent transformations of the polynomial normal form (PNF) (chapter 5. 1.). The form of equivalent transformations (chapter 5. 1., Tables 2, 3) is similar to the decomposition of an analytical function into multipliers, in a given case, by inserting similar conjuncterms with the following operation of super-gluing the variables.

Equivalent transformations of PNF (chapter 5. 1., Table 2) can be an object for comparison, to a certain extent, with the rules of minimization based on the splitting of conjuncterms in a polynomial theoretic-multiple format (PTMF) [29] (Table 7).

The peculiarity of applying the theoretical-multiple rules for splitting the PNF conjuncterms [29] is that the lower-ranked conjuncterms formed by the splitting procedure can be simplified according to the rules of equivalent transformation with other conjuncterms of the assigned function. In this case, the Boolean function will be optimized [29].

Table 7 demonstrates that the conjuncterms in the following form:

$$\begin{pmatrix} 000 \\ 111 \end{pmatrix}^{\oplus}, \begin{pmatrix} 0000 \\ 1111 \end{pmatrix}^{\oplus}, \begin{pmatrix} 0000 \\ 0111 \end{pmatrix}^{\oplus} \quad (31)$$

are not simplified directly by the method of splitting the conjuncterms.

Conjuncterms (31) can be simplified by inserting similar conjuncterms with the following operation of super-gluing the variables (chapter 5. 1., Table 2). The resulting logical expressions, after such a procedure, can be simplified according to the rules of equivalent transformation with other terms of the assigned function (chapter 5. 3., example 8). Identifying this algorithm is a reserve for increasing the hardware capabilities in minimizing the symmetrical Boolean functions by a visual-matrix form of the analytical method.

Table 6

Figurative transformation method technology

1	Binary combinatorial systems with repeated 2-(n, b)-design, 2-(n, x/b)-design
2	Verbal and figurative representation of information
3	Logical operation of super-gluing the variables
4	Logical operation of incomplete super-gluing the variables
5	Hermeneutics of logical operations on binary equivalents of logical functions
6	Figurative transformation protocols
7	Attribute of a minimum logical function
8	Minimizing Boolean functions on a complete truth table
9	Algorithm of analytical method and its automation
10	Expanding the analytical method to other logical bases
11	Algebra of equivalent transformation in the class of perfect normal forms of Schaeffer algebra functions
12	Algebra of equivalent transformation in the class of perfect implicit normal forms
13	Relatively complex algorithms for the use of logical absorption operations and super-gluing the variables
14	Stack of logical operations
15	Algorithms for simplifying a function with the procedure of inserting two identical PNF conjuncterms with the following operation of super-gluing the variables
16	Singular function
17	Algebra of equivalent transformation in the class of polynomial normal forms of Boolean functions
18	Mixed basis

Table 7

Modern equivalent transformations of conjuncterms in a polynomial format

Function	Theoretical-multiple rules for splitting the PNF conjuncterms	Equivalent transformations of MFT PNF conjuncterms
$\begin{pmatrix}000\\111\end{pmatrix}^{\oplus}$	$\left\{\left[\begin{pmatrix}00-\\0-1\\-11\end{pmatrix},\begin{pmatrix}00-\\-01\\1-1\end{pmatrix},\begin{pmatrix}0-0\\-10\\11-\end{pmatrix},\begin{pmatrix}0-0\\01-\\-11\end{pmatrix},\begin{pmatrix}-00\\10-\\1-1\end{pmatrix},\begin{pmatrix}-00\\1-0\\11-\end{pmatrix}\right]\right\}$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)$
$\begin{pmatrix}0000\\1111\end{pmatrix}^{\oplus}$	$\left\{\left[\begin{pmatrix}000-\\00-1\\0-11\\-111\end{pmatrix},\begin{pmatrix}000-\\00-1\\-011\\1-11\end{pmatrix},\begin{pmatrix}000-\\0-01\\-101\\11-1\end{pmatrix},\begin{pmatrix}000-\\0-01\\01-1\\-111\end{pmatrix},\begin{pmatrix}000-\\-001\\10-1\\1-11\end{pmatrix},\begin{pmatrix}000-\\-001\\1-01\\11-1\end{pmatrix},\right.\right.$ $\begin{pmatrix}00-0\\0-10\\-110\\111-\end{pmatrix},\begin{pmatrix}00-0\\0-10\\011-\\-111\end{pmatrix},\begin{pmatrix}00-0\\-010\\101-\\1-11\end{pmatrix},\begin{pmatrix}00-0\\-010\\1-10\\111-\end{pmatrix},\begin{pmatrix}00-0\\001-\\0-11\\-111\end{pmatrix},\begin{pmatrix}00-0\\001-\\-011\\1-11\end{pmatrix},$ $\begin{pmatrix}0-00\\-100\\110-\\11-1\end{pmatrix},\begin{pmatrix}0-00\\-100\\11-0\\111-\end{pmatrix},\begin{pmatrix}0-00\\010-\\01-1\\-111\end{pmatrix},\begin{pmatrix}0-00\\010-\\-101\\11-1\end{pmatrix},\begin{pmatrix}0-00\\01-0\\-110\\111-\end{pmatrix},\begin{pmatrix}0-00\\01-0\\011-\\-111\end{pmatrix},$ $\left.\left.\begin{pmatrix}-000\\100-\\10-1\\1-11\end{pmatrix},\begin{pmatrix}-000\\100-\\1-01\\11-1\end{pmatrix},\begin{pmatrix}-000\\10-0\\1-10\\111-\end{pmatrix},\begin{pmatrix}-000\\10-0\\101-\\1-11\end{pmatrix},\begin{pmatrix}-000\\1-00\\110-\\11-1\end{pmatrix},\begin{pmatrix}-000\\1-00\\11-0\\111-\end{pmatrix}\right]\right\}$	$\left(\overline{x_1}\oplus x_2\right)\left(\overline{x_1}\oplus x_3\right)\left(\overline{x_1}\oplus x_4\right)$
$\begin{pmatrix}0000\\0111\end{pmatrix}^{\oplus}$	$\left\{\left[\begin{pmatrix}000-\\00-1\\0-11\end{pmatrix},\begin{pmatrix}000-\\0-01\\01-1\end{pmatrix},\begin{pmatrix}00-0\\0-10\\011-\end{pmatrix},\begin{pmatrix}00-0\\001-\\0-11\end{pmatrix},\begin{pmatrix}0-00\\010-\\01-1\end{pmatrix},\begin{pmatrix}0-00\\01-0\\011-\end{pmatrix}\right]\right\}$	$\overline{x_1}\left(\overline{x_2\oplus x_3}\right)\left(\overline{x_2\oplus x_4}\right)$
$\begin{pmatrix}000-\\0110\end{pmatrix}^{\oplus}$	$\begin{pmatrix}00--\\0-1-\\0111\end{pmatrix}$	$\overline{x_1}\left(\overline{x_2\oplus x_3}\right)\left(\overline{x_2}+\overline{x_4}\right)$

MFT provides for the minimization of symmetrical Boolean functions in the main basis $\{\vee,\wedge,\neg\}$ (chapter 5. 2., examples 4–6). However, not all symmetrical functions in PPNF or PCNF are minimized in the Boole basis. In this case, one needs to try to optimize the assigned function in a polynomial basis $\{\wedge,\oplus,1\}$, using the Reed-Muller algebra [26] (chapter 5. 3., examples 7, 8).

MFT implies the analysis of a stack of logical operations [25], which is a certain analog and difference from decomposition (chapter 5. 4.).

The stack makes it possible to select a promising simplification option for the assigned function.

According to data at our disposal (examples 9 and 10), it can be noted that the results of simplification of functions by the method of figurative transformations and decomposition methods coincide but the MFT is much simpler. Minimizing by two methods for a larger, to a certain extent, number of variables would lead to their corresponding conclusions. Work [25] reports the minimization of a Boolean function by the method of figurative transformations into 64 input variables.

A method of splitting the conjuncterms can be used to simplify conjuncterms in the form $\begin{pmatrix}000-\\0110\end{pmatrix}^{\oplus}$ (Table 7) by using the logical operation of polynomial absorption of variables [26]:

$$\begin{array}{|cccc|}\hline x_1 & x_2 & x_3 & x_4\\ \hline 0 & 0 & 0 & -\\ 0 & 1 & 1 & 0\\ \hline\end{array} = \begin{vmatrix}0 & 0 & - & -\\ 0 & - & 1 & -\\ 0 & 1 & 1 & 1\end{vmatrix} =$$

$$=\overline{x_1}\,\overline{x_2}\oplus\overline{x_1}x_3\oplus\overline{x_1}x_2x_3x_4=\overline{x_1}\,\overline{x_2}\oplus\overline{x_1}x_3\overline{x_2x_4}=$$

$$=\overline{x_1}\,\overline{x_2}\oplus\overline{x_1}x_3\left(\overline{x_2}+\overline{x_4}\right)=\overline{x_1}\left(\overline{x_2}\oplus x_3\left(\overline{x_2}+\overline{x_4}\right)\right).$$

The result of such a simplification of expression $\begin{pmatrix}000-\\0110\end{pmatrix}^{\oplus}$ is 5-level logic whose cost of implementation is:

$$k_l / k_{in} = 5/4.$$

In turn, the expression $\overline{x_1}\left(\overline{x_2\oplus x_3}\right)\left(\overline{x_2}+\overline{x_4}\right)$ (Table 7), obtained from the procedure of inserting similar conjuncterms with the following operation of super-gluing the variables, represents a 3-level logic whose cost of implementation is:

$$k_l / k_{in} = 5/4.$$

The proper derivation of the model of a symmetric 4-input adder of binary codes (chapter 5. 5., system of equations 30) by the analytical method is ensured by the introduction of the apparatus of equivalent figurative transformations to minimize Boolean functions. The order of mutual arrangement of the elements in a binary matrix, the same as the algebraic approach, plays an essential role in the visual perception of two-dimensional data. The logical scheme of the 4-input symmetric adder in Fig. 3 is optimal; it has less complexity compared to the schemes in the open patents back in the USSR [38–45] related to symmetrical 4-input adders.

Table 8 gives the results of minimizing symmetrical Boolean functions borrowed from works by other authors and by the analytical method.

The peculiarity of the method of figurative transformations is that the method is based on the binary combinatoric systems with repeated 2-(*n*, *b*)-design, 2-(*n*, *x/b*)-design, which are essentially the truth tables of the given functions, for example, Tables 1, 4, 5. The specified objects 2-(*n*, *b*)-design, 2-(*n*, *x/b*)-design are a sufficient hardware resource to minimize symmetrical Boolean functions, which makes it possible to do without auxiliary objects, such as Karnaugh maps, Weich diagrams, acyclic graph, non-directed graph, cover tables, cubes, etc. The visual representation of 2-dimensional binary matrices allows the manual simplification of symmetrical Boolean functions (using a mathematical editor, for example, MathType 7.4.0 (USA), within up to 64 input variables [25] for the PDNF (PCNF) representation of a function.

Table 8

Comparative table of the examples of minimization of symmetrical Boolean functions borrowed from works by other authors and the visual-matrix form of the analytical method

Example No.	Minimization method title	Number of input variables	Minimization result	Analytical method result
3	Conjuncterm splitting method [29]	4	9 literals	7 literals
4	Analytical method [27]	3	6 literals	4 literals
5	Karnaugh map [27]	4	12 literals	9 literals
6	Analytical method [30]	3	6 literals	4 literals
8	Karnaugh map [32]	4	15 literals	8 literals
9	Decomposition method [34]	3	Minimization results coincide	
10	Decomposition method [35]	3	Minimization results coincide	
11	Karnaugh map [50]	4	6 literals	5 literals

The use of MFT to minimize symmetrical functions in the Boole basis and in the Reed-Muller basis brings, to some extent, the task of simplifying symmetrical Boolean functions to the level of a well-researched problem in the class of disjunctive-conjunctive normal forms (DCNF) of Boolean functions.

The use of an element basis of only one functionally complete system of switch functions, in a general case, does not provide conditions for obtaining an optimal combination scheme. In this regard, it is advisable to apply a mixed basis.

Example 11. Use the method of figurative transformations to simplify the Boolean function with partial symmetry, which is given in the following canonical form (32) [50].

$$f=(5,6,7,8,9,10,11). \qquad (32)$$

Solution.

Function (32) is partially symmetrical. The permutation of variables with indexes (3,4) does not change the value of function (32).

The simplification of function (32) will be performed in a conjunctive normal form [51].

$$f_{\min}=\left|\begin{array}{c|cccc}0&0&0&0&0\\1&0&0&0&1\\2&0&0&1&0\\3&0&0&1&1\\4&0&1&0&0\\12&1&1&0&0\\13&1&1&0&1\\14&1&1&1&0\\15&1&1&1&1\end{array}\right|=\left|\begin{array}{cccc}1&1&1&1\\1&1&1&0\\1&1&0&1\\1&1&0&0\\1&0&1&1\\0&0&1&1\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{array}\right|=$$

$$=\left|\begin{array}{cccc}1&1&&\\1&0&1&1\\0&0&&\end{array}\right|=\left|\begin{array}{cccc}1&1&&\\1&&1&1\\0&0&&\end{array}\right|=$$

$$=(x_1+x_2)(x_1+x_3+x_4)(\overline{x_1}+\overline{x_2})=$$

$$=(x_1\overline{x_2}+\overline{x_1}x_2)(x_1+x_3+x_4)=(x_1\oplus x_2)(x_1+x_3+x_4).$$

$f_{\min}$ in the mixed basis takes the following form (33):

$$f_{\min}=(x_1\oplus x_2)(x_1+x_3+x_4). \qquad (33)$$

Function (33) represents 2-level logic. The cost of implementing $f_{\min}$ (33) in the mixed basis is:

$$k_\theta / k_l / k_{in} = 2/5/0,$$

which is one literal less compared to [50].

The minimum function (33) remains partially symmetrical. The permutation of variables with indexes (3,4) does not change the value of the minimum function.

Limiting the use of the method of figurative transformations are those cases where the switch function is represented in a mixed basis. In this case, the function must be represented by one logical basis.

The weak side of the method in question is in its small practical application to minimize symmetrical Boolean functions, followed by the design and manufacture of appropriate computational components. The negative internal factors of the MFT are associated with additional time costs for establishing protocols for simplifying the symmetrical logical functions in the Boole basis and in the Reed-Muller basis, followed by the creation of a library of protocols that have an illustration of the corresponding figurative transformations.

The prospect of further research may be the search for new rules for the transformation of majoritarian logical functions and their minimization.

7. Conclusions

1. We have established the results of the equivalent transformations of a polynomial normal form of Boolean functions by inserting similar conjuncterms followed by the operation of super-gluing the variables. The expressions obtained for the specified transformations can be objects for comparison, to a certain extent, with the rules of minimization based on the splitting of conjuncterms in a polynomial theoretical-multiple format (PTMF). The difference between the equivalent transformations of PNF of Boolean functions by inserting similar conjunctures and known methods is that such transformations make it possible to summarize the result of minimization and derive new equivalent transformations based on the induction apparatus.

The effectiveness of the method of inserting similar conjuncterms with the following operation of super-gluing the variables to minimize symmetric Boolean functions has been confirmed by examples 3, 8 of minimizing 4-bit Boolean functions.

2. The peculiarity of minimizing symmetrical Boolean functions is that not all such functions are simplified in PPNF

or PCNF of the main basis $\{\vee,\wedge,\neg\}$. In the absence of minimization of the given function in the main basis, attempts should be made to optimize the function in the polynomial basis $\{\wedge,\oplus,1\}$, using the Reed-Muller algebra.

The difference between the method of figurative transformations is that the method is based on the binary combinatorial systems with repeated 2-(n, b)-design, 2-(n, x/b)-design, whose mathematical apparatus is a sufficient resource to minimize symmetrical Boolean functions. This makes it possible to do without auxiliary objects, such as Karnaugh maps, Weich diagrams, acyclic graph, non-directed graph, coverage tables, cubes, etc. The interpretation of the result is that there are no symmetrical logical functions (except minimal) that cannot be simplified.

The effectiveness of the method of figurative transformations to minimize symmetric Boolean functions is mainly confirmed by examples 4, 6 (the minimization of 3-bit partially symmetric Boolean functions); example 5 (minimizing the 4-bit partially symmetric Boolean function).

3. The algebraic apparatus of polynomial basis makes it possible to implement the method of figurative transformations to minimize symmetrical Boolean functions in the Reed-Muller basis $\{\wedge,\oplus,1\}$. The peculiarity of simplifying the symmetrical Boolean functions in the polynomial basis is that the function must be singular unless it is specified in another way. The difference between the minimization of symmetric polynomial functions by MFT and known methods is the existence of the procedure for inserting similar conjuncterms with the following operation of super-gluing the variables. That expands optimization options, which increases the efficiency of the procedure for minimizing symmetrical Boolean functions of PNF by the method of figurative transformations. The interpretation of the result is that the technology of simplification of polynomial functions makes it possible during the equivalent transformations to transfer from the Reed-Muller algebra to the Boole algebra, and vice versa.

An illustrative (figurative) description is visual, which makes it possible to simultaneously represent a system of relations between individual variables of the problem. Thus, the figurative form of information in the form of combinatorial objects provides a greater chance to determine the algorithm for minimizing Boolean functions. Combinatoric objects, in this case, are the 2-dimensional binary matrices 2-(n, b)-design, or incomplete 2-(n, x/b)-design, and, in essence, of the combinatoric images themselves. As a result, the verbal procedures of algebraic transformations are replaced by equivalent figurative transformations.

The effectiveness of the method of figurative transformations to minimize symmetric Boolean functions in the polynomial basis has been confirmed by examples 7, 8 (the minimization of 4-bit partially symmetric Boolean functions).

4. The results of simplifying the functions in the comparative examples by the method of figurative transformations and decomposition methods coincide but the procedure for figurative transformations is much simpler. Interpretation of the result of simplification of the Boolean function, in particular, is the presence of a method for inserting similar conjuncterms with the following operation of super-gluing the variables.

The effectiveness of the method of figurative transformations, in comparison with the decomposition methods, to minimize symmetric Boolean functions has been confirmed by example 9 (the minimization of a 4-bit partially symmetric Boolean function; example 10 (the minimization of 3-bit partially symmetrical Boolean function).

5. Proper optimization of the logical structure of the symmetrical 4-input adder of binary codes by the analytical method has been confirmed by the introduction of an apparatus of equivalent figurative transformations to minimize Boolean functions. The obtained logical scheme of the 4-input symmetrical adder is optimal; it has less complexity compared to the schemes given in the open patents of the USSR related to symmetric 4-input adders.

References

1. Shannon, C. E. (1938). A symbolic analysis of relay and switching circuits. Electrical Engineering, 57 (12), 713–723. doi: https://doi.org/10.1109/ee.1938.6431064
2. Avgul', L. B., Petrochenko, A. S. (1989). Dekompozitsiya simmetricheskih bulevyh funktsiy i bulevyh funktsiy s chastichnoy simmetriey v bazise monotonnyh funktsiy. Kibernetika i sistemniy analiz, 3, 26–40.
3. Sasao, T. (1993). FPGA Design by Generalized Functional Decomposition. Logic Synthesis and Optimization, 233–258. doi: https://doi.org/10.1007/978-1-4615-3154-8_11
4. Svirshcheva, E. A. (1988). Strukturniy sintez neizomorfnyh sistem s odnorodnymi komponentami. Kharkiv, 256. Available at: http://www.techlibrary.ru/b1/2z1c1j1r2a1f1c1a_3l.2h._2z1t1r1u1l1t1u1r1o2c1k_1s1j1o1t1f1i_1o1f1j1i1p1n1p1r1v1o2c1w_1s1j1s1t1f1n_1s_1p1e1o1p1r1p1e1o2c1n1j_1l1p1n1q1p1o1f1o1t1a1n1j._1998.pdf
5. Rytsar, B. Ye. (1996). Do formalizatsiyi symetrychnykh lohichnykh funktsiy n zminnykh. Materialy mizhnar. konf. «Suchasni problemy avtomatyzovanoi obrobky i vyrobnytstva radioelektronnykh zasobiv zastosuvannia zasobiv zviazku». Ch. 2. Lviv-Slavsk, 28–30.
6. Rytsar, B. Ye. (1999). Dekompozytsiya bulovykh funktsiy metodom q-rozbyttia. Upravlyayushchie sistemy i mashiny, 5, 29–42.
7. Povarov, G. N. (1960). O gruppovoy invariantnosti bulevyh funktsiy. Primenenie logiki v nauke i tekhnike. Moscow: Izd-vo AN SSSR. Available at: https://www.twirpx.com/file/2906517/
8. Mukhopadhyay, A. (1963). Detection of Total or Partial Symmetry of a Switching Function with the Use of Decomposition Charts. IEEE Transactions on Electronic Computers, EC-12 (5), 553–557. doi: https://doi.org/10.1109/pgec.1963.263654
9. Das, S. R., Sheng, C. L. (1971). On Detecting Total or Partial Symmetry of Switching Functions. IEEE Transactions on Computers, C-20 (3), 352–355. doi: https://doi.org/10.1109/t-c.1971.223243
10. Lupanov, O. B. (1965). Ob odnom podhode k sintezu upravlyayushchih sistem - printsipe lokal'nogo kodirovaniya. Problemy kibernetiki, 14, 31–110. Available at: http://new.math.msu.su/department/dm/dmmc/publ_4.htm

11. Avgul', L. B. (1996). Polinomial'noe razlozhenie simmetricheskih bulevyh funktsiy tablichnym metodom. Kibernetika i sistemnyy analiz, 6, 59–71.
12. Rytsar, B. E. (1997). Metod minimizatsii bulevyh funktsiy. Problemy upravleniya i informatiki, 2, 100–113.
13. Paulin, O. N., Lyahovetskiy, A. M. (2002). Model' i metod proektirovaniya mnogooperandnogo summatora na baze simmetricheskih funktsiy. Tr. mezhdunar. konf. po induktivnomu modelirovaniyu «MKIM – 2002». Lviv, 208–213.
14. Paulin, O. N., Drozd, Yu. V. (1998). O sinteze logicheskih moduley, opisyvaemyh simmetricheskimi funktsiyami. Mat-ly mezhdunar. NTK «Priborostroenie». Evpatoriya, 189–192.
15. Schnieber, M., Froehlich, S., Drechsler, R. Depth Optimized Synthesis of Symmetric Boolean Functions. Available at: http://www.informatik.uni-bremen.de/agra/doc/konf/2021_ISVLSI_SymmetricFunctions.pdf
16. Burmistrov, S. V., Panasco, O. M., Kovalska, N. V. (2018). Matrix method of parallel decomposition for minimization of symmetric boolean functions in the form of extended polynomial. Bulletin of Cherkasy State Technological University, 1, 130–135. doi: https://doi.org/10.24025/2306-4412.1.2018.162604
17. Papakonstantinou, G. (2014). A parallel algorithm for minimizing esop expressions. Journal of Circuits, Systems and Computers, 23 (01), 1450015. doi: https://doi.org/10.1142/s0218126614500157
18. Brandão, L. T. A. N., Çalık, Ç., Sönmez Turan, M., Peralta, R. (2019). Upper bounds on the multiplicative complexity of symmetric Boolean functions. Cryptography and Communications, 11 (6), 1339–1362. doi: https://doi.org/10.1007/s12095-019-00377-3
19. Zhang, J. S., Mishc, A., Brayton, R., Chrzanowska-Jeske, M. (2006). Symmetry detection for large Boolean functions using circuit representation, simulation, and satisfiability. 2006 43rd ACM/IEEE Design Automation Conference. doi: https://doi.org/10.1109/dac.2006.229269
20. Drechsler, R., Becker, B. (1995). Sympathy: fast exact minimization of fixed polarity Reed-Muller expressions for symmetric functions. Proceedings the European Design and Test Conference. ED&TC 1995. doi: https://doi.org/10.1109/edtc.1995.470414
21. Drechsler, R. (1997). Pseudo Kronecker expressions for symmetric functions. Proceedings Tenth International Conference on VLSI Design. doi: https://doi.org/10.1109/icvd.1997.568188
22. Möller, D., Molitor, P., Drechsler, R. (1995). Symmetry Based Variable Ordering for ROBDDs. Logic and Architecture Synthesis, 70–81. doi: https://doi.org/10.1007/978-0-387-34920-6_7
23. Arnold, R. F., Harrison, M. A. (1963). Algebraic Properties of Symmetric and Partially Symmetric Boolean Functions. IEEE Transactions on Electronic Computers, EC-12 (3), 244–251. doi: https://doi.org/10.1109/pgec.1963.263535
24. Rytsar, B. Y. (2003). Identification of symmetry of boolean function decomposition cloning method. 6th International Conference on Telecommunications in Modern Satellite, Cable and Broadcasting Service, 2003. TELSIKS 2003. doi: https://doi.org/10.1109/telsks.2003.1246296
25. Solomko, M. (2021). Developing an algorithm to minimize boolean functions for the visual-matrix form of the analytical method. Eastern-European Journal of Enterprise Technologies, 1 (4 (109)), 6–21. doi: https://doi.org/10.15587/1729-4061.2021.225325
26. Solomko, M., Batyshkina, I., Khomiuk, N., Ivashchuk, Y., Shevtsova, N. (2021). Developing the minimization of a polynomial normal form of boolean functions by the method of figurative transformations. Eastern-European Journal of Enterprise Technologies, 2 (4 (110)), 22–37. doi: https://doi.org/10.15587/1729-4061.2021.229786
27. Paulin, O. N., Yanko, V. G. (2014). O minimizatsii simmetricheskih bulevyh funktsiy. Konferentsiya «Modern Problems And Ways Of Their Solution In Science, Transport, Production And Education 2014». SWorld.
28. Rytsar, B. Ye. (2013). A Numeric Set-Theoretical Interpretation of Polynomial Zhegalkin. Upravlinnia systemamy i mashynamy, 1, 11–26. Available at: http://dspace.nbuv.gov.ua/handle/123456789/83125
29. Rytsar, B. Ye. (2015). New minimization method of logical functions in polynomial set-theoretical format. 1. Generalized rules of conjuncterms simplification. Upravlyayushchie sistemy i mashiny, 2, 39–57. Available at: http://dspace.nbuv.gov.ua/handle/123456789/87194
30. Yablonskiy, S. V. (1986). Vvedenie v diskretnuyu matematiku. Moscow: Nauka, 384. Available at: https://stugum.files.wordpress.com/2014/03/yablonskiy-vvedenie-v-diskretnuyu-matematiku.pdf
31. Rytsar, B. Y. (2019). A New Method for Symmetry Recognition in Boolean Functions Based on the Set-Theoretical Logic Differentiation. I. Control Systems and Computers, 4 (282), 3–13. doi: https://doi.org/10.15407/csc.2019.04.003
32. Tran, A. (1987). Graphical method for the conversion of minterms to Reed-Muller coefficients and the minimisation of exclusive-OR switching functions. IEE Proceedings E Computers and Digital Techniques, 134 (2), 93. doi: https://doi.org/10.1049/ip-e.1987.0016
33. Zakrevskiy, A. D. (1981). Logicheskiy sintez kaskadnyh skhem. Moscow, 416. Available at: https://urss.ru/cgi-bin/db.pl?lang=Ru&blang=ru&page=Book&id=272497
34. Barkalov, A. A., Krasichkov, A. A. (2002). Metody dekompozitsii bulevyh funktsiy. Naukovi pratsi DonNTU, 39, 116–121.
35. Rytsar, B. E. (2009). A new approach to the decomposition of boolean functions. 4. Non-disjoint decomposition: p,q-partition. Kibernetika i sistemniy analiz, 3, 15–41. Available at: http://dspace.nbuv.gov.ua/handle/123456789/44365
36. Rytsar, B. Ye. (2013). Rytsar, B. Ye. (2013). Minimization of logic functions system by konjuncterms parallel splitting method. Visnyk Natsionalnoho universytetu «Lvivska politekhnika». Radioelektronika ta telekomunikatsiyi, 766, 18–27. Available at: http://nbuv.gov.ua/UJRN/VNULPPT_2013_766_6

37. Riznyk, V. V., Solomko, M. T. (2017). Application of algebraic operation of super-gluing of variables for minimization of Boolean functions by combinatorial method. Technological audit and production reserves, 6 (2 (38)), 60–76. URL: http://journals.uran.ua/tarp/article/viewFile/118336/112951
38. Viter, V. V., Gur'yanov, A. V., Kozyuminskiy, V. D., Mishchenko, V. A., Tereshko, S. M. (1984). Pat. No. 1228099 SSSR. Chetyrekhvhodoviy odnorazryadniy summator. No. 3696344; declareted: 30.01.1984; published: 30.04.1986. Available at: http://patents.su/3-1228099-chetyrekhvkhodovyjj-odnorazryadnyjj-summator.html
39. Avgul', L. B., Dubovik, Yu. I., Suprun, V. P., Yakush, V. P. (1986). Pat. No. 1374216 SSSR. Chetyrekhvhodoviy odnorazryadniy summator. No. 4098636; declareted: 30.07.1986; published: 15.02.1988. Available at: https://patents.su/3-1374216-chetyrekhvkhodovyjj-odnorazryadnyjj-summator.html
40. Avgul', L. B., Suprun, V. P. (1987). Pat. No. 1429108 SSSR. Chetyrekhvhodoviy odnorazryadniy summator. No. 4195013; declareted: 17.02.1987; published: 07.10.1988. Available at: http://patents.su/3-1429108-chetyrekhvkhodovyjj-odnorazryadnyjj-summator.html
41. Anisimova, L. O., Zemtsova, N. K., Mutihina, O. I., Hlystov, A. V., Chizhuhin, G. N. (1987). Pat. No. 1524045 SSSR. Chetyrekhvhodoviy odnorazryadniy summator. No. 4238062; declareted: 04.05.1987; published: 23.11.1989. Available at: https://patents.su/2-1524045-chetyrekhvkhodovyjj-odnorazryadnyjj-summator.html
42. Avgul', L. B., Suprun, V. P. (1987). Pat. No. 1479928 SSSR. Chetyrekhvhodoviy odnorazryadniy summator. No. 4306252; declareted: 14.09.1987; published: 15.05.1989. Available at: https://patents.su/2-1479928-chetyrekhvkhodovyjj-odnorazryadnyjj-summator.html
43. Avgul', L. B., Suprun, V. P. (1988). Pat. No. 1575172 SSSR. Chetyrekhvhodoviy odnorazryadniy summator. No. 4404427; declareted: 05.04.1988; published: 30.06.1990. Available at: https://patents.su/2-1575172-chetyrekhvkhodovyjj-odnorazryadnyjj-summator.html
44. Avgul', L. B., Suprun, V. P. (1988). Pat. No. 1658145 SSSR. Chetyrekhvhodoviy odnorazryadniy summator. No. 4476179; declareted: 26.08.1988; published: 23.06.1991. Available at: https://findpatent.ru/patent/165/1658145.html
45. Chizhuhin, G. N. (1989). Pat. No. 1683007 SSSR. Chetyrekhvhodoviy odnorazryadnyy summator. No. 4746424; declareted: 02.10.1989; published: 07.10.1991. Available at: https://patents.su/2-1683007-chetyrekhvkhodovojj-odnorazryadnyjj-summator.html
46. Riznyk, V., Solomko, M. (2018). Research of 5-bit boolean functions minimization protocols by combinatorial method. Technology Audit and Production Reserves, 4 (2(42)), 41–52. doi: https://doi.org/10.15587/2312-8372.2018.140351
47. Riznyk, V., Solomko, M., Tadeyev, P., Nazaruk, V., Zubyk, L., Voloshyn, V. (2020). The algorithm for minimizing Boolean functions using a method of the optimal combination of the sequence of figurative transformations. Eastern-European Journal of Enterprise Technologies, 3 (4 (105)), 43–60. doi: https://doi.org/10.15587/1729-4061.2020.206308
48. Solomko, M., Khomiuk, N., Ivashchuk, Y., Nazaruk, V., Reinska, V., Zubyk, L., Popova, A. (2020). Implementation of the method of image transformations for minimizing the Sheffer functions. Eastern-European Journal of Enterprise Technologies, 5 (4 (107)), 19–34. doi: https://doi.org/10.15587/1729-4061.2020.214899
49. Riznyk, V., Solomko, M. (2017). Minimization of Boolean functions by combinatorial method. Technology Audit and Production Reserves, 4 (2 (36)), 49–64. doi: https://doi.org/10.15587/2312-8372.2017.108532
50. Lobanov, V. I. (2002). Russkaya logika protiv klassicheskoy (azbuka matematicheskoy logiki). Moscow: Sputnik+, 113. Available at: https://rusneb.ru/catalog/000199_000009_002095987/
51. Riznyk, V., Solomko, M. (2018). Minimization of conjunctive normal forms of boolean functions by combinatorial method. Technology Audit and Production Reserves, 5 (2 (43)), 42–55. doi: https://doi.org/10.15587/2312-8372.2018.146312