UDC 539.3
DOI: 10.15587/1729-4061.2021.243990

DETERMINING PATTERNS IN THERMOELASTIC INTERACTION BETWEEN A CRACK AND A CURVILINEAR INCLUSION LOCATED IN A CIRCULAR PLATE

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A two-dimensional mathematical model of the thermoelastic state has been built for a circular plate containing a curvilinear inclusion and a crack, under the action of a uniformly distributed temperature across the entire piece-homogeneous plate. Using the apparatus of singular integral equations (SIEs), the problem was reduced to a system of two singular integral equations of the first and second kind on the contours of the crack and inclusion, respectively. Numerical solutions to the system of integral equations have been obtained for certain cases of the circular disk with an elliptical inclusion and a crack in the disk outside the inclusion, as well as within the inclusion. These solutions were applied to determine the stress intensity coefficients (SICs) at the tops of the crack.

Stress intensity coefficients could later be used to determine the critical temperature values in the disk at which a crack begins to grow. Therefore, such a model reflects, to some extent, the destruction mechanism of the elements of those engineering structures with cracks that are operated in the thermal power industry and, therefore, is relevant.

Graphic dependences of stress intensity coefficients on the shape of an inclusion have been built, as well as on its mechanical and thermal-physical characteristics, and a distance to the crack. This would make it possible to analyze the intensity of stresses in the neighborhood of the crack vertices, depending on geometric and mechanical factors.

The study's specific results, given in the form of plots, could prove useful in the development of rational modes of operation of structural elements in the form of circular plates with an inclusion hosting a crack.

The reported mathematical model builds on the earlier models of two-dimensional stationary problems of thermal conductivity and thermoelasticity for piece-homogeneous bodies with cracks

Keywords: crack, inclusion, thermoelasticity, stress intensity coefficient, singular integral equation

Received date 28.10.2021
Accepted date 01.12.2021
Published date 21.12.2021

***How to Cite:** Zelenyak, V., Kolyasa, L., Klapchuk, M., Oryshchyn, O., Vozna, S. (2021). Determining patterns in thermoelastic interaction between a crack and a curvilinear inclusion located in a circular plate. Eastern-European Journal of Enterprise Technologies, 6 (7 (114)), 52–58. doi: https://doi.org/10.15587/1729-4061.2021.243990*

1. Introduction

Elements of many modern structures are often designed to work under the conditions of thermal heating, which contribute to the emergence of temperature stresses in them. This is typical for tools and structures in the heat and power industry. Their performance is largely determined by the level of concentration and intensity of these stresses in some areas, for example, in the neighborhood of technological heterogeneities (cracks, inclusions). At the same time, the destruction of materials is associated with the presence of sharp concentrators of stresses such as cracks. Therefore, the thermoelastic state near a crack must be studied to calculate strength in terms of destruction mechanics, which is especially important for structures made of high-strength and low-plastic materials that are exposed to the influence of various types of heat loads. Of importance are the theoretical and practical studies into the distribution of stresses in the neighborhood of stress concentrators such as cracks. At the same time, the intensity of stresses at crack vertices is expressed through stress intensity coefficients (SICs). These parameters make it possible to determine the limit value of the heat load at which a crack begins to grow and the body locally collapses.

Consequently, investigating the dependence of stress intensity coefficients on the shape of an inclusion, its mechanical and thermal characteristics, is important for strength calculation from the point of view of destruction mechanics. In particular, for the case of piece-homogeneous bodies with a crack, it is possible to reduce the stress intensity coefficients by selecting the appropriate mechanical and thermal-physical characteristics of the composite's components.

Thus, the relevance of our research is predetermined by the importance of studying the thermoelastic state of piece-homogeneous bodies with cracks for practical applications in terms of assessing the strength and durability of structural elements, as well as in theoretical terms for devising new effective methods for determining temperature stresses.

2. Literature review and problem statement

Two-dimensional thermoelasticity problems for bodies with cracks were considered earlier. Work [1] examines the thermoelasticity problem of connected dissimilar half-plates with a functionally graded layer, weakened by a pair of two offset interphase cracks. The integral Fourier transform method was used. The statement of the current problem of non-isothermal crack is reduced to two sets of singular integral equations of the Cauchy type for temperature and thermal stress fields in the connected system.

Paper [2] considers the problem of stress concentration in the neighborhood of the crack vertices for a crack of the finite length, located perpendicular to the interface of two elastic bodies – the half-plane and strip. Using the method of generalized integral transformations, the problem is reduced to solving a singular integral equation with the Cauchy nucleus. The integral equation is solved by a colocation method and by small parameters. The authors obtained values of intensity coefficients of normal stresses near the crack vertices for different combinations of geometric and physical parameters of the problem.

Work [3] reports a solution to the problem of thermoelasticity. Thermal impact on crack growth under different thermal and mechanical conditions was investigated, based on the edge method (ES-FEM); this method is more accurate than a standard finite-element method (FEM).

Paper [4] investigated the thermoelastic state in a half-plane with a rectilinear rigid inclusion under a homogeneous heat flow set on infinity. The singular integral equations (SIEs) obtained were solved by the method of orthogonal polynomials. A general case of the system of randomly oriented cracks in the half-plane was studied in [5].

A problem of the circular absolutely rigid inclusion of an arbitrary shape, which is located in the transversal-isotropic half-space under the conditions of smooth contact with the second half-space, was reduced to a system of two-dimensional singular integral equations. The asymptotic stresses in the vicinity of an inclusion were investigated; directions of the highest and lowest concentration of stresses [6] were determined.

Two-dimensional problems of thermoelasticity for bodies with cracks were also studied by the method of singular integral equations (SIEs). In particular, the authors of [7] investigated the thermoelastic state in a three-layer hollow cylinder with a crack. Temperature conditions of the first kind were set on the surfaces of the cylinder. The stress intensity coefficients were calculated at the top of an inner edge crack. Work [8] applied an SIE method to consider the problems of stationary thermal conductivity and thermoelasticity in a semi-finite plate containing an inner curvilinear crack. The plate is heated on the local area of the edge by heat flow. A similar problem was considered in [9] for the case of an edge crack in the half-plane. In work [10], the same method was applied to investigate the thermoelastic interaction between a two-component circular inclusion and a crack located in the plate under the conditions of a steady uniform temperature.

Work [11] investigated the thermoelastic state in a half-space, which is locally heated by the heat flow of its free surface and contains the inclusion and crack. In [12], a method of the functions of a complex variable was used to solve the problems of thermal conductivity and thermoelasticity with special Cauchy nuclei for bodies with the thermal cylindrical inclusion and crack.

The review of literary sources [7–12] allows us to conclude that the thermoelastic state was mostly studied for infinite and semi-infinite bodies, both homogeneous and with circular inclusions and cracks. And there are almost no solutions for finite bodies with curvilinear (in particular, elliptical) inclusions and cracks, as well as for multicomponent composite bodies with cracks. Given this, it is important to further investigate the effect of temperature on the stressed state of piece-homogeneous bodies with cracks. In particular, to construct mathematical models to determine heat loads at which a crack begins to grow, and the body is locally destroyed. The study of such models could make it possible to suggest an approach to preventing the growth of cracks, for example, by selecting the components of a piece-homogeneous plate with appropriate mechanical and thermophysical characteristics.

3. The aim and objectives of the study

The purpose of this work is to determine the two-dimensional thermoelastic state in a circular plate containing a curvilinear inclusion and a crack under the action of uniform temperature distribution across the entire piece-homogeneous plate with a crack. This would make it possible to determine the critical values of the heat load in order to prevent the growth of the crack, which could prevent the local destruction of the plate.

To accomplish the aim, the following tasks have been set:

– to build two-dimensional mathematical models of the thermoelasticity problem in the form of singular integral equations on the contour of a crack and a curvilinear inclusion to determine the perturbed thermal stresses due to the presence of the inclusion and crack;

– to obtain numerical solutions to the singular integral equations of the thermoelasticity problem in the partial case of a circular plate with an elliptical inclusion and a rectilinear crack under the conditions of uniform temperature distribution in the specified region;

– to derive and investigate the stress intensity coefficients at crack vertices, depending on the thermal characteristics of a piece-homogeneous plate, and to identify mechanical effects.

4. The study materials and methods

We have theoretically studied the two-dimensional thermoelastic state in a circular plate containing a curvilinear inclusion and a crack based on the method of the function of a complex variable. To this end, a two-dimensional mathematical model of the thermoelastic state was built, in the form of a system of two singular integral equations of the first and second kind on the contours of the crack and inclusion, respectively.

We have obtained the numerical solutions to the system of integral equations for the particular cases of a circular disk with an elliptical inclusion and a crack, heated to a steady temperature. These solutions were used to determine the stress intensity coefficients at the crack's vertices.

Graphic dependences have been built of the stress intensity coefficients on the shape of an inclusion, its mechanical and thermal characteristics, and a distance to the crack.

5. Results of studying the thermoelastic state in a circular plate with the curvilinear inclusion and crack

5. 1. Building a two-dimensional mathematical model of the thermoelasticity problem

We have considered a circular plate containing a curvilinear inclusion with contour L_1 and a crack L_2 located in the disk matrix, or in the inclusion. We believe that the contours L_n $\left(n=\overline{1,2}\right)$ have no common points. Each L_n $\left(n=\overline{1,2}\right)$ contour is assigned to the local coordinate systems $x_nO_ny_n$, whose O_nx_n axis forms an α_n angle with the Ox axis, and the O_n points determine in the xOy coordinate system the complex coordinates $z_n^0=x_n^0+iy_n^0$. The association between the coordinates of points in the area S in the local and main coordinate systems is given by the ratios $z=z_ne^{i\alpha_n}+z_n^0$, $z=x+iy$, $z_n=x_n+iy_n$.

Let there be a uniform temperature distribution across the entire piece-homogeneous circular plate with a crack $T(x,y)=T_c=\text{const}\neq 0$, which differs from the temperature T_C=0 of the relaxed initial state. This temperature does not create a perturbed temperature field at the crack's vertices, and, therefore, the perturbed thermoelastic state from the crack. Temperature stresses here occur only due to different values of the temperature coefficients of linear expansion for the inclusion and disk matrix. We shall determine these stresses from the solution to the thermoelasticity problem.

Supposing that the conditions for an ideal mechanical (the equality of stresses and movements) contact are specified on the inclusion's contour L_1:

$$\left[N(t_1)+iT(t_1)\right]^+=\left[N(t_1)+iT(t_1)\right]^-,$$

$$\left(u_1+iv_1\right)^+-\left(u_1+iv_1\right)^-=0,\quad t_1\in L_1, \tag{1}$$

the cracks' banks L_2 in the process of deformation do not come into contact and there are no forced loads on them

$$\left[N(t_2)+iT(t_2)\right]^{\pm}=0. \tag{2}$$

The plate's contour L_0 of radius R_0 is also considered free from loads.

In ratios (1), (2), there are the following designations: $N(t_n)$, $T(t_n)$ is the normal and tangent components of stresses, u_1, v_1 are the components of displacement.

Complex potentials, based on the use of known complex potentials for a homogeneous disk with cracks [13], are to be chosen for a piece-homogeneous disk in the following form

$$\Phi(z)=\Phi_1(z)+\Phi_2(z),$$

$$\Psi(z)=\Psi_1(z)+\Psi_2(z), \tag{3}$$

where

$$\Phi_1(z)=\frac{1}{2\pi}\sum_{k=1}^{2}\int_{L_k}\frac{Q_k(t_k)e^{i\alpha_k}\mathrm{d}t_k}{\zeta_k-z};$$

$$\Psi_1(z)=\frac{1}{2\pi}\sum_{k=1}^{2}\int_{L_k}\left[\frac{\overline{Q_k(t_k)}e^{-i\alpha_k}\overline{\mathrm{d}t_k}}{\zeta_k-z}-\frac{\overline{\zeta_k}Q_k(t_k)e^{i\alpha_k}\mathrm{d}t_k}{(\zeta_k-z)^2}\right];$$

$$\Phi_2(z)=\frac{1}{2\pi}\times$$

$$\times\sum_{k=1}^{2}\int_{L_k}\left\{\begin{gathered}\frac{1}{z\overline{\zeta_k}-R_0^2}\overline{\zeta_k}Q_k(t_k)e^{i\alpha_k}\mathrm{d}t_k+\\+\frac{z^2\overline{\zeta_k}-2zR_0^2+\zeta_kR_0^2}{\left(z\overline{\zeta_k}-R_0^2\right)^2}\overline{Q_k(t_k)}e^{-i\alpha_k}\overline{\mathrm{d}t_k}\end{gathered}\right\};$$

$$\Psi_2(z)=\frac{1}{2\pi}\sum_{k=1}^{2}\int_{L_k}\left\{\begin{gathered}\frac{\overline{\zeta_k^3}Q_k(t_k)e^{i\alpha_k}\mathrm{d}t_k}{\left(z\overline{\zeta_k}-R_0^2\right)^2}\times\\\times\left[1+\frac{\left(z\overline{\zeta_k}-3R_0^2\right)\left(\zeta_k\overline{\zeta_k}-R_0^2\right)}{\left(z\overline{\zeta_k}-R_0^2\right)^2}\right]\times\\\times\frac{\overline{\zeta_k}\overline{Q_k(t_k)}e^{-i\alpha_k}\overline{\mathrm{d}t_k}}{z\overline{\zeta_k}-R_0^2}\end{gathered}\right\};$$

$$|z|<R_0,\quad |\zeta_k|<R_0,\quad \zeta_k=t_ke^{i\alpha_k}+z_k^0;$$

$$Q_k(t_k)=\begin{cases}g_1(t_1), & t_1\in L_1,\\ g_2'(t_2), & t_2\in L_2.\end{cases}$$

Here, $g_1(t_1)$ is an unknown function on the inclusion's contour L_1; $g_2'(t_2)$ – unknown function on the crack's contour. Complex potentials $\Phi_1(z)$, $\Psi_1(z)$, $\Phi_2(z)$, $\Psi_2(z)$ characterize the perturbed thermally strain state caused by the inclusion.

Having satisfied, with the use of potentials (3), the second equality of boundary condition (1) on the contour of the inclusion, and boundary condition (2) on the crack's contour, we obtain a system of two singular integral equations, respectively, of the second and first kind relative to two unknown functions $Q_1(t_1)$ and $Q_2(t_2)$ on the contours L_1 and L_2

$$A_1Q_1(\tau_1)+\frac{1}{2\pi}\int_{L_1}\left[\begin{gathered}R_{11}(t_1,\tau_1)Q_1(t_1)\mathrm{d}t_1+\\+S_{11}(t_1,\tau_1)\overline{Q_1(t_1)}\,\overline{\mathrm{d}t_1}\end{gathered}\right]+$$

$$+\frac{1}{2\pi}\int_{L_2}\left[\begin{gathered}R_{12}(t_2,\tau_1)Q_2(t_2)\mathrm{d}t_2+\\+S_{12}(t_2,\tau_1)\overline{Q_2(t_2)}\,\overline{\mathrm{d}t_2}\end{gathered}\right]=P_1(\tau_1),\quad \tau_1\in L_1; \tag{4}$$

$$\frac{1}{2\pi}\int_{L_1}\left[\begin{gathered}R_{21}(t_1,\tau_2)Q_1(t_1)\mathrm{d}t_1+\\+S_{21}(t_1,\tau_2)\overline{Q_1(t_1)}\,\overline{\mathrm{d}t_1}\end{gathered}\right]+$$

$$+\frac{1}{2\pi}\int_{L_2}\left[\begin{gathered}R_{22}(t_2,\tau_2)Q_2(t_2)\mathrm{d}t_2+\\+S_{22}(t_2,\tau_2)\overline{Q_2(t_2)}\,\overline{\mathrm{d}t_2}\end{gathered}\right]=P_2(\tau_2),\quad \tau_2\in L_2,$$

where

$$R_{nk}(t_k,\tau_n)=R_{nk}^1(t_k,\tau_n)-$$

$$-e^{i\alpha_k}\left\{\begin{gathered}\frac{B_n\overline{\zeta_k}}{R_0^2-\eta\overline{\zeta_k}}-\\-C_n\left[\begin{gathered}\frac{\zeta_k\left(\overline{\eta_n}\right)^2-2\overline{\eta_n}R_0^2}{\left(R_0^2-\overline{\eta_n}\zeta_k\right)^2}+e^{-2i\alpha_n}\frac{\overline{d\tau_n}}{d\tau_n}\times\\\times\frac{2\eta_n\left(\zeta_k\overline{\zeta_k}-R_0^2\right)R_0^2+\zeta_k^2\left(\overline{\eta_n}+\overline{\zeta_k}\right)\left(\overline{\eta_n}\zeta_k-3R_0^2\right)+4\zeta_kR_0^4}{\left(R_0^2-\overline{\eta_n}\zeta_k\right)^3}\end{gathered}\right]\end{gathered}\right\};$$

$$S_{nk}\left(t_k,\tau_n\right)=S^1_{nk}\left(t_k,\tau_n\right)- -e^{-i\alpha_k}\left\{\begin{matrix}\dfrac{B_n\zeta_k}{R_0^2-\overline{\eta_n}\zeta_k}+ \\ -C_n\left[\begin{matrix}\dfrac{\overline{\zeta_k}\left(\overline{\eta_n}\right)^2-2\overline{\eta_n}R_0^2+\zeta_k R_0^2}{\left(R_0^2-\eta_n\overline{\zeta_k}\right)^2}+ \\ +\dfrac{\zeta_k^2\left(\eta_n-\zeta_k\right)}{\left(R_0^2-\overline{\eta_n}\zeta_k\right)^2}\dfrac{\overline{d\tau_n}}{d\tau_n}e^{-2i\alpha_n}\end{matrix}\right]\end{matrix}\right\};$$

$$R^1_{nk}\left(t_k,\tau_n\right)=e^{i\alpha_k}\left[\frac{1}{H_{nk}}+B_n\frac{\overline{d\tau_n}}{d\tau_n}\frac{e^{-2ia_n}}{\overline{\overline{H_{nk}}}}\right];$$

$$S^1_{nk}\left(t_k,\tau_n\right)=-C_n e^{-i\alpha k}\left[\frac{1}{\overline{\overline{H}}_{nk}}-\frac{\overline{d\tau_n}}{d\tau_n}\frac{e^{-2i\alpha k}H_{nk}}{\overline{H^2_{nk}}}\right];$$

$$H_{nk}=\zeta_k-\eta_n,\quad T_{nk}=\zeta_k-\overline{\eta}_n;$$

$$\eta_n=\tau_n e^{i\alpha_n}+z_n^0,\quad \left(k=1,2;\ \ n=1,2\right);$$

$$Q_1\left(t_1\right)=g_1\left(t_1\right);\ \ Q_2\left(t_2\right)=g'_2\left(t_2\right);$$

$$A_1=0.5\left[1+\chi_1+\Gamma_1\left(1+\chi\right)\right];$$

$$B_1=\chi_1-\Gamma_1\chi;\ \ C_1=1-\Gamma_1;\ \ B_2=1;\ \ C_2=-1;$$

$$P_1\left(\tau_1\right)=\left(\Gamma_1\beta_t-\beta_t^1\right)\cdot T_0;\ \ P_2\left(\tau_2\right)=0;$$

$$\chi=3-4\mu,\quad \beta^t=\alpha^t E_1,$$

$$\chi_1=3-4\mu_1,\quad \beta_1^t=\alpha_1^t E_1,\quad \Gamma_1=G_1\ /G.$$

Here, α^t, G, E, μ $\left(\alpha_1^t, G_1, E_1, \mu_1\right)$ are the temperature coefficient of linear expansion, the shear module, the elasticity module (Young), the Poisson coefficient of the matrix-disk (respectively, inclusion).

The system of equations (4) has a single solution for its arbitrary right-hand side, provided the following condition is met

$$\int\limits_{-1}^{1} g'_2\left(t_2\right)dt_2=0, \tag{5}$$

which ensures that movements are unambiguous when bypassing the crack's contour.

Having found the unknown functions $Q_1(t_1)$ and $Q_2(t_2)$ from the system of equations (4), (5), we then can obtain the distribution of thermal stresses throughout the entire piece-homogeneous plate with a crack; in particular, the stress intensity coefficients (SICs) K_{I}, K_{II} in the vertices of the crack are found from [13]

$$K_I^{\pm}-iK_{II}^{\pm}=\mp\lim_{t_2\to l_2^{\pm}}\left[\sqrt{2\pi\left|t_2-l_2^{\pm}\right|}Q_2\left(t_2\right)\right],$$

where indexes "–" refer to the beginning of the crack $\left(t_k=l_k^-\right)$, and "+" – its end $\left(t_k=l_k^+\right)$.

5. 2. Applying a method of mechanical quadrature to determine the stress intensity coefficients in the vertices of the crack

We consider a circular disk of radius R_0 containing an elliptical inclusion with half-axes a and b, limited by the contour L_1 with a common center; associate it with the xOy coordinate system with its origin in the center of the disk. On the segment of the axis Ox, beyond the inclusion (Fig. 1), or in the inclusion (Fig. 2), there is a crack of length $2l$, the banks of which are unloaded. There are no loads on the disk's contour L_0, and, on the inclusion's contour L_1, the conditions for an ideal mechanical (the equality of stress and displacement) contact are specified. Let the circular piece-homogeneous disc with a crack be evenly heated to a steady temperature $T_c=\text{const}\neq 0$, which differs from the temperature of the relaxed initial state. Stresses, in this case, occur only due to different values of the temperature coefficients of linear expansion for the inclusion and disk matrix. This thermoelasticity problem is reduced to a system of two singular integral equations (4), (5) on a closed (the inclusion's contour L_1) and open (the crack's contour L_2) contours, relative to two unknown functions $Q_1(t_1)$ and $Q_2(t_2)$.

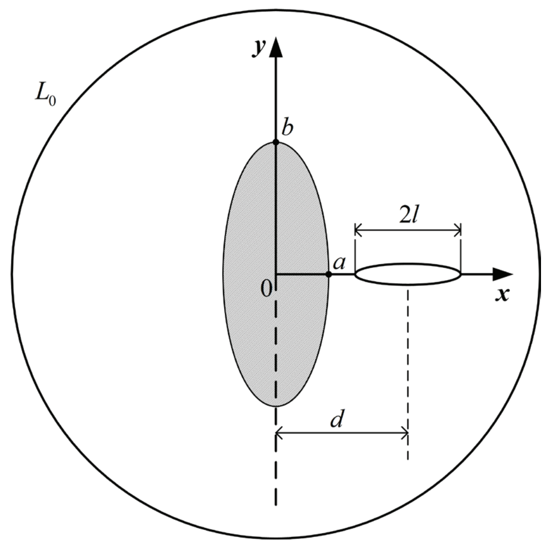

Fig. 1. Circular plate with an elliptical inclusion and a crack outside the inclusion

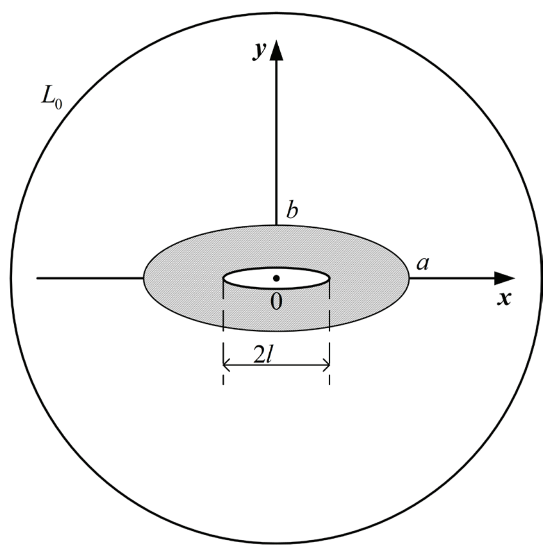

Fig. 2. Circular plate with an elliptical inclusion and a crack inside the inclusion

The solution to the system of equations (4), considering condition (5), was found numerically by the method of mechanical quadratures [14]. To this end, after replacing the variables

$$t_1=\omega(\theta)=a\cos\theta+ib\sin\theta;\quad \tau_1=\omega(\beta);\quad t_2=l\xi;\quad \tau_2=l\eta$$

the system of integral equations (4), (5) for the thermoelasticity problem is reduced to the normalized form

$$A_1\psi_1(\beta)+\int_0^{2\pi}\begin{bmatrix}R_{11}^*(\theta,\beta)\psi_1(\theta)+\\+S_{11}^*(\theta,\beta)\overline{\psi_1(\theta)}\end{bmatrix}d\theta+$$
$$+\int_{-1}^{1}\begin{bmatrix}R_{12}^*(\xi,\beta)\psi_2(\xi)+\\+S_{12}^*(\xi,\beta)\overline{\psi_2(\xi)}\end{bmatrix}d\xi=P_1^*(\beta),\qquad 0<\beta\le 2\pi;\tag{6}$$

$$\int_0^{2\pi}\begin{bmatrix}R_{21}^*(\theta,\eta)\psi_1(\theta)+\\+S_{21}^*(\theta,\eta)\overline{\psi_1(\theta)}\end{bmatrix}d\theta+$$
$$+\int_{-1}^{1}\begin{bmatrix}R_{22}^*(\xi,\eta)\psi_2(\xi)+\\+S_{22}^*(\xi,\eta)\overline{\psi_2(\xi)}\end{bmatrix}d\xi=0,\quad |\eta|<1;$$

$$\int_{-1}^{1}\psi_2(\xi)d\xi=0,\tag{7}$$

where

$$\psi_1(\theta)=Q_1(\omega(\theta));$$

$$\psi_2(\xi)=Q_2(l\xi);$$

$$R_{11}^*(\theta,\beta)=R_{11}(\omega(\theta),\omega(\beta))\omega'(\theta);$$

$$S_{11}^*(\theta,\beta)=S_{11}(\omega(\theta),\omega(\beta))\cdot\omega'(\theta);$$

$$R_{12}^*(\xi,\beta)=R_{12}(l\xi,\omega(\beta))\cdot l;$$

$$S_{12}^*(\xi,\beta)=S_{12}(l\xi,\omega(\beta))\cdot l;$$

$$R_{21}^*(\theta,\eta)=R_{21}(\omega(\theta),l\eta)\cdot\omega'(\theta);$$

$$S_{21}^*(\theta,\eta)=S_{21}(\omega(\theta),l\eta)\cdot\omega'(\theta);$$

$$R_{22}^*(\xi,\eta)=R_{22}(l\xi,l\eta)\cdot l;$$

$$S_{22}^*(\xi,\eta)=S_{22}(l\xi,l\eta)\cdot l;$$

$$P_1(\beta)=\left(\Gamma_1\beta_t-\beta_t^1\right)\cdot T_0.$$

A solution to the system of equations (6) and (7) is to be found in the class of functions, unlimited at $\xi=\pm1$, that is in the form of $\psi_2(\xi)=\frac{u_2(\xi)}{\sqrt{1-\xi^2}}$, where $u_2(\xi_k)$ is the continuous functions on the segment [–1; 1], as well as in the class of 2π-periodic continuous $\psi_1(\theta_v)$ functions.

Using Gauss quadrature formulas for a regular integral, and quadrature formulas for calculating the integral along a closed contour [13], for the system of equations (6), (7), we obtain a system of $(m+n)$ linear algebraic equations in the form

$$A_1\psi_1(\beta_s)+\frac{2}{m}\sum_{v=1}^{m}\begin{bmatrix}R_{11}^*(\theta_v,\beta_s)\psi_1(\theta_v)+\\+S_{11}^*(\theta_v,\beta_s)\overline{\psi_1(\theta_v)}\end{bmatrix}+$$
$$+\frac{1}{n}\sum_{k=1}^{n}\begin{bmatrix}R_{12}^*(\xi_k,\beta_s)u_2(\xi_k)+\\+S_{12}^*(\xi_k,\beta_s)\overline{u_2(\xi_k)}\end{bmatrix}=P_1^*(\beta_s),\qquad s=\overline{1,m};\tag{8}$$

$$\frac{2}{m}\sum_{v=1}^{m}\int_0^{2\pi}\begin{bmatrix}R_{21}^*(\theta_v,\eta_p)\psi_1(\theta_v)+\\+S_{21}^*(\theta_v,\eta_p)\overline{\psi_1(\theta_v)}\end{bmatrix}+$$
$$+\frac{1}{n}\sum_{k=1}^{n}\begin{bmatrix}R_{22}^*(\xi_k,\eta_p)u_2(\xi_k)+\\+S_{22}^*(\xi_k,\eta_p)\overline{u_2(\xi_k)}\end{bmatrix}=0,\quad p=\overline{1,n-1};$$

$$\frac{1}{n}\sum_{k=1}^{n}u_2(\xi_k)=0;\tag{9}$$

to determine m unknowns $\psi_1(\theta_v)$, $v=\overline{1,m}$ and n unknowns $u_2(\xi_k)$, $k=\overline{1,n}$.

Using the solutions to the system of equations (8), (9), we find the stress intensity coefficients

$$K_I^{\pm}-iK_{II}^{\pm}=\mp u_2(\pm1)\sqrt{l},$$

where

$$u_2(1)=-\frac{1}{n}\sum_{k=1}^{n}(-1)^k u_2(\xi_k)\operatorname{ctg}\frac{2k-1}{4n}\pi;$$

$$u_2(-1)=-\frac{1}{n}\sum_{k=1}^{n}(-1)^{k+n} u_2(\xi_k)\operatorname{ctg}\frac{2k-1}{4n}\pi.$$

The numerical values of the stress intensity coefficients K_{I}, K_{II} are real quantities that characterize the stressed-strained state in the neighborhood of the crack's vertices.

5. 3. Investigating the dependence of stress intensity coefficients on the thermal and mechanical characteristics of the components of a piece-homogeneous plate

Numerical results for the dimensionless stress intensity coefficients $F_I^{\pm}=K_I^{\pm}/K_{T_c}\left(K_{II}=0\right)$, where $K_{T_c}=T_c\cdot\beta^t\sqrt{l}$, were obtained for different values of the parameters of the problem.

Fig. 3 shows the built dependences of $F_I^{\pm}$ on parameter $\delta=l/a$ when $\chi=\chi_1=2$; $R_0/a=10$, $d/a=2$, $b/a=2$.

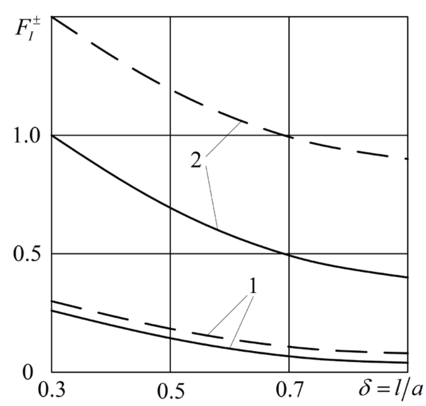

Fig. 3. Dependence of dimensionless SICs $F_I^{\pm}$ on parameter $\delta=l/a$: curves 1 – G_1/G=2,5; $\alpha_1^t/\alpha^t=4$; curves 2 – G_1/G=0,5; $\alpha_1^t/\alpha^t=4$; solid lines – right vertex $\left(F_I^{+}\right)$, dashed lines – left vertex $\left(F_I^{-}\right)$

For a crack beyond the inclusion on the line of its smaller axis, the SIC F_I^{-} in the crack top closer to the inclusion is greater than at the far F_I^{+} for different relative values of the hardness of the inclusion and the disk G_1/G. At the same time, $F_I^{\pm}$ decreases both with the growth of the crack and with an increase in the rigidity of the inclusion. In addition, $F_I^{\pm}>0$, if the temperature coefficient of linear expansion of the inclusion is greater than that of the disk $\left(\alpha_1^t > \alpha^t\right)$. If $\alpha_1^t < \alpha^t$, then $F_I^{\pm} < 0$. The banks of the crack then come into contact, which is not taken into consideration in a given model.

Fig. 4 shows the built dependences of

$$F_I = K_I / K_{T_c} \left(K_I = K_I^+ = K_I^-\right)$$

on parameter $\delta = l/a$ when $\chi = \chi_1 = 2$; $R_0/a = 10$.

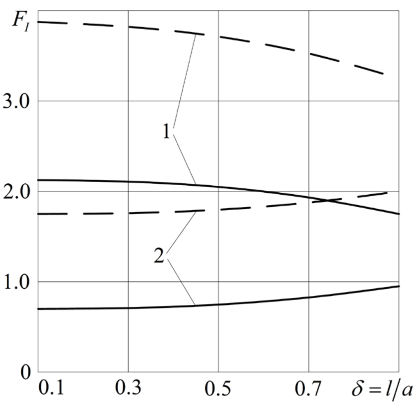

Fig. 4. Dependence of dimensionless SIC F_I on parameter $\delta = l/a$ Curves 1 – $G_1/G = 0{,}5$; $\alpha_1^t / \alpha^t = 0.5$; curves 2 – $G_1/G = 2$; $\alpha_1^t / \alpha^t = 0.5$. Solid lines correspond to $b/a = 0.5$ (the larger ellipse axis is parallel to the crack line), the dashed lines correspond to $b/a = 2$ (the larger axis of the ellipse is perpendicular to the crack line)

If the crack is in the elliptical inclusion on the line of its axis (the centers of the inclusion and crack coincide), then an increase in the relative rigidity of the inclusion and disk G_1/G leads to a decrease in SIC F_I for different shapes of the inclusion. At the same time, when the inclusion is more (less) rigid than the disk, F_I increases (decreases) with the growth of the crack. In addition, if the temperature coefficient of the linear expansion of the inclusion is less than that of the disk $\left(\alpha_1^t < \alpha^t\right)$, then $F_I > 0$. If it is larger $\left(\alpha_1^t > \alpha^t\right)$ – then $F_I < 0$; the banks of the crack are in contact, which is not considered in a given model.

6. Discussion of results of studying the interaction between a crack and an elliptical inclusion, provided that the temperature in the plate is evenly distributed

We have used a method of mechanical quadratures to numerically solve the singular integral equations. Its advantage is the fact that for fixed parameters of the problem, it makes it possible to derive a solution with a predefined accuracy. And the disadvantage is that when changing any parameter of the problem, the calculation has to be repeated again.

Other methods are also used to solve a singular integral equation. For example, the method of sequential approximations, or the method of collocations. However, both methods are not effective enough. As regards the first one, it is due to the mandatory regularization of the equation, which is quite difficult; the second one – due to the imperfection of the discrete analog of the equation.

The following restrictions are inherent: the proposed model does not take into consideration the possible contact of the crack banks. Therefore, in some cases, the stress intensity coefficient $F_I^{\pm}$ can acquire negative values, which we exclude from consideration. However, such a result can also be used to obtain a solution to the thermoelasticity problem by a superposition method. To this end, it is necessary to take into consideration the effect, in addition to the specified temperature field, of other temperature or force factors that together would not lead to contact between the crack's banks.

If it is necessary to take into consideration the contact of the banks of the crack, the problem should be stated differently, as a mixed problem on the banks of the crack. At the same time, its solution is significantly complicated, but it can also be obtained by singular integral equations.

The next limitation is the numerical method of mechanical quadratures, which is used only for inner cracks. Therefore, if the vertex of the crack reaches the connection line of the inclusion-matrix, then a quadrature method of solving the integral equation based on the Gauss-Jacobi quadrature formula [13] should be applied.

In the considered problem, the banks of the cracks do not touch. Then, according to the σ_θ-criterion (according to the hypothesis of the initial growth of the crack), critical values for the steady temperature T_{cr} can be found from the equations of the limit equilibrium [15] when the crack begins to grow and the plate is locally destroyed, using the following formula

$$T_{cr} = \frac{K_{1C}}{F_1^{\pm}}. \tag{10}$$

Here, K_{1C} is a constant that characterizes the resistance of the material to destruction and which is determined experimentally.

Based on the analysis of numerical results for SIC $F_1^{\pm}$ at fixed values of temperature coefficients of linear expansion, the disk and inclusion shear modulus, the following conclusions arise from formula (10). For a crack that is beyond the inclusion (Fig. 1), less rigid than the matrix ($G_1 < G$), the growth of the crack would begin from the left vertex (close to the inclusion) since a lower critical temperature is required there. If the inclusion is more rigid than the matrix ($G_1 > G$), then approximately simultaneously at both vertices. To start the growth of the crack in both vertices, which is inside the inclusion less rigid than the matrix ($G_1 < G$) (Fig. 3), one requires a lower critical temperature than in the case of an inclusion that is more rigid than the matrix ($G_1 > G$).

The practical value of this work relates to the possibility for a more complete consideration of the actual stressed-strained state in the piece-homogeneous structural elements with cracks that operate under conditions of various heat loads. The study's specific results, given in the form of plots, may prove useful in the development of rational modes of operation of structural elements in the form of circular plates with an inclusion hosting a crack. At the same time, it becomes possible to prevent the growth of cracks due to the appropriate selection of a composite's components with appropriate mechanical and thermophysical characteristics.

7. Conclusions

1. A two-dimensional mathematical model has been built for the problem of stationary thermoelasticity for a circular plate with a curvilinear inclusion and a crack, in the form

of a system of two singular integral equations (SIEs) of the first kind on the crack contour and the second kind on the contour of the inclusion. This makes it possible to obtain a numerical solution to an SIE system by applying the method of mechanical quadratures. The advantage of the applied method is that it makes it possible to obtain a solution to the integral equation with high accuracy compared to the asymptotic method of a small parameter, which is used only for certain types of integral equations.

2. The numerical solution to the system of singular integral equations in the partial case of a circular plate with an elliptical inclusion and a straight crack was derived. In this case, the crack is located either in the disk outside the inclusion or in the inclusion, under the effect of a uniformly distributed temperature across the entire circular plate. Using this solution helped determine the stress intensity coefficients in the vertices of the crack. Subsequently, they can be used to determine a critical value for the stable temperature in a plate at which a crack begins to grow. If the crack is out of the inclusion, less rigid than the matrix ($G_1<G$), then the growth of the crack begins from the left vertex (close to the inclusion). In this case, a lower critical temperature is required, which is determined from formula (10). If the crack is inside the inclusion that is less rigid than the matrix ($G_1<G$), then a lower critical temperature is required to start the growth than if the crack is in a more rigid inclusion than the matrix ($G_1>G$).

3. We have built graphical dependences of stress intensity coefficients on the shape of an inclusion and its characteristics. It was found that the appropriate selection of thermal-physical characteristics for an inclusion could lead to the creation of compressive or stretching normal stresses in the neighborhood of the crack vertices. This is important in terms of managing body strength within the mechanics of destruction. In addition, if the temperature coefficient of the linear expansion of an inclusion is greater than that of the disk $\left(\alpha_1^t>\alpha^t\right)$, then compression stresses occur in the neighborhood of both vertices of the crack inside the inclusion. It then ensures that the crack growth is stopped. If the crack is outside the inclusion, then compression stresses occur in the neighborhood of the crack vertices, provided $\alpha_1^t<\alpha^t$, which also causes the crack to stop growing.

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