1. Introduction

During gas field exploitation, there are cases when a well opens several gas-bearing formations simultaneously. Moreover, there may occur gas cross-flows between the formations, which is considered to be economically undesirable. Given that a wellhead pressure $p$ is the only parameter by changing which it is possible to control the production rate, it is important to know how production rates of exploited formations functionally depend on such a pressure. The pressure problem on two formations has already been considered in the author's previous studies. By this it is necessary to know the formation pressures and filtration coefficients. However, this condition often doesn't hold in practice. The present study considers the problem of determining unknown formation pressures and filtration coefficients of two gas-bearing formations opened by a single well on the basis of stationary wellhead pressure conditions and the total production rate measurements data. The problem is reduced to solving a complicated system of three nonlinear equations. The research involves an algorithm and an example of a numerical solution, with software processing of one of the possible options.

2. Literature review and problem statement

For clarity and convenience in understanding the research focus, the primary intention is to give a direct problem statement. It will allow introducing all required terminology. Next, there will be an overview of publications connected with the stated problem.

In [1, 2], attention is given to a mathematical model of a high-pressure gas flow in a gas-dynamic system consisting of a D-diameter vertical well and two horizontal gas-saturated porous formations of thicknesses $h_j$, opened at depths $L_j$, $j=1,2$ to the surface. The gas-bearing formations are isolated from each other by enclosing rocks, so they affect each other only through perforations (holes) vented in the well along the formation thickness. It is assumed that the gas flow in the formation is steady, isothermal, and radially symmetrical about the well axis. It is also assumed that the thicknesses are sufficiently small, so $h_j/L_j<<1$, and thus the through-thickness pressure variation can be neglected. The diameter $D$ is significantly smaller than the depth $L_j$; therefore, it is reasonable to assume that the gas flow in the well is one-dimensional at a resistance force per unit mass being equal to $\frac{\lambda u^2}{2D}$, where $u$ is an axial velocity and $\lambda$ is a resistance coefficient. The latter assertion is explained in [3].

The pressure usually reaches tens and hundreds of atmospheres, therefore, the gas state equation is taken to be $s = \rho z(s) RT$, where $s$ is a current pressure, $\rho$ is a gas density, $R$ is a gas constant, $T$ is an average well bore temperature, and $z(s)$ is a real gas factor.

Similar to [1, 2], below we use representation $z(s) = e^{\alpha s} + \beta s$ [4].

Here,

$$\alpha = -\ln \frac{\rho_c}{\rho_c}, \quad \beta = \frac{0.1}{P_c} \quad \text{and} \quad \kappa = 0.73 + 0.17371 \ln \frac{T}{T_c}.$$
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where \( p_c \) and \( T_c \) are the critical pressure and temperature.

Gas filtration in the j-formation complies with the nonlinear law of flow – the Darcy-Forchheimer law:

\[
\frac{ds}{dr} = -(\alpha_j + \beta_j \rho j \nu), j = 1, 2.
\]

The substantiation of this law is provided in [5] and [6]. In the present article, like in [1] and [2], the nonlinear law of flow (the Darcy-Forchheimer law) is used to account for a homogeneous porous medium in a radially symmetric gas flow. In [7], this law is investigated for a homogeneous porous medium in a 3D-dimension case. The Darcy-Forchheimer law is applied in [8] to study heterogeneous porous media.

Here, \( r \) is the current radius measured from the well axis, \( s_j = s_j(r) \) is a pressure in the j-formation, \( \rho_j = \rho_j(r) \) is a gas density, \( w_j \) is a filtration velocity, and \( \alpha_j, \beta_j \) are the constants characterizing the formation filtration capacity and the gas viscosity. In this notation, \( U = s_j(D/2) \) and \( V = s_j(D/2) \) are wellbore pressures at the depths \( L_1 \) and \( L_2 \), \( X = s_j(T) \) and \( Y = s_j(T) \) are pressures in the upper and lower formations, \( r_j \) is a supply contour radius of the j-formation, \( j = 1, 2 \). The production rate \( q_j \) of the j-formation is the gas volume under normal conditions produced from the formation per unit of time. The well production rate is \( Q = q_1 + q_2 \), under the assumptions made that \( q_1 = q_1(p) \) and \( Q = Q(p) \). Here, \( p \) is a wellhead pressure. The filtration coefficients of the formations \( a_j, b, j = 1, 2 \) are assumed to be determined as

\[
a_j = \frac{\alpha_j s_j T_j}{\pi r_j} \ln \frac{2r}{D} > 0, \quad b_j = \frac{\beta_j s_j T_j}{2\pi r_j^2} \left( \frac{2}{D} - \frac{1}{r_j} \right) > 0, \tag{1}
\]

where \( s_j \) is an atmospheric pressure, \( \rho_j \) is a gas density under atmospheric pressure, \( T_j = T_j / T_c \). These values are derived from the filtration law. For instance, let us assume that \( j = 1 \). The gas mass flowing along the well through the sidewall of the cylinder of radius \( r \) per unit of time equals \(-2\pi rh \rho_j \nu w_j(r)\). Under the formation conditions, this mass has a density \( \rho_j(r) \), a temperature \( T \), and it is subjected to a pressure \( s_j(r) \). Under steady-state filtration conditions, this mass does not change, though under normal conditions it equals \( s_j q_1 / RT \) – according to the Ideal Gas Law.

Taking into consideration that \( s_j = z(s_j) \rho_j RT \) and the following two expressions for the mass, we shall obtain

\[
w_j = \frac{q_j z(s_j) T_j}{2\pi rh_j}.
\]

By substituting the obtained expression in the filtration law and integrating it with respect to \( r \) from \( D/2 \) to \( r_j \), we shall obtain the following equation of steady filtration from the upper formation:

\[
b_a q_1^2 + a_1 q_1 = \int \frac{2ds}{z(s)} q_1 > 0.
\]

Along with this, \( b_1, a_1 \) are calculated using formula (1) at \( j = 1 \).

Let us define that \( g \) is a gravity acceleration: \( \sigma_j = \frac{gL_j}{RT} \), \( j = 1, 2 \), and \( \sigma_1 = \sigma_1 - \sigma_2 \). Let \( X_1, Y_1, X_2, Y_2 \) be the solutions of the following equations:

\[
\int \frac{z(s)ds}{s} = \sigma_1, \quad \int \frac{z(s)ds}{s} = \sigma_2.
\]

These values are uniquely determined and are effectively calculable using Newton’s method [9].

Now we are ready to make an overview of previous publications on this topic.

The stationary production rate of a well draining several gas-bearing strata is treated in [10]. There it is assumed that all strata pressures are neighbouring, a gas cross-flow between the formations is absent, and a real gas factor is a constant. These assumptions are used to obtain the approximation formula for the well production rate. The problem of a simultaneous exploitation of two formations with identical pressures is studied in [11] under an assumption that the strata summary rate is a constant and the gas cross-flow is absent. The results of [10] become specified in [12] for a case when the difference of two squares of formation and bore pressures is identical for all formations. In works [10–12]:

1) gas cross-flows in a running well bore are not taken into account;

2) the real gas factor is assumed to be a constant.

The present article is based on [1, 2] in the following respects:

1) the described limitations are overcome;

2) the turning points concept is essentially applied.

The notion of a turning point is of significant importance for further research.

The definition. The turning point of the total well production rate \( Q(p) \) is a value of the wellhead pressure \( p = P^* \), so that \( Q(P^*) = 0 \). The turning point of the production rate \( q_j(p) \) of a j-formation is such a value of the well-head pressure \( p = P^*_j \) that \( q_j(P^*_j) = 0 \).

An assertion. There always exists a \( P^* \). If \( Y > X_2 \), there also exists a \( P^*_2 \), with the inequalities \( X_1 < P^* < Y_1 < P^*_2 \) being satisfied. May \( q_j > 0 \) be determined from the solution \((V, q_j)\) of a system of equations, with \( a_2 > 0, b_2 > 0, \mu > 0 \) being specified as follows:

\[
\int \frac{z(s)ds}{s + \mu z(s)} = \sigma_3, a_2 q_2 + a_1 q_1 = \int \frac{2ds}{z(s)} \tag{3}
\]

For a \( P^*_2 \) to exist, it is necessary and sufficient that the next inequality is satisfied in the following way:

\[
\int \frac{z(s)ds}{s + \mu z(s)} \geq \sigma_3. \tag{4}
\]

Under (4), \( P^* < X_1 \). If \( q_2 \) is so large that inequality (4) does not hold, then \( P^*_2 \) does not exist, and for all \( p > 1 \) there will be \( q_1(p) < 0 \), i.e. even for a fairly small \( p \) there occurs a gas cross-flow from the lower formation into the upper one.

We argue that the existence of a \( P^*_2 \) is independent of \( a_1, b_1 \). It should also be noted that, according to the engineering system of units, \( L_1, L_2 \) are in metres, \( D \) are in centimetres, \( q \) is a \( 10^6 \text{m}^3/\text{24-hour period} \), \( T \) is a Kelvin temperature. Therefore, when the value of \( \mu \) is introduced into (3), (4) is calculable using the formula:

\[
\mu = \frac{1.3761 \pi T^2}{D^2}. \tag{5}
\]
By virtue of [1], system (3) is uniquely solvable. The systems of equations from which $P^*, P_1^*, P_2^*$ are calculated by Newton’s method are also taken into account. As shown in [1], if a $P_1^*$ exists, and $1 \leq p < P_1^*$, the production rates $q_i(p)$, $q_2(p)$ and $Q(p)$ and the well bore pressures $U$ and $V$ are connected with the formation pressures $X$ and $Y$ as well as the filtration coefficients $(a_i, b_i)$ and $(a_j, b_j)$ by the system of equations:

$$
\int_0^U \frac{sz(s)ds}{z^2 + \mu Q(z)} = \sigma_1,
$$

(6)

$$
b_{sq_1} + a_{q_1} = \int_0^U \frac{2ds}{z(s)} q_1 > 0,
$$

(7)

$$
\int_0^V \frac{sz(s)ds}{z^2 + \mu Q(z)} = \sigma_1,
$$

(8)

$$
b_{sq_2} + a_{q_2} = \int_0^U \frac{2ds}{z(s)} q_2 > 0,
$$

(9)

$$
q_1 + q_2 = Q.
$$

(10)

If the condition $P_1^* < p < P^*$ is satisfied or a $P_1^*$ does not exist, equations (6) and (8)–(10) are preserved, and (7) is replaced by the following equation:

$$
b_{sq_1} - a_{q_1} = \int_0^U \frac{2ds}{z(s)} q_1 < 0, U > X.
$$

(11)

The relations of type (6) for the first time are received and investigated in [13].

By virtue of [1], system (6)–(10) is uniquely solvable, and the description of its numerical solution algorithm is given as follows: at known $X, Y$ and $a_1, b_1, a_2, b_2$, it is possible to find $q_1, q_2, q_3, U(p), V(p)$ and $Q=Q(p)$ as functions of $p$. Hence, a solution of the direct problem is obtained. It is noteworthy that there is no close-form solution of the integrals in formulae (6)–(9); thus, it is not yet the exact solution of the system. In [2], analogous results are obtained in the case of $Y > X_2$. The difficulty of the formulation of the direct problem is that the formation pressures $X$ and $Y$ and the filtration coefficients are unknown in practice. Hence, this raises the problem of calculating the six parameters $X, Y, a_1, b_1, a_2, b_2$ with respect to six-time stationary pressure measurements of $p_i$ and a corresponding total production rate $Q_i$, where $i=1,...,6$. Thus, an inverse problem arises.

3. The purpose and objectives of the study

The purpose of this study is to show that the formation pressures $X$ and $Y$ and the filtration coefficients $(a_i, b_i)$ may be calculated by the data of $(p_i, Q_i)$ $(i = 1, 2, ..., 6)$ from the results of the direct problem solution. To achieve the outlined aim, the following tasks are formulated and solved.

1. A three nonlinear equations system with unknowns $(\xi, \theta_i, \theta_j)$ is receive on the basis of the corresponding (12) system of equations of the direct problem.

2. We use a fundamental fact: the required $(X, Y, a_1, b_1, a_2, b_2)$ may be expressed in terms of $(\xi, \theta_i, \theta_j)$; and show the parameters $(\xi, \theta_i, \theta_j)$ are changed in the bounded domain while the primary unknowns $(X, Y, a_1, b_1, a_2, b_2)$ are a priori unbounded.

3. The geometry of the domain of the parameters $(\xi, \theta_i, \theta_j)$ is investigated.

4. The calculation method for solving the received three-equations system and its software realization are developed.

5. The test calculation for checking the efficiency and precision of the suggested method is carried out.

4. Formulation of the inverse problem

If the formation pressures and filtration coefficients are unknown, then $P^*, P_1^*$ and even their existence are unknown. However, $P^*$ can be measured experimentally: it is a stabilized shut-in wellhead pressure, so $Q(P^*) = 0$. Therefore, the measurements of $p_i$ are conducted so that

$$
p_1 < p_2 < ... < p_b < P^*.
$$

As shown in [1, 2], the production rates $q_i(p)$ are monotone decreasing functions of $p$ as well as $Q(p)$, just as $U$ and $V$ are monotone increasing functions of $p$. Hence,

$$
Q_1 > Q_2 > ... > Q_b > 0, U_1 < U_2 < ... < U_b,
$$

and

$$
V_1 < V_2 < ... < V_b
$$

are satisfied for $Q_2 = Q(p)$. Considering that there is no information concerning the turning points $P^*, P_1^*$ and the relations between $X_i$ and $Y_i$ it is a cause for viewing various options of the gauge $p_i$ position with respect to the turning points $P_1^*$ and $P_2^*$ as well if one of them does not exist. It is easy to calculate that there are 13 such options, and each of them requires detailed consideration. In [14], attention is given to an option when $X_1 < Y_1, P_1^*$ exists, and the inequalities $p_1 < ... < p_b < P_1^*$ are satisfied. In [15], the considered option is when

$$
p_1 < ... < p_i < P_i^* < p_j < p_b < P^*.
$$

The present study deals with a more complicated case:

$$
p_1 < P_1^* < p_i < ... < p_b < P^*
$$

(12)

under the same assumptions made concerning $X_i, Y_i, P_i^*$.

Let us assume $\chi_i = q_i(p_i), \chi_j = q_j(p_j), i = 1, ..., 6$ as unknown formation production rates, whereas $U_i$ and $V_i$ are unknown well bore pressures at $p = p_i$. Instead of the unknown formation pressures $X$ and $Y$, let us introduce the parameters $c_1$ and $c_2$ according to the formulae:

$$
c_1 = \int_0^X \frac{2ds}{z(s)} c_2 = \int_0^V \frac{2ds}{z(s)}.
$$

Moreover, let us also suggest that

$$
g_i = \int_0^U \frac{2ds}{z(s)}, d_1 = \int_0^V \frac{2ds}{z(s)}.
$$

From the monotony of the production rates and from (12), it follows that
\( x_1 > x_2 > 0 > x_3 > \ldots > x_i \), \( i = 1, \ldots, 6 \) \hspace{1cm} (15) \\
\( y_1 > y_2 > \ldots > y_i > 0 \). \hspace{1cm} (16)

Now, from (6)–(14), it follows that the unknown production rates \( x_i \) and \( y_i \) and the filtration coefficients of the formations \( (a_i, b_i) \) and \( (a_j, b_j) \), as well as the parameters \( c_i \) and \( c_j \), and the well bore pressures \( U_i \) and \( V_i \), \( i = 1, \ldots, 6 \) (30 unknown variables in total) are connected with the measured wellhead pressures \( p_i \) and the total production rates \( Q_i \), \( i = 1, \ldots, 6 \), by the following system of 30 equations:

\[
\int \frac{sz(s)ds}{s^2 + \mu Q^2(s)} = \sigma_i, \quad (i = 1, \ldots, 6),
\]

\[
b_i x_i^2 + a_i x_i - c_i = -g_i, \quad x_i > 0, \quad (i = 1, 2),
\]

\[
b_i x_i^2 - a_i x_i + c_i = g_i, \quad x_i < 0, \quad (i = 3, \ldots, 6),
\]

\[
\int \frac{sz(s)ds}{s^2 + \mu Q^2(s)} = \sigma_i, \quad (i = 1, \ldots, 6),
\]

\[
b_i x_i^2 + a_i y_i - c_i = -d_i, \quad y_i > 0, \quad (i = 1, \ldots, 6),
\]

\[
x_i + y_i = Q_i, \quad (i = 1, \ldots, 6).
\]

Furthermore, from physical considerations as well as from (11), (15) and (16), it follows that the following inequalities are to be satisfied:

\[
a_i > 0, b_i > 0, a_i > 0, b_i > 0,
\]

\[
g_i < c_i < g_i,
\]

\[
c_i > d_i.
\]

5. Reduction of the system (17)–(22)

In equations (17), at each \( i = 1, \ldots, 6 \), the values of \( p_i \), \( x_i \), \( y_i \) and \( Q_i \) are known, so \( U_i \) can be uniquely determined from this equation. There is an efficient \( U_i \) calculation algorithm based on Newton's method. It follows that \( U_i \) is a known parameter; thus \( g_i \) is known, too. The remaining system (18)–(22) contains 24 equations with the unknowns \( a_i, b_i, a_j, b_j, c_i, c_j, x_i, y_i \) and \( V_i \), \( i = 1, \ldots, 6 \). Let us set \( x_1, x_3 \) and \( x_5 \) as the principal unknowns and express the rest through them using relations (18)–(22). Let us consider the equations (18) at \( i = 2 \) and from (19) at \( i = 4, 6 \), as well as using expressions (26)–(28) for \( b_i, a_i \) and \( c_i \), we shall obtain

\[
x_i = x_{\min}(\theta_i, \theta_j), \quad (i = 2, 4, 6). \hspace{1cm} (33)
\]

where

\[
m_i(\theta_i, \theta_j) = \frac{\psi_i(\theta_i, \theta_j) - \psi_i(\theta_j, \theta_j) + \psi_j(\theta_i, \theta_j) - \psi_j(\theta_j, \theta_j)}{2\psi_i(\theta_i, \theta_j)}, \quad (i = 4, 6). \hspace{1cm} (34)
\]

Let us also establish that

\[
\xi = \frac{x}{Q_i}, \quad t_i = \frac{Q_i}{Q_i}, \quad (i = 2, 4, 6),
\]

then from (22) and (33) it follows that

\[
y_i = Q_i(t_i - \xi m_i(\theta_i, \theta_j)), \quad (i = 2, 4, 6). \hspace{1cm} (36)
\]

Similarly, from (21) at \( i = 2, 4, 6 \), we shall obtain

\[
b_i = \frac{(d_i - d_j) y_i + (d_j - d_i) y_j + (d_i - d_j) y_i + (d_j - d_i) y_j}{(y_j - y_i)(y_j - y_i) + (y_i - y_j)(y_i - y_j)}.
\]

\[
a_i = \frac{(d_i - d_j) y_i^2 + (d_j - d_i) y_j^2 + (d_i - d_j) y_i^2 + (d_j - d_i) y_j^2}{(y_j - y_i)(y_j - y_i) + (y_i - y_j)(y_i - y_j)}.
\]

\[
c_i = \frac{y_j x_j(y_j - y_i) d_i + y_j x_j(y_j - y_i) d_i + y_i x_i(y_i - y_j) d_i + y_i x_i(y_i - y_j) d_i}{(y_j - y_i)(y_j - y_i) + (y_i - y_j)(y_i - y_j)}.
\]
According to (14), \( d_i \) is dependent on \( V_i \), which is calculated through (20) at a known \( V_i \) and is thus implicitly dependent on \( \gamma = \mu_0^2 \). By virtue of the \( z(s) \) function structure, there is no closed-form evaluation of the integral in formula (20); thus, it is not to be spoken of the exact evaluation of \( V_i \). It should be noted, though, that \( V_i \) is analytically dependent on the parameter \( \gamma \) at a point \( \gamma = 0 \). So it is natural to seek a solution \( V_i \) of the equation

\[
\int \frac{z(s)ds}{s + \gamma'z'(s)} = \sigma_3
\]

in the form of

\[
V_i = V_{o0} + \gamma V_{o1} + \frac{1}{2}\gamma^2 V_{i2} + ..., \tag{40}
\]

thus restricting ourselves to the first three addends. Alongside this, \( V_{o0} \) is determined from the equation

\[
\int \frac{z(s)ds}{s^2 + \gamma'z'(s)} = \sigma_3
\]

with respect to the parameter \( \gamma \); then let \( \gamma = 0 \). Hence, we shall obtain

\[
V_{o0} = \frac{V_{o0}}{z(V_o)} I_{1}(i),
\]

\[
V_{o1} = 2vV_{o0} I_{2}(i) + \frac{V_{o0}(1 + \alpha V_{o0})e^{-\gamma V_{o0}}}{z'(V_o)} I_{1}(i) - \frac{2V_{o0}}{z(V_o)} I_{0}(i),
\]

and is calculated by Newton's method. \( V_{o1} \) and \( V_{i2} \) are determined successively in terms of onefold and twofold differentiations of the identity

\[
\int \frac{z(s)ds}{s^2 + \gamma'z'(s)} = \sigma_3
\]

Using (40), we assume

\[
d_i(\gamma) = d_i(0) + \gamma d_i + \frac{1}{2}\gamma^2 d_{i2} + ..., \tag{41}
\]

also keeping the first three addends. Then we shall obtain

\[
d_{o0} = \int \frac{2ds}{z(s)} d_i = \frac{2V_{o0}V_{o1}}{z(V_o)},
\]

\[
d_{o1} = 2\left( \frac{e^{-\gamma V_{o0}}(1 + \alpha V_{o0})V_{i2}}{z'(V_o)} + \frac{V_{o0}V_{i2}}{z(V_o)} \right).
\]

If we set

\[
y_{o0} = d_{o0}, y_{i1} = \mu d_{i1}, y_{i2} = \frac{1}{2}\mu^2 d_{i2},
\]

then for \( d_i \) we shall obtain the following approximation:

\[
d_i = y_{o0} + y_{o1} y_{i2} + y_{i2} y_{i1}, \quad (i = 1, ..., 6). \tag{42}
\]

At \( i = 2, 4, 6 \), let us process the expression for \( y_i \) from (36) through (41) and then process the obtained expressions through formulae (37)–(39). Then we shall obtain the expressions for \( a_2, b_2 \), and \( c_2 \) of the form

\[
b_2 = \frac{b(\xi)}{\eta(\xi)}, \quad a_2 = \frac{k(\xi)}{\eta(\xi)}, \quad c_2 = \frac{l(\xi)}{\eta(\xi)}.
\]

Here, \( b(\xi), k(\xi) \), and \( l(\xi) \) are the polynomials in \( \xi \) as to degrees 5, 6, 7, and \( \eta(\xi) \) is a third-degree polynomial in \( \xi \). The coefficients of \( \eta(\xi) \) are polynomials in \( m_0(\theta, \theta) \), \( j = 2, 4, 6 \), and are not cited because of their complicatedness.

We note that \( 0 < \xi < 1 \). By virtue of the monotony of the production rates, we shall obtain

\[
y_{2} > y_{1} > y_{6}.
\]

We also note that

\[
t_i = t_i - t_{i-1}, \quad m_i(\theta, \theta_i) = m_i(\theta, \theta_i) - m_i(\theta, \theta_i) \quad (i = 2, 4, 6).
\]

From (36) and (43), it follows that

\[
0 < \xi < \min \left\{ t_1, t_2, t_3, t_4, t_5, t_6, t_{10}, t_{11}, t_{12} \right\} = \xi_r. \tag{43}
\]

From (22) at \( i = 1, 3, 5 \), we shall obtain

\[
y_{1} = Q_1(1 - \xi), \quad y_{3} = Q_3(t_2 - \theta_5), \quad y_{3} = Q_3(t_2 - \theta_5). \tag{44}
\]

Let us now finally turn to the formulae (21) at \( i = 1, 3, 5 \) by processing expressions (42) and (45) through them. Then, it is easily seen that the following relations are satisfied:

\[
M_{i}(\xi) = h(\xi)(1 - \xi)^2 + Q^2(1 - \xi)^2 Q^2 + \eta(\xi)Q^2(1 - \xi)^2 + \gamma_{i2}(1 - \xi)^3 + Q^2 \gamma_{i2}(1 - \xi)^3 = 0. \tag{45}
\]

\[
M_{i}(\xi) = Q^2(1 - \xi)^2 + \eta(\xi)(Q^2(1 - \xi)^2 + \gamma_{i2}(1 - \xi)^3 + Q^2 \gamma_{i2}(1 - \xi)^3) = 0. \tag{46}
\]

\[
M_{i}(\xi) = Q^2(1 - \xi)^2 + \eta(\xi)(Q^2(1 - \xi)^2 + \gamma_{i2}(1 - \xi)^3 + Q^2 \gamma_{i2}(1 - \xi)^3) = 0. \tag{47}
\]

\[
M_{i}(\xi) = Q^2(1 - \xi)^2 + \eta(\xi)(Q^2(1 - \xi)^2 + \gamma_{i2}(1 - \xi)^3 + Q^2 \gamma_{i2}(1 - \xi)^3) = 0. \tag{48}
\]

The left-hand sides of these relations are represented by the seven-degree polynomials in \( \xi \) with the coefficients that are polynomially dependent on \( \theta_0, \theta_0 \) and on the algebraic functions of \( m(\theta_0, \theta_0) \), \( j = 2, 4, 6 \) as well as on \( p_i, Q_i, i = 1, ..., 6 \). Thus, it is shown that if the system of equations (17)–(22) is satisfied, then the triple \( (\xi, \theta_0, \theta_0) \) satisfies the system of three equations (46)–(48) that are polynomial in the variable \( \xi \). It is to be recalled that equalities (40) and (41) provide an approximation for \( V_i \) and \( d_i \). Along with this, it turns out that if the filtration coefficients \( (a_1, b_1), (a_2, b_2) \) and the coefficients \( c_1, c_2 \), which are connected with the formation pressures \( X \) and \( Y \) by formulae (13) are expressed through the solution \( (\xi, \theta_0, \theta_0) \) of the system (46)–(48) using formulae (26)–(28) and (37)–(39). At known \( c_1 \) and \( c_2 \), the forma-
tion pressures are evaluated by Newton’s method. Thus, the solution of the inverse problem reduces to the solution of the three equation system (46)–(48). The advantage of choosing \( x_1, x_3, x_5 \) as the principal unknowns lies in the fact that, as it will be clear from the subsequent, the solution \( (\xi, \theta_3, \theta_5) \) lies within a bounded domain while the required parameters \( X, Y, a_1, \ldots, b_2 \) are a priori unbounded. Along with that, the number of the required parameters can be half. In part 6, we will show that inequalities (23)–(25) set bounds to the domain of variables \( (\theta_1, \theta_3) \), with the geometry of the bounded domain \( \Omega_0 \) being dependent on the inverse problem data.

6. The numerical solution of the inverse problem

Equations (46)–(48) mean that \( \xi \) is the common root of the three polynomials

\[
M_j(\xi, \theta_3, \theta_5), \quad j=1,2,3
\]

at a point

\( (\theta_3, \theta_5) \in \Omega_0 \).

For approximate evaluation of \( (\xi, \theta_3, \theta_5) \), the domain \( \Omega_0 \) is covered by the uniformly spaced node grid

\( (\theta_3(k), \theta_5(l)) \)

where

\[ 1 \leq k \leq N, 1 \leq l \leq N_0 \]

are all integer numbers. In each node \( (\theta_3(k), \theta_5(l)) \), the coefficients of the polynomials \( M_j(\xi) \) are determined as well as

\[
\xi_j(k,l) = \xi_j(\theta_3(k), \theta_5(l)).
\]

Such a node as \( (\theta_3(k), \theta_5(l)) \) is considered to be an approximate solution of the problem in which each of the \( M_j \)-polynomials has at least one root \( \xi_j \), \( j=1,2,3 \), and the distance between these roots on the grid of the nodes is minimal. Then

\[
\frac{\xi_1 + \xi_2 + \xi_3}{3}
\]

is taken as a corresponding approximate solution. Along with this, the satisfiability of the inequalities

\[
\begin{align*}
b_2(\xi_j, \theta_3(k), \theta_5(l)) &> 0, \\
a_2(\xi_j, \theta_3(k), \theta_5(l)) &> 0, \\
c_2(\xi_j, \theta_3(k), \theta_5(l)) &> d_4
\end{align*}
\]

is rechecked in accordance with (23), (25), and (37)–(39). It may occur that the polynomial

\[
M_j(\xi, \theta_3(k), \theta_5(l))
\]

has a set \( K_j \) of roots of power that is greater than one in the interval

\[ [0, \xi_j(\theta_3(k), \theta_5(l))] \]

Then we select one point out of the sets \( K_1, K_2 \) and \( K_3 \) and find the diameter of the set obtained. As \( \xi_j, \xi'_j, \xi''_j \), we take such elements of the sets \( K_j (j=1,2,3) \) that implement the minimum with the diameters obtained. The roots of the polynomials \( M_j \) are calculated using Newton’s method. The algorithm is processed through software.

7. The description of the range of \( (\xi, \theta_3, \theta_5) \)

Here we give the description of the domain \( \Omega_0 \) at the corner

\[
\Lambda = \{ (\theta_3, \theta_5) : \theta_3 < \theta_5 < 0 \}
\]

in which the inequalities (31) are satisfied. The denominator \( \Delta(\theta_3, \theta_5) \) in (26)–(28) becomes zero on a hyperbola \( h_0 \) with the equation

\[
\theta_3 = \theta_5 = \theta_3 - \theta_5 = \frac{2}{\theta_3 - 1}
\]

and \( \Delta(\theta_3, \theta_5) > 0 \) at the intersection \( \Lambda \) with the domain of points lying under the lower branch of a hyperbola \( h_0 \). This branch meets the bisector of the third quadrant at a point

\[
\Lambda = (1 - \sqrt{2}, 1 - \sqrt{2}),
\]

so \( \Delta(\theta_3, \theta_5) < 0 \) is in a curvilinear triangle with the vertexes at the points \( \Lambda, (0, -1), (0, 0) \).

As we establish that

\[
\nu = 1 - \frac{\lambda_3}{\lambda_5 - \lambda_1},
\]

then \( \nu > 1 \) and

\[
\frac{1 - \lambda_3}{\lambda_5 - \lambda_1} = \nu - 1.
\]

The equality \( \psi(\theta_3, \theta_5) = 0 \) is satisfied on the line

\[
\tau_1 : \theta_3 = \theta_5 + \frac{1 - \nu}{\nu}
\]

that passes through the points

\[
\theta_1 = 1 - \frac{1}{\nu}, \quad \theta_2 = 0
\]

although in (26) the numerator is >0 to the left of the line \( \tau_1 \) and it is <0 to the right of \( \tau_1 \).

7. 1. The case when \( \nu < 2 \)

Let \( \nu < 2 \), then \( 1 - \nu > -1 \). The line \( \tau_1 \) and the hyperbola \( h_0 \) have a common point

\[
\left( \theta_1 = 1 - \frac{2}{\sqrt{2}}, 1 - \sqrt{2} \right) at \Lambda.
\]

The inequality \( \psi(\theta_3, \theta_5) > 0 \) is satisfied in the subdomain of the corner \( \Lambda \) that consists of two connected components, which are described by the inequality in accordance with
1) \(-\infty < \theta_5 < \theta_4, \theta_4 < \theta_3 < \tau_1(\theta_4), \)
\[ \theta_5 < \theta_3 < \frac{1}{\sqrt{2}}, \theta_4 < \theta_3 < h_1(\theta_4), \]
2) \(-1 < \theta_3 < \theta_5, h_1(\theta_4), \theta_3 < 0, \)
\[ \theta_5 < \theta_3 < 1 - \sqrt{2}, \tau_1(\theta_4) < \theta_3 < 0. \] (49)

Now we turn to the inequality \( \psi_1(\theta, \theta_1) > 0 \). The numerator in (27) becomes zero on the branch of the hyperbola
\[ \theta_3 = \tau_1(\theta_4) = -\sqrt{\frac{\theta_3^2 - v + 1}{v}}, \theta_3 < -\sqrt{1 - v}, \]
and it is positive at points \( \Lambda \) lying below \( \tau_2 \). The line
\[ \theta_3 = \frac{\theta_1}{\sqrt{v}} \]
lying in \( \Lambda \) is an asymptote of this branch. It is directly checked that \((\theta_3, \theta_5) \in \tau_1 \), so the line \( \tau_1 \) and hyperbolas \( h_0 \) and \( \tau_2 \) meet at the only point in which the numerators \( \psi_1 \) and \( \psi_2 \) and the denominator become 0. The inequalities
\[ \psi_1(\theta, \theta_1) > 0, \psi_2(\theta, \theta_1) > 0 \]
are simultaneously satisfied in the subdomain \( \Lambda \) that consists of two connected components in accordance with

1) \(-\infty < \theta_5 < \theta_4, \theta_4 < \theta_3 < \tau_1(\theta_4), \)
2) \(\theta_5 < \theta_3 < \frac{1}{\sqrt{2}}, \theta_4 < \theta_3 < \tau_1(\theta_4), \)
\[ -\sqrt{1 - v} < \theta_3 < 1 - v, \tau_1(\theta_4) < \theta_3 < 0. \]

We note that at \( v \geq 1 \) the next inequality holds:
\[ \sqrt{2v - 1} \geq \sqrt{v - 1}, \]
so \( \theta_3 < \frac{1}{\sqrt{v - 1}}. \)

Let us consider the inequality
\[ \psi_1(\theta, \theta_1) < \lambda_{25} \]
in the domain where \( \Delta > 0 \). Then it is possible to conceive of (51) as
\[ h_1(\theta_5, \theta_3) = \frac{(\theta_3 - \theta_4 - v + 1)(\theta_3 + v - 1) + (v - 1)(v - 2)}{0}. \] (52)

The equation \( h_1(\theta_5, \theta_3) = 0 \) is the equation of a hyperbola with the asymptote \( \theta_3 - \theta_4 = v - 1, \theta_3 = 1 - v \). It is directly checked that \( h_1 \) has a common point \((0, -\sqrt{1 - v})\) with the hyperbola \( \tau_1 \) and also passes through the point \((\theta_3, \theta_5) \), so this point is common for the curves \( h_1 \), \( \tau_1 \), and \( \tau_2 \).

From what has been said, it follows that inequality (52) is satisfied in the subdomain \( \Omega \):
\[ \Omega = \{(\theta, \theta_1): \theta_3 < \theta_5 < 0, \Delta > 0, \psi_1(\theta, \theta_1) > 0, \psi_2(\theta, \theta_1) > 0\}. \]

Let us turn to the inequality
\[ \lambda_{25} < \psi_1(\theta, \theta_1). \] (53)
in the domain where \( \Delta > 0 \). Let us establish that
\[ \sigma = \frac{1 - \lambda_{25}}{\lambda_{25} - \lambda_{35}} \]
\[ \frac{\lambda_{25} - \lambda_{35}}{\lambda_{25} - \lambda_{35}} \]
\[ \frac{\sigma + 1 - v}{v} > 0. \] (54)

Inequality (53) can be perceived as
\[ F(\theta, \theta_1) = -\lambda(\theta_1 + \sigma) + (\theta_1^2 - \sigma) \theta_1 + \zeta \theta_1(\theta_1 + 1) > 0. \] (55)

From (54), it follows that \( \sigma > v - 1 \). Our prime interest is in the set
\[ \Omega_0 = \Omega \cap \{ (\theta, \theta_1) \in \Lambda, F(\theta, \theta_1) > 0 \}. \]

Taking (50) into account, it can be shown that at \( \theta_1 = -\sigma \), inequality (55) is satisfied in \( \Omega \) if \( 1 > \sigma > \sqrt{2v - 1} \), and it is not satisfied if \( \sigma < \sqrt{2v - 1} \). At \( \theta_1 \neq \sigma \), we consider \( F(\theta, \theta_1) \) as a quadratic trinomial in \( \theta_1 \) the coefficients of which are dependent on the parameters \( \theta, \sigma, \zeta \).

Let \( \theta_0(\theta_1) \) be its discriminant
\[ \theta_0(\theta_1) = (\theta_1 - \sigma)^2 + 4\zeta \theta_1(\theta_1 + \sigma)(\theta_1 + 1). \] (56)

Considering that the polynomial
\[ \theta_0(\theta_1) = \theta_1(\theta_1 + \sigma)(\theta_1 + 1) \]
is negative at
\[ \theta_1 = (-\infty, -1) \cup (-\sigma, 0) \]
and is positive at \( \sigma = (-1, -\sigma) \), it seems reasonable to say that \( \theta_0(\theta_1) > 0 \) at an open set \( G \subset (-\infty, 0) \) that contains the neighbourhoods of the points
\[ \theta_1 = -1, \theta_1 = -\sigma, \theta_1 = 0. \]

So at \( G \), the real \( \theta_1 \) roots of the denominator polynomial \( F(\theta, \theta_1) \) that are determined as
\[ \tau_1(\theta_1) = \frac{\theta_1^2 - \sigma - \sqrt{\theta_0(\theta_1)}}{2(\theta_1 + \sigma)}, \quad \tau_2(\theta_1) = \frac{\theta_1^2 - \sigma + \sqrt{\theta_0(\theta_1)}}{2(\theta_1 + \sigma)} \] (57)
and the satisfiability of inequality (55) is dependent on a mutual arrangement of \( \theta_1 \) and \( \tau_1(\theta_1), \tau_2(\theta_1) \) at a fixed \( \theta_1 \).

Many different variants for the boundaries of the domain \( \Omega \) are possible here.

7. 1. 1. Let \( 1 > \sigma > \sqrt{2v - 1} \), so the following inequalities are satisfied
\[ -1 < -\sqrt{\sigma < -\sigma} < 1 - \sqrt{2v} = \theta_1 < \sqrt{1 - v} < 0. \] (58)

At \( \theta_1 \in (-1, -\sigma) \), it is evident that
\[ \tau_1(\theta_1) > 0, \tau_2(\theta_1) < 0. \]

so inequality (55) is satisfied if \( \theta_1 < \tau_1(\theta_1) \). Fig. 1 shows the boundaries \( 1(\tau_1), 2(\tau_2) \) and \( 3(\tau_1) \) of the variables \( \theta, (\theta_1) \) and the view of a corresponding domain \( \Omega_0 \) for this variant.

We also note that the following equalities hold:
\[ \tau_1(\sqrt{\sigma}) = \tau_2(-\sqrt{\sigma}) = \tau_1(\sigma) = \lim_{\theta_1 \to -\sigma} \tau_1(\theta_1) = -\zeta. \]
However, it turns out that \( \varphi_2(\theta_1) \) can be perceived as

\[ \varphi_2(\theta_1) = (1 - \nu)(\theta_1 + \sigma)(\theta_1 - 1 + \sqrt{2v})(\theta_1 - 1 - \sqrt{2v}) \]

whence inequality (60) follows.

**Lemma 3.** At \( \theta_1 \in (-\sqrt{\sigma}, \theta_1^0) \), the following inequality holds:

\[ \tau_1(\theta_1) < \tau_2(\theta_1) \quad (61) \]

*Proof.* The curves \( \theta_1 = \tau_1(\theta_1) \) and \( \theta_1 = \tau_2(\theta_1) \) are branches of the algebraic curves

\[ \tau(\theta_1, \theta_2) = \nu \theta_2^2 - \theta_2^2 + \nu - 1 = 0 \]

and

\[ F(\theta_1, \theta_2) = -(\theta_1 + \sigma)\theta_2^2 + (\theta_2^2 - \sigma)\theta_2 + +\nu\theta_2^2 \]

Consideration of \( \nu \xi = \sigma - \nu + 1 \), it can be shown that (62) has the form of

\[ R(\tau, F, \theta_1) = -(\nu - 1)(\theta_2^2 - \sigma)(\theta_2 - 1)^2 - 2v ) \]

However,

\[ \tau(\theta_1, \theta_2) \] and \( F(\theta_1, \theta_2) \)

have common points if and only if \( R(\tau, F, \theta_1) = 0 \), i.e. only at the points \( \theta_1 = -\sqrt{\sigma} \) and \( \theta_1 = \theta_1^0 = 1 - \sqrt{2v} \). (The solutions \( \theta_1 = \sqrt{\sigma} \) and \( \theta_1 = 1 + \sqrt{2v} \) do not belong to \( \Lambda \) and are thus of no interest). The corresponding values are

\[ \theta_1 = \theta_2(\sqrt{\sigma}) = 1 - \frac{2}{\sqrt{v}} \]

From (57), it follows that \( \tau_1(-\sqrt{\sigma}) > 0 \). Since \( -\sigma < \theta_1^0 \), from the consequence of lemma 1 we shall obtain \( \tau_1(\theta_1^0) < -1 \). Consequently, the branch \( \tau_1(\theta_1) \) has no common points with the curve \( \tau_2(\theta_1) \). Hence

\[ \tau_1(-\sqrt{\sigma}) = -\sqrt{\sigma}, \quad \tau_2(\theta_1^0) = \theta_1^0 = \tau_2(\theta_1^0) \]

and these branches do not have any other common points at \( (-\sqrt{\sigma}, \theta_1^0) \). So, in order to prove inequality (61), it is enough to show that \( (\tau_2(\theta_1^0))' = 0 \). However,

\[ \tau_2(-\sqrt{\sigma}) = \frac{2}{\sqrt{v}} \cdot \frac{1}{\sqrt{v}^2 + 1} \]

It is easier to accomplish the calculation of \( \tau_2(-\sqrt{\sigma}) \) while considering that there is an identity \( F(\tau_2(\theta_1), \theta_2) = 0 \). By differentiating it with respect to \( \theta_1 \) and considering that \( \tau_1(-\sqrt{\sigma}) = -\sqrt{\sigma} \), we shall obtain...
\[
\rho \left( -\sqrt{\sigma} \right) = \frac{1 - \sqrt{v}}{1 - \sqrt{\sigma}} = \frac{\sqrt{\sigma + 1 - v}}{\sqrt{1 - \sqrt{\sigma}}}.
\]

Therefore,
\[
\sqrt{\rho \left( -\sqrt{\sigma} \right) - \rho \left( -\sqrt{\sigma} \right)} = \sqrt{\sigma + 1 - v + v - 1 - \sqrt{\sigma}},
\]
so
\[
\text{sign} \left( \sqrt{(\sigma + 1 - v) - (\sigma - (v - 1))} \right) = \text{sign} \left( \sqrt{(v - 1)(\sigma + 1 - v)^2} \right) = \text{sign} \left( \sqrt{\sigma + 1 - 2v} \right) = 1,
\]

since \( v > 1 \). This shows that the following lemma holds.

**Lemma 4.** At \( \theta < -\sigma \), there holds the following inequality:
\[
\tau_1(\theta) < \tau_2(\theta),
\]
and at \( -\sigma < \theta < \theta_1^0 \) there holds the following inequality:
\[
\tau_2(\theta) < \tau_1(\theta).
\]

From the cited results, it follows that in the domain \( \Delta > 0 \) inequality (31) is satisfied in the subdomain \( \Omega_2 \), described by the inequalities
\[
\left( -\sqrt{\sigma} < \theta < -\sigma, \tau_2(\theta) < \theta < \tau_1(\theta) \right).
\]

so the domain \( \Omega_0 \) is bounded.

We now turn to the situation in the subdomain \( \Lambda_0 \), in which \( \Delta < 0 \). In this case,
\[
\psi_3(\theta, \theta) > 0,
\]
if
\[
\theta_1 > 1 + \frac{\theta_1 - 1}{v}, \quad \psi_3(\theta, \theta) > 0,
\]
and it can be easily seen that it is automatically satisfied in the domain where
\[\Delta < 0, \psi_3(\theta, \theta) > 0 \text{ and } \psi_3(\theta, \theta) > 0.\]

What is left to consider is inequality (53). In this situation, it has the form of
\[F(\theta, \theta_1) > 0\]
and it is satisfied at
\[\theta_1^0 < \theta \text{ and } \tau_1(\theta) < \theta_1.\]

From the estimates obtained, it follows that in the domain where \( \Delta < 0 \),
\[
\psi_3(\theta, \theta, \theta) > 0, \quad \psi_3(\theta, \theta, \theta) > 0,
\]
inequality (53) is automatically satisfied. Hence, in the case under consideration, the inequalities (31) are satisfied in a bounded domain \( \Omega_2 = \Omega_1 \cup \Omega_2 \), where \( \Omega_2 \) is described by the inequalities
\[
\theta_1^0 < \theta < -\sqrt{v - 1}, \quad \tau_1(\theta) < \theta < \tau_1(\theta_1),
\]
and
\[
-\sqrt{v - 1} < \theta < v, \quad \tau_1(\theta) < \theta < 0.
\]

7.1.2. Let \( \nu > \sigma > \sigma > \nu > \sigma \)

In this case, the inequalities
\[
-\sqrt{v - 1} < \theta < -\sqrt{v - 1}, \quad -\sigma < -\sigma - \nu < -\nu < \nu - \sigma,
\]
are satisfied, and the domain \( \Omega_2 \) of the bounded and is described by the inequalities
\[
-\sqrt{v - 1} < \theta < v, \quad \tau_1(\theta) < \theta < 0.
\]

7.1.3. Let \( \theta < \sigma < \sigma < \sigma < \sigma \)

Here the domain \( \Omega_2 \) consists of two components:
\[
\left( -\sqrt{v - 1} < \theta < \theta_1^0, \quad \tau_1(\theta) < \theta < \tau_1(\theta_1) \right).
\]

The second component is described by the inequalities
\[
-\sqrt{v - 1} < \theta < -\sigma, \quad \tau_1(\theta) < \theta < 0.
\]
and
\[
-\sigma < \theta < -\sigma - \nu, \quad \tau_1(\theta) < \theta < 0.
\]

7.1.4. Let \( \nu > \sigma > \sigma > \sigma > \sigma \)

In this case, \( \Omega_2 \) consists of one connected component described by the inequalities
\[
-\sqrt{v - 1} < \theta < -\sqrt{v - 1}, \quad \tau_1(\theta) < \theta < \tau_1(\theta_1),
\]
and
\[
-\sigma < \theta < -\sigma - \nu, \quad \tau_1(\theta) < \theta < 0.
\]

7.1.5. Let \( \theta < \sigma < \sigma < \sigma < \sigma \)

As in the previous case, \( \Omega_2 \) consists of one component described by the inequalities
\[
-\sqrt{v - 1} < \theta < -\nu, \quad \tau_1(\theta) < \theta < \tau_1(\theta),
\]
and
\[
-\sigma < \theta < -\sigma - \nu, \quad \tau_1(\theta) < \theta < 0.
\]

Fig. 2 shows boundaries (1(\tau_1), 2(\tau_2) and 3(\tau_1) of the variables \( (\theta_1, \theta_2) \) and a view of the domain \( \Omega_2 \) for this variant.
and
\[ (-\sqrt{v-1} < \theta, 1 - v, \tau, \theta < 0). \]

\[ \frac{\sigma(1-(v-1)\delta)}{v}, \]

and
\[ \text{sign}(\zeta - (1+\sigma)) = \text{sign}((2v-1)\delta^2 + 2(v-1)\delta - 1). \]

However, \( \delta \) denotes the roots of the trinomial within the brackets, namely
\[ \delta_1 = \frac{1}{2v-1}, \delta_2 = -1, \]

so
\[ \delta_1 < \delta < \delta_2, \]

and
\[ \text{sign}((2v-1)\delta^2 + 2(v-1)\delta - 1) = -1, \text{ Q.E.D.} \]

In this case, \( \Omega_0 \) consists of two components described by the inequalities
\[ (-\sigma < \theta, -\sqrt{v}, \tau, \theta < 0), \]

\[ (-\sqrt{v} < \theta, 1 - v, \tau, \theta < 0). \]

**7. 2. The case when \( v > 2 \)**

Then
\[ \sigma > 1, \theta_0^1 > 0, \theta_0^2 < 0, \]

so the intersection point \((\theta_0^1, \theta_0^2)\) of the hyperbolas \( \tau_2, h_0 \) and \( h_1 \) and the line \( \tau_1 \) is in the fourth quadrant \((\theta_0^1, \theta_0^2)\), and it is thus of no interest. Now
\[ 1 - v < -\sqrt{v-1}, \]

and inequality (51) is satisfied in the subdomain \( \Lambda_0 \), in which
\[ \psi_1(\theta_0^1, \theta_0^2) > 0 \text{ and } \psi_2(\theta_0^1, \theta_0^2) > 0. \]

We then turn to inequality (53), which is now considered in the domain where \( \Delta > 0 \) and has the form of (55).

**7. 2. 2. Let \( \sigma > 2v-1, \zeta < 1 \)**

Inequality (31) is satisfied in the bounded domain \( \Omega_0 \) that consists of one connected component according to the following inequalities
\[ (-\sigma < \theta, -\sqrt{v}, \tau, \theta < 0), \]

\[ (-\sqrt{v} < \theta, 1 - v, \tau, \theta < 0). \]

\[ \text{sign}((2v-1)\delta^2 + 2(v-1)\delta - 1) = -1, \text{ Q.E.D.} \]

In this case, \( \Omega_0 \) is a bounded domain described by the inequalities
\[ (-\sigma < \theta, -\sqrt{v}, \tau, \theta < 0), \]

\[ (-\sqrt{v} < \theta, 1 - v, \tau, \theta < 0). \]

**7. 2. 1. Let \( v-1 < \sqrt{v} < 2v-1, \zeta < 1 \)**

Inequality (31) is satisfied in the bounded domain \( \Omega_0 \) that consists of one connected component according to the following inequalities
\[ \left( -\sigma < \theta, -\sqrt{v}, \tau, \theta < 0, \tau, \theta < 0 \right), \]

\[ \left( 1 - v < \theta, -\sqrt{v}, \tau, \theta < 0, \tau, \theta < 0 \right), \]

\[ \left( 1 - v < \theta, -\sqrt{v}, \tau, \theta < 0, \tau, \theta < 0 \right). \]

**7. 2. 2. Let \( \sqrt{v} > 1 < \sigma < 2v-1 \)**

In this case, \( \Omega_0 \) is a bounded domain described by the inequalities
\[ (-\sigma < \theta, -\sqrt{v}, \tau, \theta < 0, \tau, \theta < 0), \]

\[ (-\sqrt{v} < \theta, 1 - v, \tau, \theta < 0, \tau, \theta < 0). \]
\(-\sigma < \theta_i < 1 - v, \tau_i(\theta_i) < \theta_i < \tau_i(\theta_i))

\((1 - v < \theta_i < -\sqrt{\alpha, \tau_i(\theta_i) < \theta_i < 0})\)

and

\((-\sqrt{\alpha} < \theta_i < -\sqrt{v - 1, \tau_i(\theta_i) < \theta_i < 0}).\)

If \(\sqrt{\alpha} > v - 1\), then the domain is described by the same inequalities.

7.2.3. Let \(\sqrt{\alpha} < v - 1 < 2v - 1 < \sigma\)

Now \(\zeta > 1\), according to lemma 5, \(\varphi_i(\theta_i) > 0\) at \((-\infty, 0)\).

The bounded domain \(\Omega_0\) consists of one component according to the following inequalities:

\((-\sigma < \theta_i < 1 - v, \tau_i(\theta_i) < \theta_i < \tau_i(\theta_i))\),

\((1 - v < \theta_i < -\sqrt{\alpha, \tau_i(\theta_i) < \theta_i < 0})\),

and

\((-\sqrt{\alpha} < \theta_i < -\sqrt{v - 1, \tau_i(\theta_i) < \theta_i < 0}).\)

7.2.4. Let \(v - 1 < \sqrt{\alpha} < 2v - 1 < \sigma\)

Here, there is also \(\varphi_i(\theta_i) > 0\) at \((-\infty, 0)\), and \(\Omega_0\) is bounded and described by the inequalities

\((-\sigma < \theta_i < \sqrt{\alpha, \tau_i(\theta_i) < \theta_i < \tau_i(\theta_i)}\)

\((-\sqrt{\alpha} < \theta_i < 1 - v, \tau_i(\theta_i) < \theta_i < \tau_i(\theta_i))\)

and

\((1 - v < \theta_i < -\sqrt{v - 1, \tau_i(\theta_i) < \theta_i < 0}).\)

Fig. 3. A view of the domain \(\Omega_0\) with boundaries 1, 2, 4 for variant 7.2.4

Fig. 3 shows the boundaries 1(\(\tau_1\)), 2(\(\tau_2\)) and 4(\(\tau_4\)) of the variables \((\theta_i, \theta_i)\) and the view of a corresponding domain \(\Omega_0\) for this variant.

8. An example of the numerical solution of the inverse problem

Calculations were performed for the following values of the geophysical parameters: gas-methane formations

where \(L_1=1,000\) m and \(L_2=1,500\) m; a wellhead temperature \(T_1=291\) K and a bottom temperature \(T_3=310\) K; a well diameter \(D=21.6\) cm, and a resistance coefficient \(\lambda=0.023\). By virtue of [16], the critical pressure and the critical temperature are taken as \(P_c=46.95\) at \(T_c=190.55\) K. In this case,

\(\alpha=0.0451, \ \beta=0.00213, \ \sigma=0.06368, \ \sigma=0.03184, \ \mu=0.00060745.\)

The selected values of \(X, Y, a_1, b_1)\) and \(a_2, b_2)\) are consistent with the data of [17]:

\(X=90\) at, \(a_1=0.413, \ b_1=0.00062,\)

\(Y=100\) at, \(a_2=0.863, \ b_2=0.001.\) \(66\)

In this case,

\(P'=83.3226\) at and \(P'=85.7644\) at.

The wellhead pressures are taken as follows:

\(p_1=73\) at, \(p_2=79\) at, \(p_3=83.6\) at,

\(p_4=84\) at, \(p_5=85\) at, \(p_6=85.5\) at. \(67\)

By successively solving the direct problem with these data, we shall obtain the following total well production rate values:

\(Q_1=2,717.88377, \ Q_2=1,902.357766,\)

\(Q_3=730.937964, \ Q_4=578.428906,\)

\(Q_5=240.671, \ Q_6=81.983676.\) \(68\)

Let us now take the data from (67) and (68) as the principal values in the solution of the inverse problem with the grid step of 0.002. As a result of solving the inverse problem, we shall obtain the following values of the unknowns \(X, a_1, b_1\) and \(Y, a_2, b_2\):

\(X=89.9972\) at, \(a_1=0.41218, \ b_1=0.00061,\)

\(Y=99.95035, \ a_2=0.77835, \ b_2=0.00108.\) \(69\)

In the variant under consideration, \(\nu=1.168724\) and \(\sigma=1.13689\), so case 1.6 occurs. As can be seen, a proper agreement of the parameters (69) with the assumed (66) is obtained.

9. Conclusion

The research results are the following:

- System (17)-(22) of 30 equations with 30 unknowns appears in a primary formulation of the inverse problem. It is reduced to a system of three equations about \(\xi, \theta_3, \theta_5\), which are connected with the unknown rates \(q_1, q_3, q_5\) of an upper stratum.

- It happens to be essential that these equations are polynomial of the seventh degree over the first unknown \(\xi\), and their coefficients are expressed as some algebraic functions on \(\theta_3\) and \(\theta_5\).
Another essential moment is the proof of an unbounded domain where the parameters \( (\xi, \theta_3, \theta_5) \) are changed and described in the domain geometry. Per contra, the desired formation pressures and filtration coefficients \( (X, a_1, b_1) \) and \( (Y, a_2, b_2) \) are a priori unbounded.

These considerations allow applying the combination of a discretization technique to \( \theta_3 \) and \( \theta_5 \), with Newton’s method being used to solve the polynomial equation of the seventh degree, and so the system (46)–(48).

The algorithm of the numerical solution and its software realization were ready-built.

The test calculation shows the efficiency and precision of the suggested method.

This article has described a step to constructing a full algorithm for solving the problem of identifying the pressures and filtration coefficients of two gas-bearing strata opened by a single well on the basis of the wellhead pressure and production rate measurements.

References