Побудовано чисельний метод знаходження нулів функції однієї дійсної змінної за допомогою апарату некласичних мінорант та діаграм Ньютона функцій, заданих таблично. Цей метод не потребує додаткової інформації про розміщення коренів і має переваги над іншими методами пошуку нулів функцій. Функція, для якої потрібно знайти корені, може бути як гладкою, так і негладкою

Ключові слова: міноранта функції, нуль функції, поліном Чебишева, діаграма Ньютона, гладка і негладка функція

Построен численный метод нахождения нулей функции одной действительной переменной с помощью аппарата неклассических минорант и диаграмм Ньютона функций, заданных таблично. Этот метод не требует дополнительной информации о размещении корней и имеет преимущества перед другими методами поиска нулей функций. Функция, для которой нужно найти корни, может быть как гладкой, так и негладкой

Ключевые слова: миноранта функции, ноль функции, полином Чебышева, диаграмма Ньютона, гладкая и негладкая функция

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CONSTRUCTION OF A NUMERICAL METHOD FOR FINDING THE ZEROS OF BOTH SMOOTH AND NONSMOOTH FUNCTIONS

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1. Introduction

The task of finding (or localization) of the roots of algebraic and transcendental equations belongs to the important problems of applied mathematics. The need for solving such a problem arises in many fields of physics, mechanics and natural science in the broadest sense of this word. That is why this is a relevant topic today.

Finding the zeros of a function is the most important procedure when exploring and constructing different functions of dependencies, when examining continuous processes. Finding the zeros of functions comes down actually to a gradual approximation to the region in which the function becomes zero and to the study of it.

In the framework of the task on finding the roots of algebraic and transcendental equations, the following problems are applied:

1. A problem on the localization by modulo of the roots of algebraic polynomials with real or complex coefficients. For such polynomials, the formulas are derived for determining the upper and lower limits of the roots. Sufficient conditions are also established of the existence of "maximum" annular regions where there are no roots, as well as the "minimum" annular regions that have a defined number of roots.

2. A problem on the localization of the true roots of algebraic equations with real coefficients. For such equations, there are derived formulas for determining the boundaries of roots and established sufficient conditions for the existence of "maximum" intervals where there are no roots.

3. Iterative methods for refining the real roots of algebraic and transcendental equations. These methods imply that there are fairly small neighbourhoods that are known in advance, each of which contains only one root and in this neighborhood certain conditions hold. In order to find a certain root with a given accuracy, it is necessary to choose one of the points from the neighborhood, which contains the root, as the initial approximation and to employ appropriate iterative process.

2. Literature review and problem statement

Note that for finding the roots (both real and complex) of algebraic equations, there are direct methods. These methods do not require additional information about the location of the roots. These include methods of the Lobachevsky-Greffe type [1]. However, these methods do not have a wide practical application in connection with the growth to infinity of coefficients of equations, derived using the process of taking the square roots.

Great progress in the polynomial liquidation was provided by the calculation of real roots of poorly determined polynomials of high level (over 1,000). Article [2] presented the result of algorithmic complexity relating to the isolation of the roots of integer one-variant polynomials.

Paper [3] presented the classic methods for finding the zeros of functions, in particular, the Newton's method and showed the advantages and disadvantages of the methods.

Article [4] proposed an improvement to the iterative process of the Ostrovsky method, which increase the local order of convergence. However, in this method it is required to calculate a derivative of the function, as in the Newton's method, which sometimes is very difficult or long. Fairly fast methods are based on the Newton's method. Paper [5] proposed the method of fourth order based on the Newton's method. In [6], authors built an efficient method for finding the roots based on the multiplicative computations. These methods require a derivative of the function, which, as was already mentioned, is quite difficult to calculate.

Article [7] built a new method for finding the root of a function. The method is very fast and reliable, but searches for only one root at a given interval.

In [8], authors constructed an apparatus of nonclassical Newton majorants and diagrams of functions, given in the tabular form. This apparatus is used to approximate the functions, to build numerical methods for calculating certain integrals, numerical methods for solving the Cauchy problem for ordinary differential equations and systems. This apparatus is also employed for functions precise on certain classes, to optimize the smooth and nonsmooth concave functions of one and many real variables.

A nonclassical approach to building the apparatus of Newton's majorants and diagrams of functions given in the tabular form differs from the classical one practically by the point of mapping. If the nonclassical approach employs, as a point of representation, points

$$P_v\left(x_v, -\ln|a_v|\right),$$

where a_v are the coefficients of the appropriate series, then in the nonclassical case, the points of representation are taken as

$$P_v\left(x_v, -\ln\left|f\left(x_v\right)\right|\right),$$

where f(x) is the arbitrary function, set on certain interval [a,b], and points x_v satisfy the condition

$$a \le x_1 \le x_2 \le \ldots \le x_n \le b.$$

The apparatus of nonclassical Newton's minorants was used in [9] for constructing the numerical optimization methods of nonsmooth concave functions of one, two and many real variables.

In [9], authors for the first time built the apparatus of nonclassical minorants of the Newton's functions of one, two and many real variables given tabularly. The apparatus is used for the approximation of functions and the development of numerical methods of optimization of smooth and nonsmooth convex and arbitrary functions of one, two and many real variables.

In [10], authors built a numerical method for finding all points of the extremum of arbitrary smooth and nonsmooth functions of one real variable.

The theory of Newton majorants and diagrams, designed for power series of functions of one complex variable, was widely used in [11]. Further development of this theory covered the Laurent and Dirichlet series, generalized power series, and numerical sequences [12]. The theory was also employed in the power series, the Laurent and Dirichlet series of functions of two complex variables [13, 14]. Using the apparatus of the Newton majorants helped obtain important results in various areas of mathematics and cybernetics.

A question was raised: is it possible to apply the apparatus of non-classical Newton's minorant of functions, given in the tabular form, to find zeros of functions as well? However, the new method should search for zeros for arbitrary functions and would not need to separate the roots because it is a resource-intensive process. As mentioned above, such methods already exist (methods of the Lobachevsky-Greffe type), but are not commonly applied because of the infinite growth of equation coefficients. As it turned out, by using the apparatus of classical Newton's minorants, it is possible to overcome this problem, and then it is not required to calculate the derivative of the function for which zeros are to be found.

Here we consider the construction of a numerical method for finding zeros of both smooth and nonsmooth arbitrary functions of one real variable. The basis of the method is the so-called apparatus of nonclassical Newton's minorants of functions, given in the tabular form, which does not need additional information on the location of roots, as well as has many advantages over other methods for finding the zeros of functions.

3. The aim and tasks of the study

The aim of present work is to construct a numerical method for finding the zeros of a function that would not require imposed conditions both on the function and its roots.

To achieve the set aim, the following tasks are to be solved:

– to devise a new numerical method for finding the zeros of a function of both smooth and nonsmooth functions;

– to give examples and to estimate the speed of finding the zeros of functions based on the constructed method.

4. Materials and methods of research into finding the zeros of both smooth and nonsmooth functions

4. 1. The apparatus of non-classical Newton majorants and minorants of functions, given in the tabular form

Consider a function of real variable y=f(x), which is set by its values in some points x_i , i=0,1,...,n:

$$f\left(x_i\right) = y_i, \quad i = 0,\ 1,\ \ldots,n. \tag{1}$$

Assume

$$\left|y_i\right| = a_i \le M, \quad i = 0,\ 1,\ \ldots,n, \quad a_1 \cdot a_n \ne 0, \tag{2}$$

where M is a constant.

Definition 1. Point $P_i(x_i, -\ln a_i)$ with coordinates $x=x_i$, $y=-\ln a_i$, in plane xy is called *a representation point* of the value of function y=f(x) in point $x=x_i$.

Suppose that representation points P_i of the values of function y=f(x) in points x_i, i=0, 1,..., n in the xy plane have been built. From each point P_i we shall draw a half-line in the positive direction of the Oy axis, perpendicular to the Ox axis. A set of points of these half-lines will be denoted through S, and its convex shell – through C(S).

For each $x \in \left[x_0,\ x_n\right]$, the point $B_x\left(x, \chi_x\right)$ is defined, where

$$\chi_x = \inf_{(x,y)\in C(S)} y.$$

A set of points $B_x(x, \chi_x)$, $x \in [x_0, x_n]$, forms line δ_f, which limits C(S) from below. This line is a continuous, convex polyline and its equation takes the form

$$y = \chi(x),$$

$$x \in [x_0, x_n],$$

where $\chi(x) = \chi_x$.

Definition 2. The polyline δ_f, defined on the interval $[x_0, x_n]$, is called a *nonclassical Newton's diagram* of function y=f(x) on the given interval.

The Newton's diagram δ_f of function y=f(x) has the following properties:

– each vertex δ_f is in one of the representation points P_i of the value of function y=f(x) in point x_i, i=0, 1,..., n;

– each representation point P_i, i=0, 1,..., n, is on δ_f or above it.

Denote

$$M_f(x) = \exp(-\chi(x)), \quad x \in [x_0, x_n].$$

Then for each x_i, i=0, 1,..., n, equality holds

$$|f(x_i)| = a_i \le M_f(x_i).$$

In fact, from the construction of δ_f, it follows that

$$-\ln|f(x_i)| \ge \chi(x_i)$$

or

$$|f(x_i)| \le \exp(-\chi(x_i)) = M_f(x_i).$$

In addition,

$$M_f(x_0) = |f(x_0)|, \quad M_f(x_n) = |f(x_n)|.$$

Definition 3. Function $y=M_f(x)$, defined on the interval $[x_0, x_n]$, is called the *Newton's majorant* of function y=f(x) on the given interval.

Let

$$M_f(x_i) = T_i, \quad i = 0, 1, ..., n.$$

Definition 4. Magnitudes

$$R_i = \left(\frac{T_{i-1}}{T_i}\right)^{\frac{1}{x_i - x_{i-1}}} \quad (i = 1, 2, ..., n; \quad R_0 = 0)$$

and

$$D_i = \frac{R_{i+1}}{R_i} \quad (i = 1,\ 3,\ ...,\ n-1;\ D_0 = D_n = \infty)$$

are called, respectively, the i-th numerical slope and the i-th deviation from the Newton's diagram δ_f.

Definition 5. If the representation point P_i, i=0,1,..., n, is in vertex δ_f, then index i is called the *vertex index*, and if it is in δ_f, it is the *diagram index* δ_f. Indices i=0 and i=n refer to the vertex indices.

A set of all vertex indices will be denoted as I, and the set of diagram indexes as G. It is obvious that $I \subset G$ and $T_i = a_i$ for all $i \in G$.

The Newton's diagram is constructed in Fig. 1 for the function, assigned in nine points.

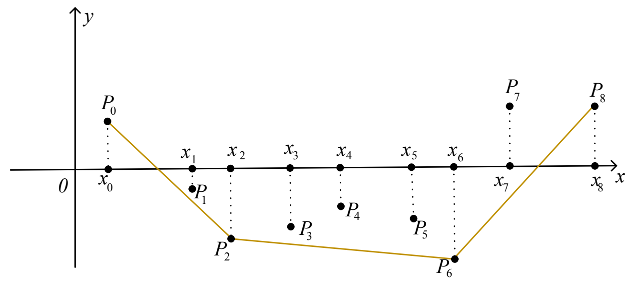

Fig. 1. Newton's diagram for the function assigned in nine points

Assume that representation points P_i of the value of function y=f(x) in points x_i, i=0, 1,..., n in the xy plane have been built. From each point P_i, we shall draw a half-line in the negative direction of the Oy axis, perpendicular to the Ox axis. A set of points of these half-lines will be denoted as S, and its convex shell through C(S). For each $x \in [x_0, x_n]$, we shall define point $D_x(x, \chi_x)$, where

$$\chi_x = \sup_{(x,y)\in C(S)} y.$$

The set of points $D_x(x, \chi_x)$, $x \in [x_0, x_n]$, form a line δ_f, which limits C(S) on the top. This line is a continuous, concave polyline and its equation takes the form

$$y = \chi(x), \quad x \in [x_0, x_n],$$

where

$$\chi(x) = \chi_x.$$

Denote

$$m_f(x) = \exp(-\chi(x)), \quad x \in [x_0, x_n].$$

Then for each x_i, i=0,1,...,n, the following equation holds

$$m_f(x_i) \le |f(x_i)| = a_i.$$

In fact, from the construction of δ_f, it follows that

$$-\ln|f(x_i)| \le \chi(x_i)$$

or

$$|f(x_i)| \ge \exp(-\chi(x_i)) = m_f(x_i).$$

In addition,

$$m_f(x_0) = |f(x_0)|, \quad m_f(x_n) = |f(x_n)|.$$

Definition 6. The function $y=m_f(x)$, defined on the interval $[x_0, x_n]$, is called the nonclassical Newton's minorant of function y=f(x) on the given interval, and the polyline δ_f is its diagram.

Fig. 2 shows the Newton's diagram of minorant of function, assigned in nine points

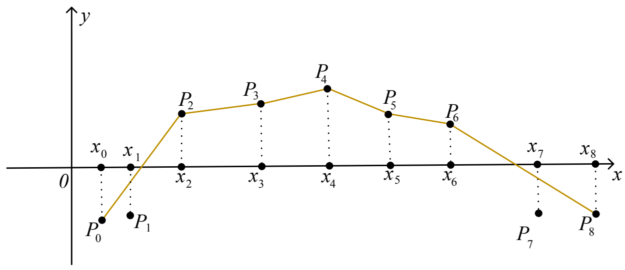

Fig. 2. Newton's diagram of minorant of function assigned in nine points

The diagram δ_f of Newton's minorant of function y=f(x) has the following properties:

1) each vertex δ_f is in one of the representation points P_i of the value of function y=f(x) in point x_i, i=1,2,..., n;

2) each representation point P_i, i=0,1,..., n, is on δ_f or below it.

Let

$$m_f(x_i)=t_i,\quad i=0,1,\ldots,n.$$

Definition 7. Magnitudes

$$r_i=\left(\frac{t_{i-1}}{t_i}\right)^{\frac{1}{x_i-x_{i-1}}}\quad (i=1,2,\ldots,n;\quad r_0=\infty)$$

and

$$d_i=\frac{r_{i+1}}{r_i}\quad (i=1,2,\ldots,n-1;\quad d_0=d_n=0)$$

are called, respectively, the i-th numerical slope and the i-th deviation from diagram δ_f of the Newton's minorant.

Let f(x) is the concave function on the interval [a, b]. Choose on the interval [a, b] a system of points x_0, x_1,...,x_n, where

$$x_k=x_0+kh\ (k=0,1,\ldots,n),\ x_0=a,\ h=\frac{b-a}{n},$$

and find the value of function y=f(x) in these points. Let

$$f(x_i)=c_i,\quad i=0,1,\ldots,n.$$

Since f(x) is a concave function on the interval [a, b], then the numerical slopes in the diagram of the Newton's majorant, built by values of the function in points x_0, x_1,...,x_n, are determined by formula

$$R_k=\left(\frac{c_{k-1}}{c_k}\right)^{\frac{1}{h}}\ (k=1,2,\ldots,n;\quad R_0=0).$$

In this case

$$R_1<R_2<\ldots<R_n.$$

Deviations D_k in the diagram of the Newton's majorant will satisfy the condition

$$D_k>1\ \ (k=1,2,\ldots,n-1;\quad D_0=D_n=\infty).$$

If, for some index k (0<k<n), the conditions $R_k\le 1$, $R_{k+1}>1$, hold, then point x_k with accuracy $\varepsilon<h$ is the point of the maximum of function f(x).

Let f(x) is a convex function on the interval [a, b]. We select in a similar way a system of points x_0, x_1,...,x_n on the interval [a, b], where

$$x_k=x_0+kh\ \ (k=0,1,\ldots,n),$$

$$x_0=a,\quad h=\frac{b-a}{n},$$

and find the value of function y=f(x) in these points. Let

$$f(x_i)=c_i,\quad i=0,1,\ldots,n.$$

Since f(x) is a convex function on the interval [a, b], then the numerical slopes in the diagram of the Newton's minorant, built by the values of function in points x_0, x_1,...,x_n, are determined by formula

$$r_k=\left(\frac{c_{k-1}}{c_k}\right)^{\frac{1}{h}}\ (k=1,2,\ldots,n;\quad r_0=\infty).$$

In this case

$$r_1>r_2>\ldots>r_n.$$

Deviations d_k in the diagram of the Newton's minorant will satisfy the condition

$$0<d_k<1\ \ (k=1,2,\ldots,n-1;\quad d_0=d_n=0).$$

If, for some index k (0<k<n), the conditions $r_k\ge 1$, $r_{k+1}<1$ hold, then point x_k with accuracy $\varepsilon<h$ is the point of a minimum of function f(x).

5. Results of research into finding the zeros of both smooth and nonsmooth functions

5. 1. Algorithm of the method for finding the zeros of both smooth and nonsmooth functions

Assume one has to find all the roots (zeros) of function f(x) on the given interval [a, b], that is, we shall search for the solution of equation f(x)=0. If f(x)=0, then |f(x)|=0. Let

$$\hat{f}(x)=1+|f(x)|,$$

$$\hat{f}(x_i)=a_i\ \ (i=0,1,\ldots,n).$$

We shall construct representation points $P_i(x_i,-\ln a_i)$ of the values of function $y=\hat{f}(x)$ in points x_i, $i=0,1,\ldots,n$, in the xy plane. From each point P_i, we shall draw a half-line in the negative direction of the Oy axis, perpendicular to the Ox axis.

The algorithm of the method consists of four steps. The first step is to check whether points x=a and x=b are the roots of function f(x). If point x=a is the root of function f(x), then we take i=0, x_i=a and proceed to the fourth step. Otherwise, proceed to the second step, at which we compute

$$\tilde{r}_k = \left(\frac{a_{k-1}}{a_k}\right)^{\frac{1}{h}},\ k=1,2,\ldots,n-1$$

until for some k=I the following conditions are satisfied

$$\tilde{r}_i \ge 1,\ \tilde{r}_{i+1} \le 1.$$

If, in this case, $|f(x_i)|<h$, then point x_i with accuracy h is taken as zero of function f(x) and we proceed to the fourth step. If condition $|f(x_i)|<h$ is not met, then we proceed to the third step.

At the third step we compute

$$\tilde{r}_{i+l} = \frac{a_i}{a_{i+l}},\quad l=1,2,\ldots,n-i-1$$

until for some l the following conditions are satisfied

$$\tilde{r}_{i+l} \ge 1,\ \tilde{r}_{i+l+1} \le 1,\ |f(x_{i+l})|<h. \tag{3}$$

If conditions (1) are fulfilled, then point x_{i+l} is zero of function f(x) with accuracy h and we proceed to the fourth step. If the conditions (1) are not satisfied at any l, then function f(x) has no zeros, unless x=a and x=b.

At the fourth step, we select as a starting point the point x_i, found at some of the previous steps. $\tilde{r}_{i+k}$ is calculated by formula:

$$\tilde{r}_{i+k} = \frac{a_i}{a_{i+k}},\ k=1,2,\ldots n-i-1.$$

Then if

$$\left|1-\tilde{r}_{i+k}\right| \le h,\ \left|1-\tilde{r}_{i+k+1}\right| > h,$$

then point x_{i+l} is taken as the root of function f(x) with accuracy h.

Since the algorithm is employed once at each partition, then finding the zeros of function f(x) by the above algorithm must take O(n) time units.

Theorem 1. Let x^* is the first zero point of function f(x). If function f(x) is continuous and has consistent derivative in the vicinity of zero, then it is always possible to find this point with precision $\left|f(x)\right|<h$ by the above algorithm.

Example 1. Find zeros of the function:

$$f(x)=x^2=9. \tag{4}$$

$f(4)$ on the interval [–5; 5]. Function graph (4) is shown in Fig. 3.

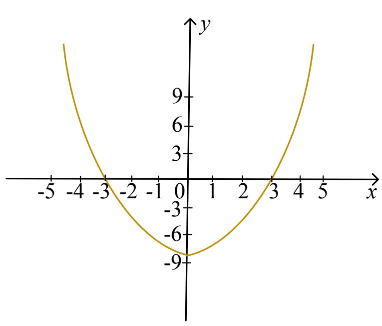

Fig. 3. Function graph (4)

Assume

$$h=0.01,\ [a,b]=[-5;5],$$

$$n=1000,\ x_i=a+ih,\ i=0,\ldots,1000.$$

First, we modify function (1). We have:

$$\hat{f}(x)=1+\left|f(x)\right|=1+\left|x^2-9\right|. \tag{5}$$

Construct representation points of function (4). Let

$$\hat{f}(x_i)=a_i\ \ (i=0,1,\ldots,1000).$$

We have

$$\tilde{r}_k = \left(\frac{a_{k-1}}{a_k}\right)^{\frac{1}{h}},\ k=1,2,\ldots,1000.$$

Fig. 4 shows function graph (5).

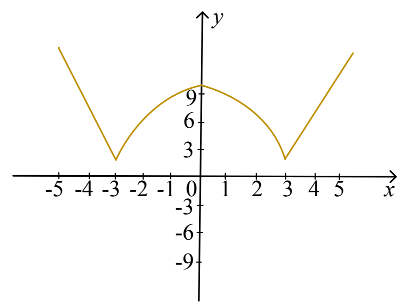

Fig. 4. Function graph (5)

At the first step of our algorithm we check whether points x=a or x=b are the roots of function f(x). We receive

$$f(a)=f(-5)=16,\ \ f(b)=f(5)=16.$$

Thus, points x=a and x=b are not the roots of function f(x).

At the second step, we receive:

$$\tilde{r}_1 = \left(\frac{a_0}{a_1}\right)^{\frac{1}{h}} = \left(\frac{17}{16.9001}\right)^{\frac{1}{0.01}} = 1.8029,$$

$$\tilde{r}_2 = \left(\frac{a_1}{a_2}\right)^{\frac{1}{h}} = \left(\frac{16.9001}{16.8004}\right)^{\frac{1}{0.01}} = 1.807.$$

Since $\tilde{r}_1>1, \tilde{r}_2>1$, then we compute $\tilde{r}_3,\tilde{r}_4,\ldots$ until for some l (l>1) the following conditions are fulfilled: $\tilde{r}_l \ge 1,\ \tilde{r}_{l+1} \le 1.$

We receive:

$$\tilde{r}_{200} = \left(\frac{a_{199}}{a_{200}}\right)^{\frac{1}{h}} = \left(\frac{1.0601}{1}\right)^{\frac{1}{0.01}} = 342.518,$$

$$\tilde{r}_{201} = \left(\frac{a_{200}}{a_{201}}\right)^{\frac{1}{h}} = \left(\frac{1}{1.0599}\right)^{\frac{1}{0.01}} = 0.00298.$$

Now we check condition $|f(x_{200})|<h.$ We obtain:

$$f(x_{200}) = (-3)^2 - 9 = 0 < 0.01.$$

Therefore, we received the first root of function f(x) at x=–3. Proceed to the fourth step. At the fourth step, we shall receive one more zero of function:

$$\tilde{r}_{800} = \frac{a_{200}}{a_{800}} = \frac{1}{1} = 1,$$

$$|1 - \tilde{r}_{800}| = 0 \le 0.01,$$

$$\tilde{r}_{801} = \frac{a_{200}}{a_{801}} = \frac{1}{1.0601} = 0.9433,$$

$$|1 - \tilde{r}_{801}| = 0.0567 > 0.01.$$

And, accordingly, it took O(1,000) units of time.

Example 2. One has to find zeros for the Chebyshev polynomial at n=5:

$$T_5(x) = 16x^5 - 20x^3 + 5x. \qquad (6)$$

We know that the Chebyshev polynomials of the n-th power have n roots on the interval [–1; 1]. That is, there should be 5 roots. Function graph (6) is shown in Fig. 5.

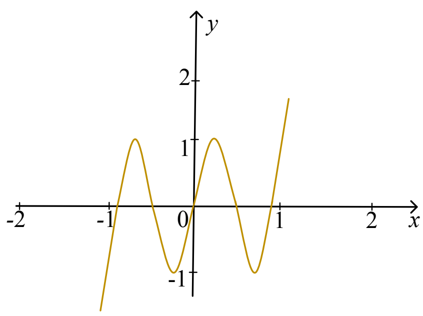

Fig. 5. Function graph (6)

Assume

$$h = 0.001, \; [a, b] = [-1; 1],$$

$$n = 2000, \; x_i = a + ih, \; i = 0, \ldots, 2000.$$

First, we shall modify function (6). We receive:

$$\hat{f}(x) = 1 + |f(x)| = 1 + |16x^5 - 20x^3 + 5x|. \qquad (7)$$

Construct representation points of function (7). Let

$$\hat{f}(x_i) = a_i \; (i = 0, 1, \ldots, 2000).$$

We receive:

$$\tilde{r}_k = \left(\frac{a_{k-1}}{a_k}\right)^{\frac{1}{h}}, \; k = 1, 2, \ldots, 2000.$$

Fig. 6 shows function graph (7).

At the first step of our algorithm we check whether points x=a or x=b are the roots of function f(x). We receive

$$f(a) = f(-1) = -1,$$

$$f(b) = f(1) = 1.$$

Thus, points x=a and x=b are not the roots of function f(x).

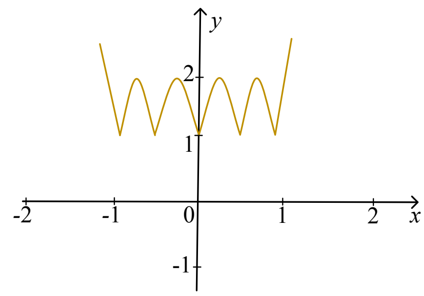

Fig. 6. Function graph (7)

At the second step, we receive:

$$\tilde{r}_1 = \left(\frac{a_0}{a_1}\right)^{\frac{1}{h}} = \left(\frac{2}{1.975}\right)^{\frac{1}{0.001}} = 290332.53,$$

$$\tilde{r}_2 = \left(\frac{a_1}{a_2}\right)^{\frac{1}{h}} = \left(\frac{1.975}{1.95}\right)^{\frac{1}{0.001}} = 340791.38.$$

Since

$$\tilde{r}_1 > 1, \tilde{r}_2 > 1,$$

then we compute $\tilde{r}_3, \tilde{r}_4, \ldots$ until for some l (l>1) the following conditions are fulfilled

$$\tilde{r}_l \ge 1, \; \tilde{r}_{l+1} \le 1.$$

We receive:

$$\tilde{r}_{49} = \left(\frac{a_{48}}{a_{49}}\right)^{\frac{1}{h}} = \left(\frac{1.0153}{1.0009}\right)^{\frac{1}{0.001}} = 1598442.19,$$

$$\tilde{r}_{50} = \left(\frac{a_{49}}{a_{50}}\right)^{\frac{1}{h}} = \left(\frac{1.0009}{1.017}\right)^{\frac{1}{0.001}} = 0.0000001.$$

Check now the condition $|f(x_{49})| < h$. We receive:

$$f(x_{49}) = 16(-0.951)^5 - 20(-0.951)^3 +$$
$$+5(-0.951) = 0.0009 < 0.001.$$

Therefore, we received the first root of function f(x) at $x = -0.951$. Proceed to the fourth step. At step 4, we obtain four more zeros of the functions, which are given in Table 1.

Table 1

The roots of function (5) after the fourth step

$i+k$	x_{i+k}	$\tilde{r}_{i+k}$	$\tilde{r}_{i+k+1}$
412	–0.588	0.99957	0.0000001
1000	0	1.0009	0.9959
1588	0.588	0.99958	1.0
1951	0.951	1	0.9858

And, accordingly, it took O(2000) units of time.

The advantages of the method constructed are:

– no need to first isolate the roots, then refine them, as in the classical methods;

– a function may be both smooth and nonsmooth, both discontinuous and discrete;

– simplicity and visual representation of the method.

As one can see, the method does not require additional information about location of the roots, thus it can gain a widespread application in numerical analysis. Nevertheless, it is necessary to note an obvious shortcoming of the method that is associated with the need to run many steps, the number of which depends on the accuracy required. A direction for further research may involve solving the systems of nonlinear equations, as well as solving a Cauchy problem using the apparatus of nonclassical Newton's minorants of functions given in the tabular form.

6. Conclusions

1. We constructed a numerical method for finding the zeros of both smooth and nonsmooth functions. Underlying the method is the so-called apparatus of nonclassical Newton's minorant and diagrams of functions, given in the tabular form. This method does not require the isolation of roots, in contrast to other well-known methods of finding the zeroes of functions. We also managed to overcome a problem of the growth of equation coefficients, typical for methods of the Lobachevsky-Greffe type, since here we do not employ the process of taking the square roots.

2. The examples are given of finding the zeros of functions, as well as the estimation of speed to find these zeros. As is shown, the speed depends on the accuracy required when finding them. However, by using modern computers, it will not be a problem. Therefore, the method constructed might gain a widespread application in the problems on finding the zeros of functions.

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