Розглянута транспортна задача лінійного програмування з проміжними центрами, коли значення пропускної здатності проміжних центрів не задані. Поставлена задача розрахунку раціонального розподілу сумарної пропускної здатності з метою мінімізації середнього значення транспортних витрат. Запропоновані ефективні методи розв'язання цієї задачі

Ключові слова: транспортна задача с проміжними центрами, розрахунок пропускної здатності проміжних центрів

Рассмотрена транспортная задача линейного программирования с промежуточными центрами, когда значения пропускных способностей промежуточных центров не заданы. Поставлена задача расчета рационального распределения суммарной пропускной способности с целью минимизации среднего значения транспортных расходов. Предложены эффективные методы решения этой задачи

Ключевые слова: транспортная задача с промежуточными центрами, расчёт пропускных способностей промежуточных центров

UDC 517.8
DOI: 10.15587/1729-4061.2017.103950

CALCULATION OF THROUGHPUTS OF INTERMEDIATE CENTERS IN THREE-INDEX TRANSPORTATION PROBLEMS

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1. Introduction

In practice, a problem often arises with drawing up a plan for transporting homogeneous productы from production centers to consumption centers using vehicles of various types. Implementation of this plan would ensure minimum transportation costs. It is clear that it is impossible to solve such a problem based on the classical transport theory since in the case under consideration the cost of transporting a unit of product depends not only on the mutual location of production and consumption centers but also on the type of transport. Also, additional restrictions on the quantity of the product transported by vehicles of a given type are added to the usual transport constraints.

This problem, as shown in [1], is a three-index transport problem and is solved by the known method of potentials [1, 2]. On the other hand, when planning transportation, a situation occurs when the products are delivered to consumers not directly but through intermediate centers (warehouses, secondary processing enterprises, etc.). The problem of optimizing the transportation plan corresponding to this situation is called a transportation problem with intermediate centers. This problem, apparently, was first formulated in [3] as a general transport-type problem Then it was redefined as a multi-index transportation problem in [4], and finally, it was formulated as a transportation problem with intermediate stations in [5]. Necessity of taking into account intermediate centers and possible differences in assignment of their throughputs make the task nontrivial. In this case, nature of distribution of the overall throughput of the system of intermediate centers significantly affects magnitude of total transportation costs. This circumstance determines urgency of the problem.

2. Literature review and problem statement

Let us consider the known approaches to this problem solution. The model of this problem has the following form.

Introduce the following notation:

$i \in I = \{1, 2, ..., m\}$: the numbers of product manufacturers;

$j \in J = \{1, 2, ..., n\}$: the numbers of product consumers;

$k \in K = \{1, 2, ..., r\}$: the numbers of intermediate centers;

C_{ik}: the average cost of transporting a unit of product from the i-th producer to the k-th intermediate center;

C_{jk}: the average cost of transporting a unit of product from the k-th intermediate center to the j-th consumption center;

X_{ik}: the planned volume of transportation from the i-th producer to the k-th intermediate center;

Xkj: the planned volume of transportation from the k-th intermediate center to the j-th consumption center;

a_i: the total volume of products intended for transportation from the i-th production center, $i \in I$;

d_k: the permissible volume of transportation through the k-th intermediate center, $k \in K$;

b_j: demand for products at j-th consumption center, $j \in J$.

In this notation, the problem of rational organization of transportation is formulated as follows: find sets $\{X_{ik}\}$, $\{X_{kj}\}$ minimizing the total average cost of transportation

$$R=\sum_{i=1}^{m}\sum_{k=1}^{r}C_{ik}X_{ik}+\sum_{k=1}^{r}\sum_{j=1}^{n}C_{kj}X_{kj} \quad (1)$$

and satisfying the following constraints:

$$\sum_{k=1}^{r}X_{ik}=a_i,\ \ X_{ik}\ge 0,\ \ i\in I,\ \ k\in K; \quad (2)$$

$$\sum_{k=1}^{r}X_{kj}=b_j, X_{kj}\ge 0, j\in J, k\in K; \quad (3)$$

$$\sum_{i=1}^{m}X_{ik}=\sum_{j=1}^{n}X_{kj}\le d_k, k\in K. \quad (4)$$

Fulfillment of conditions (2)–(4) ensures taking out of all products, full satisfaction of consumer demand and absence of accumulation of products, which were not taken out from intermediate centers.

Constraints (4) are system-organizing constraints. At the same time, if the volume of traffic through intermediate centers is not limited, then, as it will be shown, the formulated problem is reduced to a three-index transportation problem. On the other hand, if the set $\{d_k\}$ is given, and

$$\sum_{i=1}^{m}a_i=\sum_{k=1}^{r}d_k=\sum_{j=1}^{n}b_j=d, \quad (5)$$

then the initial problem is divided into two independent subproblems:

a) find a set $\{X_{ik}\}$, minimizing

$$R_1=\sum_{i=1}^{m}\sum_{k=1}^{r}C_{ik}X_{ik} \quad (6)$$

and satisfying constraints (2) and also

$$\sum_{i=1}^{m}X_{ik}=d_k,\ \ k=1,2,...,r; \quad (7)$$

b) find a set $\{X_{kj}\}$, minimizing

$$R_2=\sum_{k=1}^{r}\sum_{j=1}^{n}C_{kj}X_{kj} \quad (8)$$

and satisfying constraints (3) and also

$$\sum_{j=1}^{n}X_{kj}=d_k,\ \ j=1,2,...,n. \quad (9)$$

The canonical formulation of problem (1)–(5) leads to the three-index problem in the following way [1], [4–6]. Introduce

X_{ikj}: the planned volume of traffic from the i-th producer to the j-th consumption center through the k-th intermediate center;

$C_{ikj}=C_{ik}+C_{kj}$: the total average cost of transporting a unit of product from the i-th producer to the j-th consumption center thru the k-th intermediate center.

In this case, the problem reduces to finding a set $\{X_{ikj}\}$, minimizing

$$R=\sum_{i=1}^{m}\sum_{k=1}^{r}\sum_{j=1}^{n}C_{ikj}X_{ikj} \quad (10)$$

and satisfying constraints:

$$\sum_{k=1}^{r}\sum_{j=1}^{n}X_{ikj}=a_i,\ \ i\in I, \quad (11)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}X_{ikj}\le d_k,\ \ k\in K, \quad (12)$$

$$\sum_{i=1}^{m}\sum_{k=1}^{r}X_{ikj}=b_j,\ \ j\in J, \quad (13)$$

$$X_{ikj}\ge 0,\ \ i\in I,\ \ k\in K,\ \ \ j\in J. \quad (14)$$

The obtained problem is a three-index triplanar transportation problem.

For the exact solution of the obtained problem (10)–(14), the method of potentials [1, 2] is used. The solution process consists of a preliminary stage and a finite number of similar iterations constituting the main stage of the algorithm work [1, 7–10]. At the preliminary stage, initial reference plan is determined. One of the known methods is used, for example, the method of sequential distribution. Each iteration of the main stage consists of two steps. At the first step, optimality of the reference plan obtained in the previous iteration (or at the preliminary stage) is determined. If the plan is optimal, then the solution process ends. Otherwise, the second iteration step is performed in which a new reference plan is obtained which is closer to the optimal one.

The main drawback of this method is that the complexity of the computational procedure for solving triplanar problems by this method grows very rapidly with increase in the problem dimensionality. For real values of m, n, r, the number of variables in the problem is of the order 10^6 and the number of constraints is $(10^2–10^3)$. In this case, the problem dimensionality determined by the product of the number of variables by the number of constraints is of the order $10^8–10^9$.

Approximate methods for solving this problem are described in [1, 10–14]. The need to develop fairly simple approximate methods for solving transportation problems is due to the following circumstances. First, numerical values of the parameters of real transportation problems, as a rule, are not accurate due to random errors in their measurement and estimation and due to the possible inadequacy of the linear model. Besides, the model inaccuracies can result from the influence of unaccounted factors. In a situation where initial data of the problem are random variables, the transportation plan obtained as a result of an exact solution for these data may appear to be not optimal. Secondly, laboriousness of solving multidimensional transportation problems by exact methods grows very rapidly with increase in their dimensionality. In this case, dimensionality of many real problems is so great that it is impossible to obtain their exact solution in an acceptable time even with the use of modern computing machinery.

Note that all of the above should not be understood in such a way that approximate methods are intended just for manual problem solution. An effective approximate algorithm, as a rule, allows one to obtain a reference plan close to optimal one and therefore can be used at the initial stage of work of the exact algorithm in machine solution significantly reducing total time for obtaining the result.

Approximate methods of solving the triplanar transport problem considered in [1] are based on an effective choice of the leading index element in the transportation costs matrix. The simplest of them is the method of the minimal element in the section. The method is a modification of the method of sequential distribution to construct an initial reference plan. In this case, the section in which the leading element is sought can be one-dimensional (the method of the minimum element in a row, column), two-dimensional or even three-dimensional.

In the well-known formulations of the transportation problem with intermediate centers, two variants of formulation of constraints (4) are considered for which constraints (5) are fulfilled [9–13] or not fulfilled [14–16]. A natural complicating generalization of the original problem arises if the set $\{d_k\}$ was not specified but its components must satisfy the constraints (5) for all admissible plans $\{X_{ikj}\}$.

3. The aim and objectives of the study

The study objective was to develop a method for solving a transportation problem with intermediate centers with unspecified throughputs.

In accordance with this objective, the study tasks were formulated:

– development of a multi-step iterative procedure for calculating throughputs of intermediate centers;

– development of a one-step procedure for direct calculation of throughputs.

4. Materials and methods of research. Development of a method for calculating throughput of intermediate centers

Let values of the components of the sets $\{a_i\}$, $\{b_j\}$, $\{C_{ik}\}$, $\{C_{kj}\}$, $i\in I$, $j\in J$, $k\in K$ are specified. It is required to find a set $\{d_k\}$, $k\in K$, the use of which in task (6)–(9) provides a minimum average transportation cost. The possible approach to finding a set is as follows.

Introduce an arbitrary set $\{d_1, d_2,..., d_r\}$ of admissible volume values for intermediate centers with their components satisfying the constraints

$$\sum_{k=1}^{r} d_k = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \tag{15}$$

$$d_k \geq 0, \quad k = 1,2,...,r. \tag{16}$$

Next, solve the problem of finding the transportation plans in the statement (6)–(9). Let $\{X^*_{ik}\}$, $\{X^*_{kj}\}$ is the corresponding optimal plan which corresponds to the value of the total cost of transportation equal to

$$R^* = \sum_{i=1}^{m}\sum_{k=1}^{r} C_{ik}X^*_{ik} + \sum_{k=1}^{r}\sum_{j=1}^{n} C_{kj}X^*_{kj}. \tag{17}$$

It is clear that by enumerating for $\{d_1, d_2,..., d_r\}$ and taking into account (15), (16), one can find a set of admissible volumes of intermediate centers for which value R^* will be minimal.

This path is ineffective since the number of enumerations increases rapidly with increase in the number of intermediate centers and tightening of the requirements for accuracy of the solution obtained. In this connection, consider an optimization procedure that constructively leads to the desired set $\{d_k\}$.

Use the Nelder-Mead method. Introduce matrix I $\dim I = r\times(r+1)$, of the form:

$$I = \begin{pmatrix} d_0 & \hat{d}_1 & \hat{d}_2 & \dots & \hat{d}_2 \\ d_0 & \hat{d}_2 & \hat{d}_1 & \dots & d_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d_0 & \hat{d}_2 & d_2 & \dots & \hat{d}_1 \end{pmatrix}.$$

A certain admissible distribution of the volume values $\{d_k^{(j)}\}$, $j=1, 2,..., r+1$, for the intermediate centers corresponds to each column of this matrix. Solutions of a pair of transportation problems: T_1 – {(6), (2), (7)} and T_2 – {(8), (3), (9)}, determining the value of the total cost of transportation (17) correspond to this distribution. Specify initial distributions as follows:

$$d_0 = \frac{d}{r},$$

$$\hat{d}_1 = \frac{d}{r\sqrt{r+1}}\left(\sqrt{r+1}+r-1\right), \quad \hat{d}_2 = \frac{d}{r\sqrt{d+1}}\left(\sqrt{r+1}-1\right).$$

Next, assign coordinates of some point in a r-dimensional quotient space to the elements of each column of matrix I. These points define vertices of a convex polyhedron.

Parameters d_0, d_1, d_2 are chosen so that the sum of the admissible volume values corresponding to each column of the matrix I is equal to the total volume of the transported product and the distances between the polyhedron vertices are equal to each other.

Indeed,

$$\hat{d}_1 + (r-1)\,\hat{d}_2 = \frac{d}{r(r+1)}\left[\sqrt{r+1}+r-1+(r-1)\sqrt{r+1}-(r-1)\right] =$$

$$= \frac{dr\sqrt{r+1}}{r(r+1)} = d,$$

$$R\left(\left\{d_k^{(j_1)}\right\},\left\{d_k^{(j_2)}\right\}\right) = \sqrt{2}\left(\hat{d}_1-\hat{d}_2\right) = \frac{d}{\sqrt{2}\sqrt{r+1}},$$

$$j_1 = 1,2,...,r, \quad j_2 = 2,3,...,r+1.$$

The optimized objective function of the problem is determined by expression (17) the value of which is calculated from the results obtained in solving problems (6), (2), (7) and (8), (3), (9).

Block diagram of the problem solution is shown in Fig. 1.

The problem solution results in a set $\{d^*_k\}$, Its use ensures obtaining of optimal plans of transportation from producers to intermediate centers (plan $\{X^*_{ik}\}$) and from intermediate centers to consumers (plan $\{X^*_{kj}\}$).

The proposed technique for solving the problem of rational organization of transportation from suppliers to consumers through a system of intermediate centers can be effectively used in problems of low dimensionality. The central part of this technique is the Nelder-Mead algorithm which includes an independent solution of 2(r+1) transport problems (r is the number of intermediate centers) in each step. This circumstance combined with the extremely slow convergence of the optimization procedure, which is inherent to the Nelder-Mead method leads to a catastrophically rapid increase in total time for solving the problem.

Reduction of this time can be achieved using the approach based on the theory of duality. As in any linear programming problem, dual problems correspond to the above problems T_1 – {(6), (2), (7)} and T_2 – {(8), (3), (9)} [2, 17].

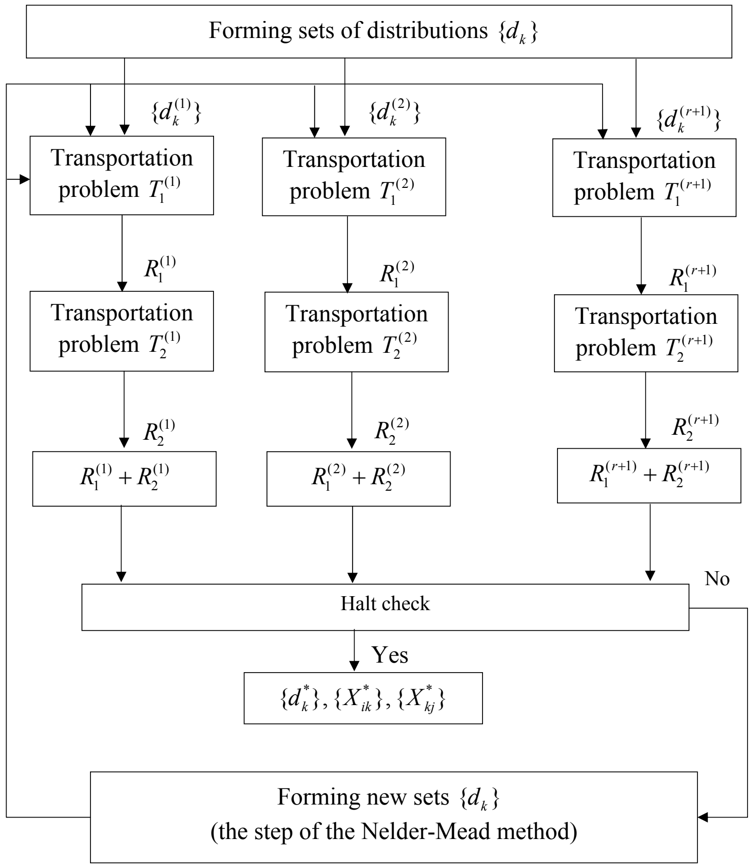

Fig. 1. Block diagram of the problem solution

In this case, the problem conjugate to T_1 has the following form: find sets $\{U_i\}$, $\{V_k\}$, i=1, 2,..., m, k=1, 2,..., r, that maximize the function

$$L_1(U,V)=\sum_{i=1}^{m}a_iU_i+\sum_{k=1}^{r}d_kV_k \tag{18}$$

and satisfy constraints

$$U_i+V_k\le C_{ik},\quad i=1,2,..,m,\ k=1,2,...,r. \tag{19}$$

The following problem corresponds to the problem T_2: find sets $\{Z_k\}$, $\{W_j\}$, k=1, 2,..., r, j=1, 2,..., n, that maximize the function

$$L_2(Z,W)=\sum_{k=1}^{r}d_kZ_k+\sum_{j=1}^{n}b_jW_j \tag{20}$$

and satisfy the constraints

$$Z_k+W_j\le C_{kj},\quad k=1,2,..,r,\quad j=1,2,...,n. \tag{21}$$

The dual problems (18), (19) and (20), (21) always have a solution. Moreover, if $X_1^*=\{x_{ik}^*\}$, $\{U_i^*\}$, $\{V_k^*\}$ are the optimal plans of problems (6), (2), (7) and (18), (19), then

$$\sum_{i=1}^{m}C_{ik}x_{ik}^*=\sum_{i=1}^{m}a_iU_i^*+\sum_{k=1}^{r}d_kV_k^*. \tag{22}$$

Similarly, if $X_2^*=\{x_{kj}^*\}$, $\{Z_k^*\}$, $\{W_j^*\}$ are the optimal plans of problems (8), (3), (9) and (20), (21), then

$$\sum_{k=1}^{r}\sum_{j=1}^{n}C_{kj}x_{kj}^*=\sum_{k=1}^{r}d_kZ_k^*+\sum_{j=1}^{n}b_jW_j^*. \tag{23}$$

Besides, the following correlations are fulfilled for the optimal plans of the formulated dual problems,

$$U_i^*+V_k^*\le C_{ik},\quad i=1,2,..,m,\quad k=1,2,...,r, \tag{24}$$

$$U_i^*+V_k^*=C_{ij},\ \text{if}\ x_{ik}^*>0, \tag{25}$$

$$Z_k^*+W_j^*\le C_{kj},\quad k=1,2,..,r,\quad j=1,2,...,n, \tag{26}$$

$$Z_k^*+W_j^*=C_{kj},\ \text{if}\ x_{kj}^*>0. \tag{27}$$

It follows that solutions of direct problems can be obtained from solution of dual problems. Indeed, let $\{U_i^*\}$, $\{V_k^*\}$ be solution of the dual problem. Substitute this solution in (24) and find sets $I=\{i^*\}$ and $K=\{k^*\}$, for which inequalities (24) are fulfilled as equalities. Then, in accordance with (25), these sets uniquely determine the set $\left\{X_{i^*k^*}\right\}$ the components of which are positive and determine solution of the direct problem (6), (2), (7). A similar operation is performed with a set $\{Z_k^*\}$, $\{W_j^*\}$.

The iterative procedure for finding D vector using the Nelder-Mead method is organized as follows. An array of sets $\{d_k^{(1)}\},\{d_k^{(2)}\},...,\{d_k^{(r+1)}\}$ is formed each of which contains r components, A pair of problems (6), (2), (7) and (8), (3), (9) is solved for each set $\{d_k^{(s)}\}$.

Effectiveness of the plan corresponding to the set $\{d_k^{(s)}\}$ is estimated by the sum

$$R^{(s)}=\sum_{i=1}^{m}a_iU_i+\sum_{k=1}^{r}d_k^{(s)}V_k+\sum_{k=1}^{r}d_k^{(s)}Z_k+\sum_{j=1}^{n}b_jW_j.$$

Next, the step of the Nelder-Meade procedure is performed.

This option is better than the previous one for the following reasons.

1. Dimensionalities of the dual problems (18)–(19) and (20)–(21) are smaller than those of the corresponding direct problems with respect to the number of variables, e. g. m+k<mk, k+n<kn.

2. Problems (18)–(19) and (20)–(21) are one-index unlike the two-index problems (6), (2), (7) and (8), (3), (9).

3. The techniques for solving dual problems are simpler than the corresponding techniques for direct problems since they are solved by the ordinary simplex method in one stage. Direct problems are structurally more complex since they contain an unpredictable number of iterations, each iteration solving the problem of forming a plan and its improvement.

Finally, consider another, seemingly most effective, approach to solving the problem of finding a rational set of throughputs of intermediate centers.

The idea of the proposed technique is as follows. The cheapest route (i–k–j) is sought for each pair (i-th supplier – j-th consumer). As a result, an ordinary two-index transportation problem of delivering products from the supplier system to the consumer system arises. Solution of this problem contains an optimal set of transportation volumes in this system. Now it only remains to totalize for each intermediate center the volumes of supplies passing through this center.

Describe the procedure for obtaining solution in more detail.

Step: There are the following matrices

$C_1=(c_{ik})$: the cost of transporting a product unit from suppliers to intermediate centers,

$C_2=(c_{kj})$: the cost of transporting a product unit from the intermediate centers to consumers.

To find routes requiring minimum delivery costs, introduce a special operation of matrix "multiplication". Perform operation of "multiplication" for two square matrices A and B of the same dimensionality applying the rule:

$$C = A \otimes B,$$

$$c_{ij} = \min(c_{i1}+c_{1j}, c_{i2}+c_{2j}, \ldots, c_{in}+c_{nj}) = c_{ik_{ij}j} = \hat{c}_{ij}. \qquad (28)$$

Here

$$k_{ij} = \arg\min_k \{c_{ik}+c_{kj}\}. \qquad (29)$$

$\hat{c}_{ij}$: minimum cost of delivery of a product unit from the i-th supplier to the j-th consumer.

To each pair (i, j), correlation (29) assigns number of the k_{ij}, station thru which it is expedient to make deliveries from i to j. This correspondence is defined by formula $S(ij)=k_{ij}$.

As a result, usual two-index transportation problem is obtained: find a set $X=(x_{ij})$, that minimizes the total transportation cost

$$L(X) = \sum_{i=1}^{m}\sum_{j=1}^{n} \hat{c}_{ij} x_{ij} \qquad (30)$$

and satisfies the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,2,\ldots,m, \qquad (31)$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,2,\ldots,n, \qquad (32)$$

$$x_{ij} \ge 0, \quad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n. \qquad (33)$$

Let $X^* = (x^*_{ij})$ be solution of problem (30)–(33).

Next, introduce the following set for each intermediate center k:

$$M_k = \left\{(i,j) : x^*_{ij} > 0, S(i,j) = k_{ij}\right\}. \qquad (34)$$

Then, for each k_{ij}, calculate the total volume of cargo transported through this intermediate center when implementing the plan X*:

$$d_k = \sum_{(i,j)\in M_k} x^*_{ij}, \quad k = 1,2,\ldots,r. \qquad (35)$$

The resulting set determines rational values of throughputs of the intermediate centers.

Example.

Consider a system consisting of two production centers, three of consumption centers and three intermediate centers.

Set $a_1 = 20,\ a_2 = 30,\ b_1 = 16,\ b_2 = 24,\ b_3 = 10.$

The balance condition is fulfilled:

$$\sum_{i=1}^{2} a_i = \sum_{j=1}^{3} b_j = 50.$$

Diagrammatic representation of the corresponding transportation system together with intermediate centers is shown in Fig. 2.

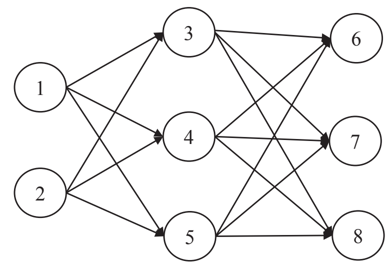

Fig. 2. Graphic model of a transportation system

Introduce matrix of costs of transportation from suppliers to intermediate centers and further to consumers:

C =

	1	2	3	4	5	6	7	8
1	0	∞	3	6	4	∞	∞	∞
2	∞	0	5	3	8	∞	∞	∞
3	∞	∞	0	∞	∞	6	4	5
4	∞	∞	∞	0	∞	2	7	6
5	∞	∞	∞	∞	0	6	5	2
6	∞	∞	∞	∞	∞	0	∞	∞
7	∞	∞	∞	∞	∞	∞	0	∞
8	∞	∞	∞	∞	∞	∞	∞	0

In this matrix, station pairs marked with symbol ∞, are the stations for which direct transition is impossible.

Now, use (28) to calculate matrix $\hat{C} = (\hat{c}_{ij})$ of minimum costs of transportation from suppliers to consumers:

$\hat{C} = C \otimes C =$

	1	2	3	4	5	6	7	8
1	0	∞	3	6	4	∞	∞	∞
2	∞	0	5	3	8	∞	∞	∞
3	∞	∞	0	∞	∞	6	4	5
4	∞	∞	∞	0	∞	2	7	6
5	∞	∞	∞	∞	0	6	5	2
6	∞	∞	∞	∞	∞	0	∞	∞
7	∞	∞	∞	∞	∞	∞	0	∞
8	∞	∞	∞	∞	∞	∞	∞	0

$\otimes$

	1	2	3	4	5	6	7	8
1	0	∞	3	6	4	∞	∞	∞
2	∞	0	5	3	8	∞	∞	∞
3	∞	∞	0	∞	∞	6	4	5
4	∞	∞	∞	0	∞	2	7	6
5	∞	∞	∞	∞	0	6	5	2
6	∞	∞	∞	∞	∞	0	∞	∞
7	∞	∞	∞	∞	∞	∞	0	∞
8	∞	∞	∞	∞	∞	∞	∞	0

=

	1	2	3	4	5	6	7	8
1	0	∞	3	6	4	$_4 8$	$_3 7$	$_5 6$
2	∞	0	5	3	8	$_4 5$	$_3 9$	$_4 9$
3	∞	∞	0	∞	∞	6	4	5
4	∞	∞	∞	0	∞	2	7	4
5	∞	∞	∞	∞	0	6	5	2
6	∞	∞	∞	∞	∞	0	∞	∞
7	∞	∞	∞	∞	∞	∞	0	∞
8	∞	∞	∞	∞	∞	∞	∞	0

,

$$k_{16}=4,\ k_{17}=3,\ k_{18}=5,$$
$$k_{26}=4,\ k_{27}=3,\ k_{28}=4.$$

This matrix elements determining minimum cost of transporting a unit of cargo from suppliers to consumers also contain (in the bottom left corner) the number of the intermediate center thru which corresponding best route passes.

The following is two examples of calculating the matrix elements:

$$\begin{aligned} c_{16} &= \min\{c_{11}+c_{16}, c_{12}+c_{26}, c_{13}+c_{36}, c_{14}+ \\ &+c_{46}, c_{15}+c_{56}, c_{16}+c_{66}, c_{17}+c_{76}, c_{18}+c_{86}\} = \\ &= \min\{\infty,\infty,3+6,6+2,4+6,\infty,\infty\} = \\ &= \min\{9,8,10\} = 8,\ k_{16}=4, \end{aligned}$$

$$\begin{aligned} c_{17} &= \min\{c_{11}+c_{17}, c_{12}+c_{27}, c_{13}+c_{37}, c_{14}+ \\ &+c_{47}, c_{15}+c_{57}, c_{16}+c_{67}, c_{17}+c_{77}, c_{18}+c_{87}\} = \\ &= \min\{\infty,\infty,3+4,6+7,4+5,\infty,\infty\} = \\ &= \min\{7,13,9\} = 7,\ k_{17}=3. \end{aligned}$$

In the obtained matrix $\hat{C}$, a submatrix C_0 containing minimum transportation costs from suppliers to consumers taking into account intermediate centers is of interest for further solution:

$$C_0 = \begin{array}{c|c|c|c|} & 6 & 7 & 8 \\ \hline 1 & {}_4 8 & {}_3 7 & {}_5 6 \\ \hline 2 & {}_4 5 & {}_3 9 & {}_4 9 \\ \hline \end{array}.$$

Now the problem is reduced to finding a set $X=(x_{ij})$, i=1, 2, j=6, 7, 8, minimizing

$$L(X)=\sum_{i=1}^{2}\sum_{j=6}^{8} C_{ij}x_{ij}$$

and satisfying the constraints

$$\sum_{j=6}^{8} x_{1j}=20,\quad \sum_{j=6}^{8} x_{2j}=30,$$

$$\sum_{i=1}^{2} x_{i6}=16,\quad \sum_{i=1}^{2} x_{i7}=24,\quad \sum_{i=1}^{2} x_{i8}=10.$$

Solution of this priblem:

$$X=\begin{pmatrix} 0 & 10 & 10 \\ 16 & 14 & 0 \end{pmatrix}.$$

Use this solution to calculate values of throughput of intermediate centers using formulas (34), (35).

$$M_3=\{(1,7),(2,7)\},$$

$$M_4=\{(2,6)\},$$

$$M_5=\{(1,8)\}.$$

$$d_3=\sum_{(i,j)\in M_3} x_{ij}=x_{17}+x_{27}=24,$$

$$d_4=\sum_{(i,j)\in M_4} x_{ij}=x_{26}=16,$$

$$d_5=\sum_{(i,j)\in M_5} x_{ij}=x_{18}=10.$$

Solution is over.

5. Discussion of results of development of calculation methods. Estimation of effectiveness of the proposed methods for distribution of the intermediate center throughputs

Thus, a transportation problem with a system of producers, consumers and intermediate centers the throughputs of which was not specified has been set and solved.

At the same time, in accordance with the study tasks, two alternative approaches were proposed for constructing a procedure for calculating throughputs of intermediate centers. The first approach uses the iterative Nelder-Mead method. A set of ordinary two-index transportation problems is solved in each step of this method. The second approach is based on a preliminary finding of the shortest routes for each supplier-consumer pair. Realization of this idea allowed authors to reduce this problem to a common two-index problem.

Comparison of effectiveness of the proposed methods for calculating rational values of throughput of the intermediate centers convincingly demonstrates advantages of the second approach. It is easier to implement and it has significantly faster performance. However, its most important advantage is the possibility of extending this approach to the case where the system of intermediate centers contains not one but several intermediate layers. The obtained results can be directly used in the practice of large enterprises implementing a closed cycle of the production process (production of products – delivery of products to points of sale – consumption of products).

The expedient direction of further studies consists in taking into account real uncertainty in estimating transportation costs. This problem was first formulated in [18] and almost simultaneously in [19] for a simple case when the random cost values are distributed according to the normal law. More adequate possibilities for specifying uncertainty are associated with the use of fuzzy mathematics [20, 21] as well as imprecise mathematics [22, 23]. Finally, another variant of uncertainty arises if the random costs of transportation are specified by the values of their mathematical expectations and variances. The way to overcome the problem in this case is to find the worst density of the cost distribution [24] and a minimax solution of the problem.

6. Conclusions

1. An iterative procedure was proposed for calculating rational values of throughput of intermediate centers. The method provides an accurate solution of the problem which minimizes total costs. A feature of this procedure is application of the decomposition principle which allows the original three-index problem to be reduced to a set of ordinary two-index problems.

2. A method has been developed for direct calculation of distribution of throughput. It is based on a preliminary search for the shortest routes for each supplier-consumer pair. The method can be used to solve transportation problems with several layers of intermediate centers.

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