29. Pawlak, Z. Rough sets [Text] / Z. Pawlak // International Journal of Computer & Information Sciences. – 1982. – Vol. 11, Issue 5. – P. 341–356. doi: 10.1007/bf01001956
30. Raskin, L. Fuzzy models of rough mathematics [Text] / L. Raskin, O. Sira // Eastern-European Journal of Enterprise Technologies. – 2016. – Vol. 6, Issue 4 (84). – P. 53–60. doi: 10.15587/1729-4061.2016.86739
31. Raskin, L. G. Prikladnoe kontinual'noe lineynoe programmirovanie [Text] / L. G. Raskin, I. O. Kirichenko, O. V. Seraya. – Kharkiv, 2013. – 293 p.

UDC 519.6
DOI: 10.15587/1729-4061.2017.108327

CONSTRUCTION OF INTERPOLATION METHOD FOR NUMERICAL SOLUTION OF THE CAUCHY'S PROBLEM

R. Bihun
Postgraduate student*
E-mail: bigunroman@ukr.net
G. Tsehelyk
Doctor of Physical and Mathematical Sciences, Professor, Head of Department*
E-mail: kafmmsep@lnu.edu.ua
*Department of Mathematical modeling of socio-economic processes
Ivan Franko National University of Lviv
Universytetska str., 1, Lviv, Ukraine, 79000

Побудовано інтерполяційний чисельний метод розв'язування задачі Коші для звичайних диференціальних рівнянь першого порядку за допомогою апарату некласичних мінорант та діаграм Ньютона функцій, заданих таблично. Цей метод дає точніші результати від методу Ейлера у випадку опуклої функції. Доведено обчислювальну стійкість методу, тобто похибка початкових даних не нагромаджується. Також показано, що метод має другий порядок точності

Ключові слова: міноранта Ньютона, диференціальні рівняння, задача Коші, діаграма Ньютон, опукла функція

Построен интерполяционный численный метод решения задачи Коши для обыкновенных дифференциальных уравнений первого порядка с помощью аппарата неклассических минорант и диаграмм Ньютона функций, заданных таблично. Этот метод дает более точные результаты по сравнению с методом Эйлера в случае выпуклой функции. Доказана вычислительная устойчивость метода, то есть погрешность начальных данных не накапливается. Также показано, что метод имеет второй порядок точности

Ключевые слова: минорант Ньютона, дифференциальные уравнения, задача Коши, диаграмма, выпуклая функция

1. Introduction

Cauchy's problem is one of the main problems in the theory of differential equations, which comes down to finding a solution (integral) of the differential equation that satisfies initial conditions (original data).

Over many years, a numerical solution of the Cauchy's problem has been the focus of attention by scientists as it is widely used in different areas of science and technology. That is why there are a large number of developed methods for it. In spite of this, however, new methods are being devised, some of them with better properties than those preceding.

Cauchy's problem usually emerges during analysis of the processes predetermined by the differential law and original state. Mathematical notation of such equations is an equation and the initial condition.

The difference between the boundary-value problems and the Cauchy's problem is that the region over which the desired solution should be determined is not specified in the latter in advance. However, the Cauchy's problem can be considered as one of the boundary-value problems.

2. Literature review and problem statement

Numerical methods of the Cauchy's problem solution are divided into 3 groups [1]:

– one-point;
– multipoint (methods of prediction and correction);
– methods with automatic choice of integration step.

The one-point methods include methods that have certain common features, such as:

1. Underlying all one-point methods is the function decomposition into Taylor's series, which preserves members that have h in a power to k inclusive. An integer k is called the order of the method. Error on a step has an order of $k+1$.

2. All one-point methods do not require a valid computation of derivatives, because only the function itself is calculated, however, one may require its values in some intermediate points. This entails, of course, additional cost of time and effort.

3. In order to receive information in a new point, it is necessary to have data only from the previous point. This property can be called "self-starting". A capability to "self-start" makes it possible to easily change the magnitude of step h.

4. Compared with the one-point methods, *the methods of prediction and correction* possess a number of special features [2].

1) To implement the methods of prediction and correction, it is necessary to have information about several of the previous points (they do not belong to the "self-starting" methods), which is why, in order to obtain additional information, it is necessary to apply the one-point method. If in the process of solving differential equations by the method of prediction and correction the step changes, then one has typically to switch over temporarily to the one-point method.

2) One-point methods and methods of prediction and correction provide approximately the same accuracy of the results. However, the latter, in contrast to the former, make it possible to estimate only an error in a step. For this reason, when employing the one-point methods, the magnitude of step h is typically chosen slightly smaller than it is required, which is why the methods of prediction and correction prove to be the most effective.

3) When applying the Runge-Kutta method [2] of the fourth-order accuracy, at each step one has to calculate four values of the functions, but for the convergence of the method of prediction and correction of the same order of accuracy, it is often sufficient to have two values of the function. That is why methods of prediction and correction require almost twice less computing time than the Runge-Kutta methods of comparable accuracy.

To solve differential equation $y' = f(x,y)$ by a numerical method means to find for the assigned sequence of arguments $x_0, x_1, \ldots, x_n$ and y_0 such values of $y_0, y_1, \ldots, y_n$ so that $y_i = F(x_i)$ $(i = 1, 2, \ldots, n)$ and $F(x_0) = y_0$. Thus, the numerical methods make it possible, instead of deriving function $y = F(x)$, to receive a table of the values of the given function for the assigned sequence of arguments. The magnitude $h = x_k - x_{k-1}$ is called *a step of integration*.

Graphically, numerical solution [3] is a sequence of short straight-line segments, by which analytical solution $y = F(x)$ of the equation is approximated (a piecewise-linear approximation).

There are methods of differential transformation (MDT) of solution to the Cauchy's problem. Basic definitions and fundamental theorems of a one-dimensional MDT and its suitability for different types of differential and integral-differential equations are given in [4].

A reliable, yet very simple, numerical method to solve different cases of a singular Cauchy-type integral equation is developed in [5]. For this purpose, first Bernstein polynomials are derived, which are used to approximate a solution of the given singular integral equation. This, however, leads to solving the system of linear algebraic equations (SLAE), which sometimes is difficult to resolve.

Article [6] examines numerical solution of the class of systems of singular integral Cauchy's equations with constant coefficients. The proposed procedure consists of two main stages: the first is to consider a modified problem, equivalent to the original under appropriate conditions, the second is to bring its solution using a vector of polynomial functions. But the solution comes down to solving the linear systems.

By applying the Haar functions [7], it is possible to receive a solution with a very small error, more accurate in some cases than the solution derived by the second order Runge-Kutta method. But the function must be superimposed with certain conditions.

3. The aim and objectives of the study

The goal of present work is to construct a numerical method to solve the Cauchy's problem for ordinary first order differential equations, which would yield more accurate results than the classical methods. The new method should not require solving a system of linear algebraic equations and should not require superimposing of conditions on the function.

To accomplish the goal, the following tasks have been set:

– to develop a new interpolation numerical method, employing the apparatus of non-classical Newton's minorants in order to solve the Cauchy's problem for ordinary first order differential equations;

– to prove the computational stability and convergence of the method;

– to assess accuracy of the solution and error of the method.

4. Materials and methods for examining a solution to the Cauchy's problem

4. 1. Formulae of the minorant type for the approximated calculation of definite integrals

There are a number of approaches in order to construct formulae for the approximated calculation of definite integrals: replacing an integrand function with interpolation polynomial [8], the use of Bernoulli numbers and polynomials [9]. In practice, however, most often used are quadrature formulae of the interpolation type. A question then arose: is it at all possible to employ the apparatus of non-classical minorants of Newton's functions, assigned in a tabular form, to solve the Cauchy's problem as well? In this case, the new method would not have to solve SLAE since this is a cumbersome process. In [10], the apparatus of non-classical minorants of Newton's functions, assigned in a tabular form, was used for finding zeros of the function.

Assume that it is required to compute a definite integral

$$J = \int_a^b f(x)\,\mathrm{d}x. \tag{1}$$

Without loss of generality, we shall consider that $f(x) > 0$ for all $x \in [a,b]$. Let us build for function $y = f(x)$ a Newton's minorant [11] by two points $x_1 = a$, $x_2 = b$. We obtain

$$m_f(x) = \left(A^{b-x} B^{x-a}\right)^{\frac{1}{b-a}}, \tag{2}$$

where $A = f(a)$, $B = f(b)$. We shall replace at interval $[a,b]$ function $f(x)$ with Newton's minorant $m_f(x)$ [12]. We receive

$$J = \int_a^b f(x)\,\mathrm{d}x = \int_a^b m_f(x)\,\mathrm{d}x + R(f), \tag{3}$$

where $R(f)$ is the remainder. Calculate

$$S = \int_a^b m_f(x)\,\mathrm{d}x = \int_a^b \left(A^{b-x} B^{x-a}\right)^{\frac{1}{b-a}} \mathrm{d}x. \tag{4}$$

At $A \neq B$ (if $A = B$, then $S = A(b-a) = B(b-a)$) we obtain:

$$S=\left(\frac{A^b}{B^a}\right)^{\frac{1}{b-a}}\int_a^b\left(\frac{B}{A}\right)^{\frac{x}{b-a}}dx=\left(\frac{A^b}{B^a}\right)^{\frac{1}{b-a}}\frac{b-a}{\ln(B)-\ln(A)}\left(\frac{B}{A}\right)^{\frac{x}{b-a}}\Big|_a^b=$$
$$=\left(\frac{A^b}{B^a}\right)^{\frac{1}{b-a}}\frac{b-a}{\ln(B)-\ln(A)}\left(\left(\frac{B}{A}\right)^{\frac{b}{b-a}}-\left(\frac{B}{A}\right)^{\frac{a}{b-a}}\right)=$$
$$=\left(\frac{A^b}{B^a}\right)^{\frac{1}{b-a}}\frac{b-a}{\ln(B)-\ln(A)}\left(\frac{B}{A}\left(\frac{B}{A}\right)^{\frac{b}{b-a}}-\left(\frac{B}{A}\right)^{\frac{a}{b-a}}\right)=$$
$$=\left(\frac{A^b}{B^a}\right)^{\frac{1}{b-a}}\frac{b-a}{\ln(B)-\ln(A)}\left(\frac{B}{A}\right)^{\frac{a}{b-a}}\frac{B-A}{A}=(b-a)\frac{B-A}{\ln(B)-\ln(A)}. \quad (5)$$

Thus,

$$\int_a^b f(x)dx=(b-a)\frac{f(b)-f(a)}{\ln(f(b)/f(a))}+R(f). \quad (6)$$

The resulting formula is called *a small formula of the minorant type* [11] for approximated calculation of definite integrals.

Let us prove that

$$\lim_{\frac{f(b)}{f(a)}\to 1}\frac{f(b)-f(a)}{\ln(f(b)/f(a))}=f(a). \quad (7)$$

Actually,

$$\lim_{\frac{f(b)}{f(a)}\to 1}\frac{f(b)-f(a)}{\ln(f(b)/f(a))}=\lim_{\frac{f(b)}{f(a)}\to 1}\frac{1}{\ln\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{f(b)-f(a)}}}=$$
$$=\frac{1}{\lim_{\frac{f(b)}{f(a)}\to 1}\ln\left(\frac{f(b)}{f(a)}\right)^{\frac{1}{f(b)-f(a)}}}=\frac{1}{\lim_{\frac{f(b)}{f(a)}\to 1}\ln\left(1+\frac{f(b)-f(a)}{f(a)}\right)^{\frac{1}{f(b)-f(a)}}}=$$
$$=\frac{f(a)}{\lim_{\frac{f(b)}{f(a)}\to 1}\ln\left(1+\frac{f(b)-f(a)}{f(a)}\right)^{\frac{f(a)}{f(b)-f(a)}}}=f(a). \quad (8)$$

If function $f(x)\in C[a,b]$ on interval $[a,b]$ s satisfies the Lipschitz condition with constant L and within this interval $f'(x)$ does not change a sign, then

$$|R(f)|=\int_a^b|f(x)-m_f(x)|\mathrm{d}x\le$$
$$\le\int_a^b|f(b)-f(a)|\mathrm{d}x\le L\int_a^b|(b-a)|\mathrm{d}x\le L(b-a)^2. \quad (9)$$

In order to record the constructed formula of the minorant type for calculating definite integrals, we shall split the interval of integration $[a,b]$ into n equal parts of length

$$h=\frac{b-a}{n}$$

with points

$$x_i=a+ih,\quad i=0,1,\ldots,n,$$

where $x_0=a$, $x_n=b$, and at every interval

$$[x_i,x_{i+1}],\quad i=0,1,\ldots,n-1,$$

we shall substitute function $f(x)$ with Newton's minorant $m_f(x)$. We receive

$$J=\int_a^b f(x)\mathrm{d}x=\sum_{i=0}^{n-1}\int_a^b f(x)\mathrm{d}x=\sum_{i=0}^{n-1}\int_a^b m_f(x)\mathrm{d}x+R_n(f). \quad (10)$$

Since at $f(x_i)\ne f(x_{i+1})$

$$\int_{x_i}^{x_{i+1}}m_f(x)\mathrm{d}x=(x_{i+1}-x_i)\frac{f(x_{i+1})-f(x_i)}{\ln\left(\frac{f(x_{i+1})}{f(x_i)}\right)}, \quad (11)$$

then

$$\int_a^b f(x)\mathrm{d}x=h\sum_{i=0}^{n-1}\frac{f(x_{i+1})-f(x_i)}{\ln\left(\frac{f(x_{i+1})}{f(x_i)}\right)}+R_n(f). \quad (12)$$

The resulting formula is called *a combined formula of the minorant type* [11] for an approximate calculation of definite integrals.

We shall estimate the remainder $R_n(f)$ for the constructed formula of the minorant type. Since

$$R_n(f)=\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}(f(x)-m_f(x))\mathrm{d}x, \quad (13)$$

then

$$|R_n(f)|\le\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f(x)-m_f(x)|\mathrm{d}x. \quad (14)$$

If we assume that function $f(x)$ on interval $[a,b]$ satisfies the Lipschitz condition with constant L and on each of the intervals $[x_i,x_{i+1}]$ is monotonous, then

$$|R_n(f)|\le\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f(x_{i+1})-f(x_i)|\mathrm{d}x\le$$
$$\le L\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}(x_{i+1}-x_i)\mathrm{d}x. \quad (15)$$

Hence

$$|R_n(f)|\le L\frac{(b-a)^2}{n}. \quad (16)$$

Thus, the following theorem holds.

Theorem 1. If function $f(x)$ on interval $[a,b]$ satisfies the Lipschitz condition with constant L and on each of the intervals $[x_{i-1},x_i]$, $i=1,2,\ldots,n$ is monotonous, the formula (12) holds and the evaluation of accuracy is performed (16). If function $f(x)$ is convex on interval $[a,b]$, then the inequality holds

$$\int_b^a f(x)\mathrm{d}x\le\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}m_f(x)\mathrm{d}x. \quad (17)$$

Theorem 2. If $f(x)\in C^2[a,b]$ and $f(x)>0, x\in[a,b]$, then

$$\left|R_n(f)\right| \le \frac{h^2}{8}\max_{x\in[a,b]} f(x)\max_{x\in[a,b]}\left|\frac{d^2}{dx^2}\ln(f(x))\right|, \tag{18}$$

that is, a quadrature formula (12) will be of second-order accuracy.

The quadrature formula constructed has the same complexity and the same order of accuracy (in case the conditions of Theorem 1 or 2 are fulfilled) compared with the quadrature formula of trapezoids. At the same time, if function $f(x)$ takes the form

$$f(x) = b_i \exp(c_i x), x \in [x_{i-1}, x_i], \quad i = 1,2,\ldots,n,$$

then the quadrature formula of the minorant type produces the exact value of the desired integral. This is the advantage of the method.

5. Results of research into an interpolation numerical method to solve the Cauchy's problem for ordinary first order differential equations

5. 1. Algorithm of the method to solve the Cauchy's problem for ordinary first order differential equations

Consider the Cauchy's problem for ordinary first order differential equation

$$y' = f(x,y), \tag{19}$$

$$y(x_0) = y_0. \tag{20}$$

Let us assume that the solution to this problem should be found on certain interval $[x_0, x_0 + a]$, where a>0. In this case, we shall consider that in region *D*, which contains a rectangle

$$R = \left\{x_0 \le x \le x_0 + a, \left|y - y_0\right| \le b\right\}$$

function $f(x,y)$ is continuous and satisfies the Lipschitz condition by *y*, that is,

$$\left|f(x,y_1) - f(x,y_2)\right| \le L\left|y_1 - y_2\right|,$$

where *L* is some constant while (x,y_1) and (x,y_2) are any two points of region *D*. Choose on interval $[x_0, x_0 + a]$, a system of points $x_0, x_1, \ldots, x_n$, where

$$x_k = x_0 + kh \ (k = 0,1,\ldots,n), \quad h > 0, \quad x_n \le x_0 + a.$$

Then, by employing the apparatus of non-classical minorants and diagrams of the Newton's functions, assigned in a tabular form, we shall build interpolation numerical method for solving the problem (19), (20). That is, we shall develop a method for finding the approximated values $y_0, y_1, \ldots, y_n$ of the exact solution $y = y(x)$ to the problem (19), (20) in points $x_0, x_1, \ldots, x_n$.

Let us clarify the issue of convergence, accuracy and computational stability of the techniques.

Let $y = y(x)$ be the desired solution to the problem (19), (20). By substituting it into equation (19), we shall obtain the identity

$$y'(x) = f(x, y(x)). \tag{21}$$

Integrate this identity at each of the intervals $[x_i, x_{i+1}]$ $(i = 0,1,\ldots,n-1)$. We receive

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(x, y(x))\mathrm{d}x. \tag{22}$$

Without reducing generality, we shall assume that

$$f(x, y(x)) > 0$$

for all

$$x \in [x_0, x_0 + a].$$

Construct for function $f(x, y(x))$ Newton's minorant $m_f(x)$ by two points

$$\left(x_i, f(x_i, y(x_i))\right) \text{ and } \left(x_{i+1}, f(x_{i+1}, y(x_{i+1}))\right).$$

We receive

$$m_f(x) = \left(A_i^{x_{i+1}-x} B_i^{x-x_i}\right)^{\frac{1}{h}}, \tag{23}$$

where $A_i = f(x_i, y(x_i))$, $B_i = f(x_{i+1}, y(x_{i+1}))$. Next, we shall substitute integrand function $f(x, y(x))$ in (22) with a Newton's minorant. We obtain

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} m_f(x)\mathrm{d}x + R_{i+1},$$

where is the remainder. Then, by computing integral at $A_i \ne B_i$, we receive

$$y(x_{i+1}) = \\ = y(x_i) + h\frac{f(x_{i+1}, y(x_{i+1})) - f(x_i, y(x_i))}{\ln\left(f(x_{i+1}, y(x_{i+1})) / f(x_i, y(x_i))\right)} + R_{i+1}. \tag{24}$$

At $A_i = B_i$

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + R_{i+1}.$$

Note that on the basis of superior boundary

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

we receive

$$\lim_{h\to 0}\frac{f(x_{i+1}, y(x_{i+1})) - f(x_i, y(x_i))}{\ln\left(f(x_{i+1}, y(x_{i+1})) / f(x_i, y(x_i))\right)} = \\ = f(x_i, y(x_i)). \tag{25}$$

In fact,

$$\lim_{h\to 0}\frac{f(x_{i+1}, y(x_{i+1})) - f(x_i, y(x_i))}{\ln\left(\dfrac{f(x_{i+1}, y(x_{i+1}))}{f(x_i, y(x_i))}\right)} = \\ = \frac{f(x_i, y(x_i))}{\lim\limits_{h\to 0}\ln\left(1 + \dfrac{f(x_{i+1}, y(x_{i+1})) - f(x_i, y(x_i))}{f(x_i, y(x_i))}\right)^{\frac{f(x_i, y(x_i))}{f(x_{i+1}, y(x_{i+1})) - f(x_i, y(x_i))}}}. \tag{26}$$

Since at $h \to 0$

$$\frac{f\left(x_{i+1}, y\left(x_{i+1}\right)\right) - f\left(x_i, y\left(x_i\right)\right)}{f\left(x_i, y\left(x_i\right)\right)} \to 0,$$

then it means that the boundary (25) holds.

Thus, in order to find the approximate values $y_0, y_1, \ldots, y_n$ of solution

$$y = f\left(x\right)$$

to the problem (12), (16), we shall obtain formula

$$y_{i+1} = y_i + h\frac{f\left(x_{i+1}, y_{i+1}\right) - f\left(x_i, y_i\right)}{\ln\left(f\left(x_{i+1}, y_{i+1}\right) / f\left(x_i, y_i\right)\right)}$$

$$\left(i = 0, 1, \ldots, n-1\right), \tag{27}$$

where $y_0 = y\left(x_0\right)$.

If

$$f\left(x_{i+1}, y_{i+1}\right) = f\left(x_i, y\left(x_i\right)\right)$$

for certain *i*, then, as it follows from (25), y_{i+1} will be searched for by formula

$$y_{i+1} = y_i + hf\left(x_i, y_i\right). \tag{28}$$

Theorem 3. If in region *D*, which contains a rectangle

$$R = \{x_0 < x < x_0 + a, \left|y - y_0\right| \le b\},$$

function $f\left(x, y\right)$ is continuous, satisfies the Lipschitz condition at variable *y* with constant *L* and

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| = \left|\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right| \le N < \infty, \tag{29}$$

where *N* is some constant, then the approximated values $y_1, y_2, \ldots, y_n$, found from formula (7), at $h \to 0$ uniformly, relative to *x*, converge to the exact solution

$$y = y\left(x\right)$$

to the problem (19), (20).

Theorem proving. First of all, note that, similar to (25), the following boundary holds

$$\lim_{h \to \infty} \frac{f(x_{k+1}, y_{k+1}) - f(x_k, y_k)}{\ln(f(x_{k+1}, y_{k+1}) / f(x_k, y_k))} = f(x_k, y_k). \tag{30}$$

Let

$$\varepsilon_k = y_k - y\left(x_k\right)$$

be an error of the approximated value of solution $y = y\left(x\right)$ to the problem (19), (20) in point $x = x_k$. Then the error gain at the (*k*+1)-th step will be equal to

$$\Delta\varepsilon_k = \varepsilon_{k+1} - \varepsilon_k = \left(y_{k+1} - y_k\right) - \left(y\left(x_{k+1}\right) - y\left(x_k\right)\right) =$$

$$= h\frac{f\left(x_{k+1}, y_{k+1}\right) - f\left(x_k, y_k\right)}{\ln\left(f\left(x_{k+1}, y_{k+1}\right) / f\left(x_k, y_k\right)\right)} - \int_{x_k}^{x_{k+1}} f\left(x, y\left(x\right)\right)\mathrm{d}x. \tag{31}$$

Since

$$\int_{x_k}^{x_{k+1}} f(x, y(x))\,\mathrm{d}x = (x - x_{k+1}) f(x, y(x))\Big|_{x_k}^{x_{k+1}} -$$

$$- \int_{x_k}^{x_{k+1}} (x - x_{k+1}) \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x =$$

$$= hf\left(x_k, y(x_k)\right) - \int_{x_k}^{x_{k+1}} (x - x_{k+1}) \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x, \tag{32}$$

then

$$\Delta\varepsilon_k = h\left(\frac{f\left(x_{k+1}, y_{k+1}\right) - f\left(x_k, y_k\right)}{\ln\left(\dfrac{f\left(x_{k+1}, y_{k+1}\right)}{f\left(x_k, y_k\right)}\right)} - f\left(x_k, y\left(x_k\right)\right)\right) -$$

$$- \int_{x_k}^{x_{k+1}} \left(x - x_{k+1}\right)\frac{df}{dx}\mathrm{d}x. \tag{33}$$

We can write based on (30)

$$\frac{f\left(x_{k+1}, y_{k+1}\right) - f\left(x_k, y_k\right)}{\ln\left(\dfrac{f\left(x_{k+1}, y_{k+1}\right)}{f\left(x_k, y_k\right)}\right)} = f\left(x_k, y_k\right) - \delta_k\left(h\right),$$

where $\delta_k\left(h\right) \to 0$ at $h \to 0$. That is why

$$\frac{f\left(x_{k+1}, y_{k+1}\right) - f\left(x_k, y_k\right)}{\ln\left(\dfrac{f\left(x_{k+1}, y_{k+1}\right)}{f\left(x_k, y_k\right)}\right)} - f\left(x_k, y_k\right) =$$

$$= f\left(x_k, y_k\right) - f\left(x_k, y(x_k)\right) - \delta_k\left(h\right). \tag{34}$$

By employing the Lipschitz condition for function $f\left(x, y\right)$ along *y*, we receive

$$\left|f\left(x_k, y_k\right) - f\left(x_k, y(x_k)\right)\right| \le L\left|y_k - y(x_k)\right| = L\left|\varepsilon_k\right|. \tag{35}$$

And based on condition (29), we obtain

$$\left|\int_{x_k}^{x_{k+1}} (x - x_{k+1}) \frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x\right| \le N \int_{x_k}^{x_{k+1}} \left|x - x_{k+1}\right| \mathrm{d}x = \frac{Nh^2}{2}. \tag{36}$$

Thus,

$$\left|\Delta\varepsilon_k\right| \le Lh\left|\varepsilon_k\right| + \frac{1}{2}Nh^2 + h\left|\delta_k(h)\right|. \tag{37}$$

From inequality

$$\left|\varepsilon_{k+1} - \varepsilon_k\right| \ge \left|\varepsilon_{k+1}\right| - \left|\varepsilon_k\right|$$

it follows that

$$\left|\varepsilon_{k+1}\right| - \left|\varepsilon_k\right| \le Lh\left|\varepsilon_k\right| + \frac{1}{2}Nh^2 + h\left|\delta_k(h)\right|$$

or

$$\left|\varepsilon_{k+1}\right| \le (1 + Lh)\left|\varepsilon_k\right| + \frac{1}{2}Nh^2 + h\left|\delta_k(h)\right|.$$

We have derived a recurrent formula for the estimation of error at the (k+1)-th step via the error at the k-th step. Substituting this formula with k=0.1,..., n–1, we receive

$$|\varepsilon_1| \le (1+Lh)|\varepsilon_0| + \frac{1}{2}Nh^2 + h|\delta_0(h)| \quad (38)$$

or

$$|\varepsilon_1| \le \frac{1}{2}Nh^2 + h|\delta_0(h)|, \quad (39)$$

since $\varepsilon_0 = y_0 - y(x_0) = 0;$

$$|\varepsilon_2| \le (1+Lh)|\varepsilon_1| + \frac{1}{2}Nh^2 + h|\delta_1(h)| \le$$
$$\le \frac{1}{2}Nh^2((1+Lh)+1) + h\left((1+Lh)|\delta_0(h)| + |\delta_1(h)|\right)$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$$
$$|\varepsilon_n| \le \frac{1}{2}Nh^2\left((1+Lh)^{n-1} + (1+Lh)^{n-2} + \cdots + 1\right) +$$
$$+h\left((1+Lh)^{n-1}|\delta_0(h)| + (1+Lh)^{n-2}|\delta_1(h)| + \cdots + |\delta_{n-1}(h)|\right).$$

Assume

$$\max_{0\le k\le n-1}|\delta_k(h)| = \delta(h).$$

Then

$$|\varepsilon_n| \le \left(\frac{1}{2}Nh^2 + h\delta(h)\right)\cdot\left((1+Lh)^{n-1} + (1+Lh)^{n-2} + \cdots + 1\right)$$

or

$$|\varepsilon_n| \le \frac{1}{L}\left(\frac{1}{2}Nh + \delta(h)\right)\cdot\left((1+Lh)^n - 1\right).$$

Because at u>0, the inequality $e^u > 1+u$, holds, then

$$|\varepsilon_n| \le \frac{1}{L}\left(\frac{1}{2}Nh + \delta(h)\right)\cdot\left(e^{nLh} - 1\right).$$

If one considers that

$$nh = x_n - x_0 \le a \text{ and } \delta(h) = Ch,$$

where C is some constant, then we finally receive

$$|\varepsilon_n| \le \frac{h}{L}\left(\frac{1}{2}N + C\right)\cdot\left(e^{aL} - 1\right). \quad (40)$$

It follows from here that at $h \to 0$, regardless of x, we have $|\varepsilon_n| \to 0$. This means that the approximates values $y_1, y_2, ..., y_n$ at $h \to 0$ uniformly relative to x converge to the exact solution $y = y(x)$. *The theorem is proven.*

It follows from the proven theorem that the method possesses first order of accuracy relative to h.

Formula (24) is actually the equation for finding y_{i+1}. That is why, in order to compute y_{i+1}, we shall construct an iterative process

$$y_{i+1}^{k+1} = y_i + h\frac{f\left(x_{i+1}, y_{i+1}^k\right) - f\left(x_i, y_i\right)}{\ln\left(f\left(x_{i+1}, y_{i+1}^k\right)/f\left(x_i, y_i\right)\right)}$$

$$(i = 0,1,\ldots,n-1), \quad (41)$$

where $y_{i+1}^{(0)}$ is the chosen zero approximation, for example, by the Euler method:

$$y_{i+1}^{(0)} = y_i + hf\left(x_i, y_i\right),\ y_0 = y\left(x_0\right).$$

Accordingly, if for some i and k

$$f\left(x_{i+1}, y_{i+1}^{(k)}\right) = f\left(x_i, y_i\right)$$

is satisfied, then at this step $y_{i+1}^{(k+1)}$ will be searched for by formula

$$y_{i+1}^{(k+1)} = y_i + hf\left(x_i, y_i\right). \quad (42)$$

We shall find at which condition this iterative process converges. An attribute of the end of the iterative process is the condition: $\left|y_{i+1}^{(k+1)} - y_{i+1}^{(k)}\right| < \varepsilon.$ After this, we accept

$$y_{i+1} \approx y_{i+1}^{(k+1)} \quad (i = 0,1,\ldots,n-1).$$

Consider

$$|y_{i+1}^{(k+1)} - y_{i+1}| =$$
$$= h\left|\frac{f(x_{i+1}, y_{i+1}^{(k)}) - f(x_i, y_i)}{\ln\left(f(x_{i+1}, y_{i+1}^{(k)})/f(x_i, y_i)\right)} - \frac{f(x_{i+1}, y_{i+1}) - f(x_i, y_i)}{\ln\left(f(x_{i+1}, y_{i+1})/f(x_i, y_i)\right)}\right| =$$
$$= h\left|\frac{f(x_i, y_i) - f(x_{i+1}, y_{i+1}^{(k)})}{\ln\left(f(x_i, y_i)/f(x_{i+1}, y_{i+1}^{(k)})\right)} - \frac{f(x_i, y_i) - f(x_{i+1}, y_{i+1})}{\ln\left(f(x_i, y_i)/f(x_{i+1}, y_{i+1})\right)}\right|.$$

Since at $h \to 0$

$$\frac{f(x_i, y_i) - f(x_{i+1}, y_{i+1}^{(k)})}{f(x_{i+1}, y_{i+1}^{(k)})} \to 0,$$

$$\frac{f(x_i, y_i) - f(x_{i+1}, y_{i+1})}{f(x_{i+1}, y_{i+1})} \to 0,$$

then

$$\frac{f(x_i, y_i) - f(x_{i+1}, y_{i+1}^{(k)})}{\ln\left(f(x_i, y_i)/f(x_{i+1}, y_{i+1}^{(k)})\right)} = f(x_{i+1}, y_{i+1}^{(k)}) - \delta_{i+1}^{(k)}(h),$$

$$\frac{f(x_i, y_i) - f(x_{i+1}, y_{i+1})}{\ln\left(f(x_i, y_i)/f(x_{i+1}, y_{i+1})\right)} = f(x_{i+1}, y_{i+1}) - \delta_{i+1}(h),$$

where $\delta_{i+1}^{(k)}(h) \to 0$ and $\delta_{i+1}(h) \to 0$ at $h \to 0$. That is why

$$|y_{i+1}^{(k+1)} - y_{i+1}| =$$
$$= h\left|\left(f(x_{i+1}, y_{i+1}^{(k)}) - f(x_{i+1}, y_{i+1})\right) - \left(\delta_{i+1}^{(k)}(h) - \delta_{i+1}(h)\right)\right| \le$$
$$\le h\left|f(x_{i+1}, y_{i+1}^{(k)}) - f(x_{i+1}, y_{i+1})\right| + h\bar{\delta}_{i+1}^{(k)}(h),$$

where

$$\bar{\delta}_{i+1}^{(k)}(h) = \left|\delta_{i+1}^{(k)}(h) - \delta_{i+1}(h)\right|.$$

By employing the Lipschitz condition at variable y for $f(x, y)$, we receive

$$|y_{i+1}^{(k+1)} - y_{i+1}| \le hL|y_{i+1}^{(k)} - y_{i+1}| + h\bar{\delta}_{i+1}^{(k)}(h).$$

Hence, at $k = 0,1,\ldots n-1$, we obtain

$$\left| y_{i+1}^{(1)} - y_{i+1} \right| \le hL \left| y_{i+1}^{(0)} - y_{i+1} \right| + h\bar{\delta}_{i+1}^{(0)}(h),$$

$$\left| y_{i+1}^{(2)} - y_{i+1} \right| \le hL \left| y_{i+1}^{(1)} - y_{i+1} \right| + h\bar{\delta}_{i+1}^{(1)}(h) \le$$
$$\le (hL)^2 \left| y_{i+1}^{(0)} - y_{i+1} \right| + h\left(hL\bar{\delta}_{i+1}^{(0)}(h) + \bar{\delta}_{i+1}^{(1)}(h) \right)$$
. . . . .
$$\left| y_{i+1}^{(n)} - y_{i+1} \right| \le (hL)^n \left| y_{i+1}^{(0)} - y_{i+1} \right| +$$
$$+ h\left((hL)^{n-1}\, \bar{\delta}_{i+1}^{(0)}(h) + (hL)^{n-2}\, \bar{\delta}_{i+1}^{(1)}(h) + \cdots + hL\bar{\delta}_{i+1}^{(n-2)}(h) + \bar{\delta}_{i+1}^{(n-1)}(h) \right).$$

Assume

$$\max_{0 \le m \le n-1} \bar{\delta}_{i+1}^{(m)}(h) = \tilde{\delta}_{i+1}(h).$$

Then

$$\left| y_{i+1}^{(n)} - y_{i+1} \right| \le (hL)^n \left| y_{i+1}^{(0)} - y_{i+1} \right| +$$
$$+ h\tilde{\delta}_{i+1}(h) \left((hL)^{n-1} + (hL)^{n-2} + \cdots + hL + 1 \right)$$

or

$$\left| y_{i+1}^{(n)} - y_{i+1} \right| \le (hL)^n \left| y_{i+1}^{(0)} - y_{i+1} \right| + h\tilde{\delta}_{i+1}(h) \frac{1-(hL)^n}{1-hL}.$$

It follows from what we received that at $hL < 1$ the iterative process converges.

Because a Newton's minorant consists of convex arcs, this method produces more accurate results than known two-point methods, in the case when the function $f(x, y(x))$ is convex.

6. Discussion of results: a study of computational stability of the method

Let us consider an issue of computational stability of the given method.

Let $\tilde{y}_0$ be an approximated value of exact initial value y_0 while ε'_0 is the absolute error of the initial approximation, that is,

$$\varepsilon'_0 = \left| \tilde{y}_0 - y_0 \right|.$$

Then, instead of formula (22) to calculate the approximated values of solution $y = y(x)$ in points $x_1, x_2, ..., x_n$, we shall receive formula

$$\tilde{y}_{i+1} = \tilde{y}_i + h \frac{f(x_{i+1}, \tilde{y}_{i+1}) - f(x_i, \tilde{y}_i)}{\ln\left(f(x_{i+1}, \tilde{y}_{i+1}) / f(x_i, \tilde{y}_i) \right)}$$

(i=0.1,...,n–1).

If we denote

$$\varepsilon'_i = \left| \tilde{y}_i - y_i \right| \quad (i=0.1,..., n),$$

then

$$\varepsilon'_{i+1} = \left| \tilde{y}_{i+1} - y_{i+1} \right| =$$
$$= \left| (\tilde{y}_i - y_i) + h\left(\frac{f(x_{i+1}, \tilde{y}_{i+1}) - f(x_i, \tilde{y}_i)}{\ln\left(f(x_{i+1}, \tilde{y}_{i+1}) / f(x_i, \tilde{y}_i) \right)} - \frac{f(x_{i+1}, y_{i+1}) - f(x_i, y_i)}{\ln\left(f(x_{i+1}, y_{i+1}) / f(x_i, y_i) \right)} \right) \right| \le$$
$$\le \varepsilon'_i + h\left| \frac{f(x_{i+1}, \tilde{y}_{i+1}) - f(x_i, \tilde{y}_i)}{\ln\left(f(x_{i+1}, \tilde{y}_{i+1}) / f(x_i, \tilde{y}_i) \right)} - \frac{f(x_{i+1}, y_{i+1}) - f(x_i, y_i)}{\ln\left(f(x_{i+1}, y_{i+1}) / f(x_i, y_i) \right)} \right|.$$

Since, based on superior boundary

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = \mathrm{e}.$$

We obtain

$$\lim_{h \to 0} \frac{f(x_{i+1}, \tilde{y}_{i+1}) - f(x_i, \tilde{y}_i)}{\ln\left(f(x_{i+1}, \tilde{y}_{i+1}) / f(x_i, \tilde{y}_i) \right)} = f(x_i, \tilde{y}_i),$$

$$\lim_{h \to 0} \frac{f(x_{i+1}, y_{i+1}) - f(x_i, y_i)}{\ln\left(f(x_{i+1}, y_{i+1}) / f(x_i, y_i) \right)} = f(x_i, y_i),$$

and then

$$\frac{f(x_{i+1}, \tilde{y}_{i+1}) - f(x_i, \tilde{y}_i)}{\ln\left(f(x_{i+1}, \tilde{y}_{i+1}) / f(x_i, \tilde{y}_i) \right)} = f(x_i, \tilde{y}_i) - \tilde{\delta}_i(h),$$

$$\frac{f(x_{i+1}, y_{i+1}) - f(x_i, y_i)}{\ln\left(f(x_{i+1}, y_{i+1}) / f(x_i, y_i) \right)} = f(x_i, y_i) - \delta_i(h),$$

where $\delta_i(h) \to 0$ at $h \to 0$. That is why

$$\varepsilon'_{i+1} \le \varepsilon'_i + h\left| \left(f(x_i, \tilde{y}_i) - f(x_i, y_i) \right) + \left(\delta_i(h) - \tilde{\delta}_i(h) \right) \right|.$$

Function $f(x, y)$ satisfies the Lipschitz condition at variable y with constant L, which is why

$$\varepsilon'_{i+1} \le \varepsilon'_i + Lh\left| \tilde{y}_i - y_i \right| + h\left| \delta_i(h) - \tilde{\delta}_i(h) \right|$$

or

$$\varepsilon'_{i+1} \le (1+Lh)\, \varepsilon'_i + h\bar{\delta}_i(h),$$

where

$$\bar{\delta}_i(h) = h\left| \delta_i(h) - \tilde{\delta}_i(h) \right|.$$

But $\bar{\delta}_i(h) \to 0$ at $h \to 0$. Then at $i = 0,1,\ldots,n-1$ we receive By denoting

$$\varepsilon'_1 \le (1+Lh)\, \varepsilon'_0 + h\bar{\delta}_0(h);$$
$$\varepsilon'_2 \le (1+Lh)\, \varepsilon'_1 + h\bar{\delta}_1(h) \le (1+Lh)^2 \varepsilon'_0 + h\left((1+Lh)\, \bar{\delta}_0(h) + \bar{\delta}_1(h) \right);$$
$$\varepsilon'_3 \le (1+Lh)\, \varepsilon'_2 + h\bar{\delta}_2(h) \le (1+Lh)^3 \varepsilon'_0 + h\left((1+Lh)^2 \bar{\delta}_0(h) + (1+Lh)\, \bar{\delta}_1(h) + \bar{\delta}_2(h) \right);$$
. . . . .
$$\varepsilon'_n \le (1+Lh)^n \varepsilon'_0 + h\left((1+Lh)^{n-1} \bar{\delta}_0(h) + (1+Lh)^{n-2} \bar{\delta}_1(h) + \cdots + \bar{\delta}_{n-1}(h) \right).$$

$$\max_{0 \le i \le n-1} \bar{\delta}_i(h) = \delta(h).$$

We shall obtain

$$\varepsilon'_n \le (1+Lh)^n \varepsilon'_0 +$$
$$+ h\delta(h) \left((1+Lh)^{n-1} + (1+Lh)^{n-2} + \cdots + 1 \right)$$

or

$$\varepsilon'_n \le (1+Lh)^n \varepsilon'_0 + \frac{1}{L} \delta(h) \left((1+Lh)^n - 1 \right).$$

Considering that at $u > 0$ the inequality $\mathrm{e}^u > 1+u$, holds, then

$$\varepsilon_n' \le e^{Lnh}\varepsilon_0' + \frac{1}{L}\delta(h)\left(e^{Lnh}-1\right)$$

or

$$\varepsilon_n' \le e^{La}\varepsilon_0' + \frac{1}{L}\delta(h)\left(e^{La}-1\right).$$

We have on the basis of the derived inequality that the error of initial data is not piled up, that is, the method possesses computational stability.

Example.

It is required to find a numerical solution to the Cauchy's problem

$$y' = e^{2x} + e^x - 2ye^x + y^2,\ y(0) = 0.5 \quad (42)$$

in interval [0;1]. Exact solution

$$y^* = e^x - \frac{1}{x+2}$$

is obtained by introducing a new variable and by taking into account the original condition. Let us also compare the solution found by using the new method with the Euler method, and the Runge-Kutta method of fourth order. The solution to the problem is given in Table 1.

Table 1

Solution to problem (42) obtained by different methods

No.	x	y^m	y^E	y^{R-K}	y^*
0	0	0.5	0.5	0.5	0.5
1	0.02	0.52515	0.525	0.52515	0.52515
2	0.04	0.55062	0.55031	0.55061	0.55061
3	0.06	0.5764	0.57594	0.5764	0.5764
4	0.08	0.60252	0.6019	0.60252	0.60252
5	0.1	0.62898	0.6282	0.62898	0.62898
6	0.12	0.6558	0.65485	0.6558	0.6558
7	0.14	0.68299	0.68187	0.68298	0.68298
8	0.16	0.71055	0.70926	0.71055	0.71055
9	0.18	0.73851	0.73704	0.7385	0.7385
10	0.2	0.76686	0.76522	0.76686	0.76686
11	0.22	0.79563	0.79381	0.79563	0.79563
12	0.24	0.82483	0.82282	0.82482	0.82482
13	0.26	0.85446	0.85227	0.85445	0.85445
14	0.28	0.88454	0.88216	0.88453	0.88453
15	0.3	0.91508	0.91251	0.91508	0.91508
16	0.32	0.9461	0.94334	0.94609	0.94609
17	0.34	0.9776	0.97464	0.9776	0.9776
18	0.36	1.00961	1.00645	1.0096	1.0096
19	0.38	1.04212	1.03876	1.04212	1.04212
20	0.4	1.07516	1.07159	1.07516	1.07516
21	0.42	1.10874	1.10496	1.10874	1.10874
22	0.44	1.14288	1.13888	1.14287	1.14287
23	0.46	1.17758	1.17335	1.17757	1.17757
24	0.48	1.21285	1.20841	1.21285	1.21285
25	0.5	1.24873	1.24406	1.24872	1.24872
26	0.52	1.28521	1.28031	1.2852	1.2852
27	0.54	1.32231	1.31717	1.32231	1.32231
28	0.56	1.36005	1.35467	1.36005	1.36005
29	0.58	1.39845	1.39282	1.39844	1.39844
30	0.6	1.43751	1.43164	1.4375	1.4375
31	0.62	1.47725	1.47113	1.47725	1.47725
32	0.64	1.5177	1.51132	1.51769	1.51769
33	0.66	1.55886	1.55221	1.55885	1.55885
34	0.68	1.60075	1.59384	1.60074	1.60074
35	0.7	1.64339	1.6362	1.64338	1.64338
36	0.72	1.68679	1.67933	1.68679	1.68679
37	0.74	1.73097	1.72323	1.73097	1.73097
38	0.76	1.77596	1.76793	1.77596	1.77596
39	0.78	1.82176	1.81344	1.82176	1.82176
40	0.8	1.8684	1.85977	1.8684	1.8684
41	0.82	1.91589	1.90696	1.91589	1.91589
42	0.84	1.96426	1.95501	1.96425	1.96425
43	0.86	2.01351	2.00395	2.01351	2.01351
44	0.88	2.06368	2.0538	2.06368	2.06368
45	0.9	2.11478	2.10457	2.11478	2.11478
46	0.92	2.16682	2.15628	2.16682	2.16682
47	0.94	2.21984	2.20896	2.21985	2.21985
48	0.96	2.27386	2.26262	2.27386	2.27386
49	0.98	2.32888	2.31729	2.32889	2.32889
50	1	2.38495	2.37299	2.38495	2.38495

In Table 1, y^* is the exact solution, y^m is the solution obtained using the new interpolation method of the minorant type (over 2 iterations), y^E is the solution obtained applying the Euler's method, and y^{R-K} is the solution obtained employing the Runge-Kutta method. Thus, the new interpolation method produces a solution that is very close to the exact solution.

As one can see, it is necessary to run a lot of iterations in order to solve the Cauchy's problem by the new method, which is the main shortcoming of the method.

6. Conclusions

1. We developed and constructed an interpolation numerical method to solve the Cauchy's problem for ordinary first order differential equations. Underlying the method is the so-called apparatus of non-classical minorants and diagrams of Newton's functions, assigned in a tabular form. In the case of a convex function, this method produces more accurate results than the Euler's method. The method also does not require solving the systems of linear algebraic equations or superimposing additional conditions on the equations.

2. The order of accuracy is proven, as well as computational stability, convergence of the new method, and an error of approximated value is estimated by the new method. The method has a second order of accuracy, similar to the Euler's method, yet produces more accurate results in the case of a convex function.

References

1. Verzhbitskiy, V. M. Chislennyye metody (matematicheskiy analiz i obyknovennyye differentsialnyye uravneniya) [Text] / V. M. Verzhbitskiy. – Moscow: Vysshaya shkola, 2001. – 382 p.
2. Süli, E. An Introduction to Numerical Analysis [Text] / E. Süli, D. Mayers. – Cambridge University Press, 2003. – 435 p.
3. Zadachyn, V. M. Chyselni metody [Text] / V. M. Zadachyn. – Kharkiv: Vyd. KhNEU im. S. Kuznetsya, 2014. – 180 p.
4. Abdulkawi, M. Solution of Cauchy type singular integral equations of the first kind by using differential transform method [Text] / M. Abdulkawi // Applied Mathematical Modelling. – 2015. – Vol. 39 Issue 8. – P. 2107–2118. doi: 10.1016/j.apm.2014.10.003
5. Setia, A. Numerical solution of various cases of Cauchy type singular integral equation [Text] / A. Setia // Applied Mathematics and Computation. – 2014. – Vol. 230. – P. 200–207. doi: 10.1016/j.amc.2013.12.114
6. De Bonis, M. C. Numerical solution of systems of Cauchy singular integral equations with constant coefficients [Text] / M. C. De Bonis, C. Laurita // Applied Mathematics and Computation. – 2012. – Vol. 219 Issue 4. – P. 1391–1410. doi: 10.1016/j.amc.2012.08.022
7. Lukomskii, D. S. Solution of Cauchy Problem for Equation First Order Via Haar Functions [Text] / D. S. Lukomskii, S. F. Lukomskii, P. A. Terekhin // Izvestiya of Saratov University New Series, Series: Mathematics Mechanics Informatics. – 2016. – Vol. 16, Issue 2. – P. 151–159. doi: 10.18500/1816-9791-2016-16-2-151-159
8. Lyashenko, M. Ya. Chysel'ni metody [Text] / M. Ya. Lyashenko, M. S. Holovan. – Kyiv: «Lybid'», 1996. – 285 p.
9. Varenych, I. I. Vyshcha matematyka: matematychnyy analiz, dyferentsialni rivnyannya [Text] / I. I. Varenych. – Kyiv: DiaSoft, 2008. – 267 p.
10. Bihun, R. R. Construction of a numerical method for finding the zeros of both smooth and nonsmooth functions [Text] / R. R. Bihun, G. G. Tsehelyk // Eastern-European Journal of Enterprise Technologies. – 2017. – Vol 2, Issue 4 (86). – P. 58–64. doi: 10.15587/1729-4061.2017.99273
11. Bihun, R. R. Formulas of minorant type to approximate computing definite integrals [Text] / R. R. Bihun, G. G. Tsehelyk // Scientific light. – 2017. – Vol. 1, Issue 7. – P. 76–79.
12. Bihun, R. R. Numerical Method for Finding All Points of Extremum of Random as Smooth and NonSmooth Functions of One Variable [Text] / R. R. Bihun, G. G. Tsehelyk // Global Journal of Science Frontier Research: F Mathematics & Decision Sciences. – 2015. – Vol. 15 Issue 2. – P. 87–93.