1. Introduction

Bulk carbon-graphite materials are widely used as high-temperature thermal insulators and heating elements in resistance furnaces of the electrode-manufacturing industry [1, 2]. To account for the dependence of physical properties of bulk materials on pressure in the course of numerical analysis of the thermal-electric state of high-temperature furnace equipment of electrode production [3, 4], it is necessary to know pressure distribution in the layers of these materials. Failure to take this dependence into account may lead to significant errors in the results of numerical analysis during development of new equipment and when working out the norms of its operation. It is known that the information on pressure distribution in bulk materials in the form of the mean hydrostatic pressure can be obtained from the solution to a nonlinear problem on plasticity by the Drucker-Prager yield criterion [5, 6]. That is why the task to improve algorithms for solving the problem on mechanical behavior of bulk materials is a relevant issue.

2. Literature review and problem statement

An analysis of data from the scientific literature revealed that the Drucker-Prager model had become wide-spread in the numerical studies into elastic-plastic state of loose and brittle materials:

- examining the strength of concrete with different composition [7–9];
- the process of rock destruction [10, 11];
- a mechanism of compaction of pharmaceutical powder materials [12, 13];
- the process of formation of metal-powder-like products [14–17].

Paper [7] describes the use of the extended Drucker-Prager yield model during numerical simulation of the behavior of reinforced concrete. Numerical experiments were performed employing the proprietary software ABAQUS.

Article [8] describes experimental studies into determining the limits of strength at compression of cylindrical concrete samples during uniaxial compression. The method is given for graphical determination of the Drucker-Prager criteria for concrete during its uniaxial loading.

In order to obtain the yield surface by Drucker-Prager for samples made of different grades of concrete, paper [9] experimentally determined coefficients of cohesion and the angles of inner friction.

Article [10] presented a model of mechanical behavior of rocks with different porosity, which is based on the combination of the Drucker-Prager boundary surface
with an elliptical surface. The ratios are given that describe changes in the surface of the boundary state and in the coefficient of dilatation during an irreversible deformation of the medium. The authors applied the non-associative law of currents.

Paper [11] considered a model of deformation of rocks that is a generalization of the Hill model of anisotropic plasticity, on the one hand, and the Drucker-Prager model, on the other hand. Underlying the model is the non-associative law of plastic current with strengthening for an anisotropic body, which takes into account the impact of volumetric stresses. The authors considered a rather general case of the combination of isotropic and translational strengthening.

Article [12] describes numerical and experimental studies into the pressing process of pharmaceutical powder materials. Numerical studies were performed using the Drucker-Prager model. The ABAQUS software was employed.

In [14], numerical studies are reported for the pressing process of aluminum oxide powder applying a modified Drucker-Prager model. Numerical experiment was performed using the ABAQUS software application.

The authors of [15], by employing the method of finite elements, carried out research into processes of plastic deformation of metallic powder materials. It is shown that the ABAQUS software package enables obtaining more accurate results for the Drucker-Prager model than when using the software DEFORM and ANSYS/LS-DYNA.

Papers [16, 17] report physical and numerical experiments of the pressing process of a mixture of metallic powders. Numerical study is represented in an axisymmetric 3D statement using the method of finite elements. The calculations were carried out applying a modified Drucker-Prager model implemented in the software ABAQUS.

Certain shortcomings of the considered studies that address application of the Drucker-Prager model for loose and brittle materials include the following:
- a lack of complete mathematical formulation of the problem and the solving algorithm;
- there are formulae only to determine the yield criterion, equivalent deformations and stresses, results and analysis of numerical studies, etc.;
- the calculations are carried out using the commercial software ABAQUS, employing which requires purchasing an appropriate license.

Thus, the studies examined above lack a complete mathematical statement of the problem and a solving algorithm, which makes it more difficult to clearly understand the issue that is being explored. That is why the promising directions of research are considered to be:
- improvement of existing algorithms for solving the problem on plastic behavior of bulk materials;
- development of the appropriate software code and its verification.

### 3. The aim and objectives of the study

The aim of present work is to improve algorithmic approaches to solving a nonlinear problem on the mechanical behavior of bulk materials by the Drucker-Prager yield criterion. This will make it possible to minimize requirements for computer resources.

To accomplish the set aim, the following tasks have been solved:
- to formulate a mathematical model for the elastic-plastic behavior of isotropic bulk material;
- to improve a procedure for solving numerically a problem on the mechanical state of bulk material based on the reverse-mapping algorithm;
- to compare data obtained by numerical experiments with the data acquired employing commercial software products.

### 4. Materials and methods for examining the elastic-plastic state of bulk material

According to the incremental theory of plasticity, a mathematical model of the elastic-plastic behavior of an isotropic bulk material includes the equilibrium equations, a generalized Hooke’s law, and geometrical equations [5, 6, 18]:

\[
\begin{align*}
\sigma_{ij} + \rho \dot{h} &= 0; \\
\sigma_{i} &= \frac{E}{1 + \nu} \dot{\varepsilon}_{i} + \frac{\nu}{1 - 2\nu} \delta_{ij} \dot{\varepsilon}_{kk} - \sigma_{eq}^{DP}; \quad i, j = 1, 2, 3, \\
\dot{\varepsilon}_{i} &= \frac{1}{2} (\dot{u}_{ij} + \dot{u}_{ji}) = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p},
\end{align*}
\]

where \( \sigma_{ij} \) are the components of symmetric tensor of stress gain of the 2nd rank, \( \rho \) is the density, \( \frac{E}{1 + \nu} \) is the modulus of elasticity during axisymmetric compression, \( \nu \) is the Poisson’s ratio; \( \delta_{ij} \) is the Kronecker symbol; \( \sigma_{eq}^{DP} \) are the components of strain tensor of initial stress, \( \sigma_{ij}^{e} \) and \( \sigma_{ij}^{p} \) are the elastic and plastic components of strain tensor of total deformations, respectively; \( \varepsilon_{ij}, \dot{u}_{ij} \) are the components of displacement gain vector, \( m \).

When employing a criterion of the onset of the Drucker-Prager yield, a condition for the yield of bulk material (plasticity function) is written in the following form [5, 6, 18]:

\[
F(\sigma_{y}, c, \phi) = \sigma_{n0p} - \varepsilon_{y}(c, \phi),
\]

where \( F \) is a function of the bulk material surface yield; \( \sigma_{n0p} = 3\alpha(\phi)\sigma_{y} + \frac{1}{2} s_{y} s_{y} \) is the equivalent stress by Drucker-Prager, \( \sigma_{y} \); \( s_{y} = \sigma_{y} - \frac{1}{3} \delta_{kl} \sigma_{kl} \) are the components of deviatoric stress tensor, \( \sigma_{y} \); \( \sigma_{n} = \frac{1}{3} \delta_{kl} \sigma_{kl}, \) Pa;
\( \sigma_s(c, \phi) \) is the yield limit of bulk material, Pa; \( c \) is the adhesion force between granules of bulk material, Pa; \( \phi \) is the angle of internal friction, or the angle of native repose of the bulk material, rad.

If we assume that the surface of fluidity by Drucker-Prager streamlines the surface of fluidity by Mohr-Coulomb \[5, \ 6\], the expressions for \( \sigma_s(c, \phi) \) and \( \alpha(\phi) \) take the form:

\[
\sigma_s(c, \phi) = \frac{6c \cos \phi}{\sqrt{3}(3 - \sin \phi)} \quad \text{(1)}
\]

\[
\alpha(\phi) = \frac{2 \sin \phi}{\sqrt{3}(3 - \sin \phi)} \quad \text{(2)}
\]

Initial conditions for (1), (2):

\[
\bar{\sigma}^0_{ij} = 0, \ i, j = 1, 2, 3. \quad \text{(3)}
\]

Boundary conditions for (1), (2):

- displacement vector gain

\[
\bar{u} \big|_{t=0} = 0, \ i = 1, 2, 3. \quad \text{(4)}
\]

where \( S \) is the surface (or a point of the surface), on which the displacement is set, m:\

- symmetry

\[
n_i \bar{u} \big|_{t=0} = 0, \ i = 1, 2, 3. \quad \text{(5)}
\]

where \( n_i \) are the components of external normal vector to the body surface; \( S_n \) is the surface of body symmetry, m:\

are the tensors of the 2nd rank, which are derived from functional relations (2) and (9) by a stress tensor, respectively; \( \hat{\mathbf{C}} \) is the tensor of elastic constants of the 4th rank, Pa; \( \hat{\mathbf{I}} \) is the singular tensor of the 2nd rank; \( \hat{\mathbf{e}}^p \) is the tensor of trial elastic deformations at each step of loading.

Considering (8) and (10), formula (7) for the loading step \( k+1 \) can be rewritten in the form

\[
\sigma_{ij}^{k+1} = \sigma_{ij}^p - \Delta \lambda^k \mathbf{C}_{ijkl} m_{kl} \left( \sigma_{ij}^{k+1} \right) \quad \text{(7)}
\]

or

\[
\hat{\mathbf{e}}^{k+1} = \hat{\mathbf{e}}^p - \Delta \lambda^k \hat{\mathbf{C}} \hat{\mathbf{m}} \left( \hat{\mathbf{e}}^{k+1} \right) \quad \text{(8)}
\]

where \( \Delta \lambda^k \) is the scalar associative multiplier \( \Delta \lambda \) or plasticity coefficient (8) in the absence of strengthening is determined from formula

\[
\Delta \lambda = \frac{n \cdot \mathbf{C} \cdot \hat{\mathbf{e}}^p}{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}}} \quad \text{(9)}
\]

where

\[
\hat{\mathbf{n}} = \frac{\partial F}{\partial \mathbf{e}} = \hat{\mathbf{a}} + \frac{1}{2} \frac{\dot{\mathbf{s}}}{\sqrt{2}} \quad \text{and}
\]

\[
\hat{\mathbf{m}} = \frac{\partial G}{\partial \sigma} = \beta \hat{\mathbf{I}} + \frac{1}{2} \frac{\dot{\mathbf{s}}}{\sqrt{2}} \quad \text{are the tensors of the 2nd rank, which are derived from functions (2) and (9) by a stress tensor, respectively;} \quad \hat{\mathbf{C}} \quad \text{is the tensor of elastic constants of the 4th rank,} \quad \hat{\mathbf{I}} \quad \text{is the singular tensor of the 2nd rank;} \quad \hat{\mathbf{e}}^p \quad \text{is the tensor of trial elastic deformations at each step of loading.}
\]

Considering (8) and (10), formula (7) for the loading stage \( k+1 \) can be rewritten in the form

\[
\sigma_{ij}^{k+1} = \sigma_{ij}^p - \Delta \lambda^k \mathbf{C}_{ijkl} m_{kl} \left( \sigma_{ij}^{k+1} \right) \quad \text{(7)}
\]

or

\[
\hat{\mathbf{e}}^{k+1} = \hat{\mathbf{e}}^p - \Delta \lambda^k \hat{\mathbf{C}} \hat{\mathbf{m}} \left( \hat{\mathbf{e}}^{k+1} \right) \quad \text{(8)}
\]

where \( \Delta \lambda^k \) is the scalar associative multiplier \( \Delta \lambda \) or plasticity coefficient (8) in the absence of strengthening is determined from formula

\[
\Delta \lambda = \frac{n \cdot \mathbf{C} \cdot \hat{\mathbf{e}}^p}{\hat{\mathbf{n}} \cdot \hat{\mathbf{m}}} \quad \text{(9)}
\]

where

\[
\hat{\mathbf{n}} = \frac{\partial F}{\partial \mathbf{e}} = \hat{\mathbf{a}} + \frac{1}{2} \frac{\dot{\mathbf{s}}}{\sqrt{2}} \quad \text{and}
\]

\[
\hat{\mathbf{m}} = \frac{\partial G}{\partial \sigma} = \beta \hat{\mathbf{I}} + \frac{1}{2} \frac{\dot{\mathbf{s}}}{\sqrt{2}} \quad \text{are the tensors of the 2nd rank, which are derived from functions (2) and (9) by a stress tensor, respectively;} \quad \hat{\mathbf{C}} \quad \text{is the tensor of elastic constants of the 4th rank,} \quad \hat{\mathbf{I}} \quad \text{is the singular tensor of the 2nd rank;} \quad \hat{\mathbf{e}}^p \quad \text{is the tensor of trial elastic deformations at each step of loading.}
\]

Formula (11) yields a mapping of the trial stress tensor \( \hat{\sigma}^p \) in the direction of the yield surface. That is why this method of integration, which is built on the inverse Euler method, acquired the name of the return-mapping algorithm [5, 6].

The system of equations (11) taking into account the symmetry of stress tensor has 7 unknowns, in particular, 6 independent components \( \hat{\sigma}^p \) and \( \Delta \lambda \). That is why, in order to receive uniqueness, the systems of equations (11) are to be supplemented with the scalar equation (2) in the form of a requirement that the condition of yield is satisfied at the end of a loading stage

\[
F(\hat{\mathbf{e}}^{k+1}, \Delta \lambda) = 0. \quad \text{(12)}
\]

The non-linear system of equations (11), (12) can be rewritten in the form of discrepancies. In this case, it is required to pass over to a six-dimensional space considering the symmetry of stress and deformation tensors. This makes it possible to replace tensors of the 2nd rank \( \hat{\sigma}^{k+1}, \hat{\mathbf{e}}^{k+1}, \hat{\mathbf{n}}, \hat{\mathbf{m}} \) with corresponding vectors \( \hat{\mathbf{e}}^{k+1}, \sigma^{k+1}, \mathbf{n}, \mathbf{m} \) with six components. Thus, rather than using a tensor of the 4th rank, one can employ elastic constants tensor of the second rank with dimensionality 6x6:

\[
\begin{align*}
\mathbf{r}_s &= \mathbf{r}_s - \Delta \lambda \mathbf{D}^s \mathbf{m} \left( \sigma^{k+1} \right); \\
\mathbf{r}_p &= F \left( \sigma^{k+1}, \Delta \lambda^{k+1} \right).
\end{align*}
\]
In order to solve a system of nonlinear equations (13), a
Newton's method is typically applied whose iterative pro-
duce is recorded as follows:
\[
\begin{pmatrix}
\sigma_{ij}^{(k)} \\
\Delta \lambda_{ij}^{(k)}
\end{pmatrix}
= \begin{pmatrix}
\sigma_{ij}^{(k-1)} \\
\Delta \lambda_{ij}^{(k-1)}
\end{pmatrix}
+ 
\begin{pmatrix}
\frac{\partial \sigma_{ij}}{\partial \sigma} \\
\frac{\partial \sigma_{ij}}{\partial \Delta \lambda}
\end{pmatrix}
\begin{pmatrix}
\Delta \lambda_{ij}
\end{pmatrix},
\tag{14}
\end{equation}
where
\[
\frac{\partial \sigma_{ij}}{\partial \sigma} = I + \Delta \lambda_{ij} D_{ij}^{el} - \frac{\partial m}{\partial \sigma} \left( \sigma_{ij}^{(k)} \right);
\]
\[
\frac{\partial \sigma_{ij}}{\partial \Delta \lambda} = D_{ij}^{el} \cdot \mathbf{n} \left( \sigma_{ij}^{(k)} \right);
\]
\[
\frac{\partial \sigma_{ij}}{\partial \Delta \lambda} = 0.
\]
Here, index \(k\) refers to the step of loading while index \(j\) –
to the number of iteration by the Newton's method.
At each iteration step by the Newton's method (14) it is
advisable to not find the inverse matrix but solve a system of
linear algebraic equations (SLAE) using the Gauss exclusion
method. This makes it possible to significantly reduce the
number of arithmetic operations, by about 3 times, where \(n\) is the
dimensionality of SLAE.
For \(k=1\), one solves an ordinary elastic problem relative
to total displacements provided the boundary conditions
(4)–(6) are assigned, which determine the trial stresses.
Next, for the part of the layer of a bulk material that is in
the elastically-plastic state, one determines the gains of
plastic deformation and the tensor of elastic tension from
solution (14) and finds initial stresses from formula
\[
\sigma^{(1)} = \Delta \lambda^{(0)} D^{el} \cdot \mathbf{m}^{(0)}.
\tag{15}
\]
The next steps of integration (1), (2) for \(k>1\) are performed
only with a load by initial stresses (15), (16), that is,
without taking into account external and gravitational loads
and at boundary conditions (5). In this case, one solves the
elastic problem and determines the gains of displacements
\(\Delta u^k\) and refines the values of total displacements, which
determine the new trial stresses. Then, from solution (14), one
determines a new value for the gain of plastic deformation and
the elastic tension for the body part that is in the
elastic-plastic state. Next, in order to perform the next step
of loading, one finds the gains of initial stresses from formula
\[
\sigma^{(k)} = \Delta \lambda^{(k)} D^{el} \cdot \mathbf{m}^{(k)} - \sigma^{(k-1)}.
\tag{16}
\]
The criterion for the end of computations can be, for example,
fulfillment of condition
\[
|\Delta u^k| \leq \delta_u, \quad \text{or} \quad |\varepsilon_{ij}^{(k)}| \leq \delta_v.
\]
New trial stresses \(\sigma^{(k)}\) in the algorithm of solving the problem can also be determined through the previous values of
\(\sigma^{(k-1)}\) and the gains of elastic deformations \(\Delta \epsilon^k\), which are found via \(\Delta u^k\), from formula
\[
\sigma^{(k)} = \sigma^{(k-1)} + D^{el} \cdot \Delta \epsilon^k \left( \Delta u^k \right).
\tag{17}
\]
In order to determine \(\Delta \epsilon^k\), at each step of integration by
time, we use a gain of initial stresses in the form
\[
\sigma^{(k)} = \Delta \lambda^{k-1} D^{el} \cdot \mathbf{m}^{(k-1)}.
\tag{18}
\]
Total plastic deformations are determined from formula
\[
\epsilon^{(k)} = \epsilon^{(k-1)} + \Delta \lambda^k \cdot \mathbf{m}^k.
\tag{19}
\]
In the case of the associative law of plastic flow, at
\(\alpha = \beta\) (\(\gamma = \phi\)), fluidity functions \(F\) (2) and \(G\) (9) converge. Then \(m = n\), and the direction of gain in plastic deformation
at current becomes normal to the yield surface. In
this case, it is necessary to replace \(m\) with \(n\) in formulas
(6)–(19). In other words, the problem is somewhat simpli-
fied. Further algorithm for solving the problem with the
associative law of flow is the same as in the case for the
non-associative one.
In order to numerically implement the proposed algo-
rithm, we used a finite element method (FEM) and the
high-level programming language Fortran employing the
integrated development environment Compaq Visual For-
tran [19]. In this case, a global matrix of SLAE is built in
the ribbon form and SLAE is solved by the Gaussian method
taking into account the ribbon form.

5. Results of numerical studies into elastic-plastic
state of bulk material

Testing of the developed programming code for solv-
ing a problem on the elastic-plasticity of bulk material
was performed on the example of a model material, char-
acterized by the associative law of current at different
values of the angles of native repose. Construction of the
tetrahedron grid was executed in the CAD-system for
grid generation Gmsh [20].
Test. A problem on the elastic-plasticity of a bulk mate-
rial using a classic model of Drucker-Prager. The estimated
area is three-dimensional, ... of a cone with radius
\[
r = \sqrt{x^2 + y^2} = 0.34 \, \text{m}
\]
and height \(z = 0.3 \, \text{m}\). Physical properties of the bulk ma-
terial: apparent density \(\rho = 2000 \, \text{kg/m}^3\), modulus of elas-
ticity \(E=4000 \, \text{Pa}\), Poisson's ratio \(\nu = 0.45\), adhesion force
between granules of the bulk material \(c=100 \, \text{Pa}\), an-
nge of native repose \(\varphi = 15, 10, 5\). Load: gravitation \(g =
9.81 \, \text{m/s}^2\). The associative law of current \(\alpha = \beta\). Boundary
conditions: fixing on the \(xOy\) plane – \(u_{z=0} = 0\), symmetry
along the \(xOz\) and \(yOz\) planes. That is, the terms of the prob-
lem, as well as the solution, correspond to a two-diimen-
sional axisymmetric problem.
Results of solving the problem, as well as their com-
parison with the data obtained using the software ANSYS
Mechanical APDL [21] for an axisymmetric geometry, are
given in Table 1.
Results of numerical simulation of the problem on plas-
ticity of bulk material with the use of the developed software
are shown in Fig. 1.
To visualize the results of calculations, we applied the
open-source graphic package for interactive visualization of
ParaView [22].
Comparison of solutions to the problem on plasticity of bulk material, obtained by employing our own programming code, and using the software ANSYS Mechanical APDL

<table>
<thead>
<tr>
<th>Type of solution</th>
<th>$u_r$, m</th>
<th>$\sigma_{eqM}$, Pa</th>
<th>$\sigma_{eqDP}$, Pa</th>
<th>$\varepsilon_{eqM}^{el}$</th>
<th>$\varepsilon_{eqM}^{pl}$</th>
<th>$\sigma_m$, Pa</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi=5^\circ$, $\sigma_y=473.89$ Pa</td>
<td>0.0970368</td>
<td>27.735–1000.69</td>
<td>–</td>
<td>0.0106–0.250173</td>
<td>0.0370229</td>
<td>–1002.03–259.964</td>
</tr>
<tr>
<td>$\phi=10^\circ$, $\sigma_y=482.81$ Pa</td>
<td>0.050676</td>
<td>7.409–1182.76</td>
<td>–</td>
<td>0.00733–0.29569</td>
<td>0.16233</td>
<td>–939.98–177.104</td>
</tr>
<tr>
<td>$\phi=15^\circ$, $\sigma_y=488.27$ Pa</td>
<td>0.051986</td>
<td>0.7–1342.2</td>
<td>–</td>
<td>0.00751–0.33525</td>
<td>0.05423</td>
<td>–876.238–151.593</td>
</tr>
</tbody>
</table>

Note: $N$ – number of nodes; $\text{El}$ – number of finite triangular or tetrahedron elements; $u_r$ – resultant displacement, m; $\sigma_{eqM}$ – equivalent stress by von Mises, Pa; $\sigma_{eqDP}$ – equivalent stress by Drucker-Prager, Pa; $\varepsilon_{eqM}^{el}$ – equivalent elastic deformation by von Mises; $\varepsilon_{eqM}^{pl}$ – equivalent plastic deformation by von Mises; $\sigma_m$ – hydrostatic pressure, Pa; $\phi$ – angle of native repose, degrees; $\sigma_y$ – yield limit of material, Pa

6. Discussion of results of numerical simulation of the elastic-plastic state of bulk material

An analysis of comparing the results showed that the data from modeling by using the developed software coincide with the numerical solutions, which were obtained by employing the commercial software ANSYS Mechanical APDL [21]. In this case, the maximal value of error for such magnitudes as $u_r$, $\sigma_{eqM}$, $\varepsilon_{eqM}^{el}$ and $\sigma_m$ is within 0.25–1.71%, for $\varepsilon_{eqM}^{pl}$ is 2.1–5.3%, which is sufficient enough to perform engineering calculations.

As a result of verification of the modified algorithm for solving a problem (1)–(7), it was established that:

- solving a linearized system of equations (14) at each step of iterations using the Gauss method instead of determining the inverse matrix makes it possible to reduce the number of arithmetic operations by about $3n^2$ for each plastic finite element;
- results of solving the problem with initial stresses via absolute values of displacements in line with (15), (16) coincide with the results obtained through the gains in displacements according to (17)–(19) under conditions of the test problem.

We consider the following benefits of present work:

- all the stages are presented in solving the problem, starting with mathematical statement, procedure of solving, all the way to numerical implementation and verification, which is a substantial methodological advantage compared to [5–17];
- modification of the return-mapping algorithm.

The shortcomings of the work are, probably, the following:

- the study is performed only for a classic model of Drucker-Prager with the associative law of current;
- the software is designed only for the application of linear tetrahedron finite elements;
- a lack of comparison of the results of numerical simulation with an experiment and the data obtained by using the ABAQUS software package.

The research results are useful when running a numerical analysis of physical fields of electro-thermal equipment.

Table 1

<table>
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<th>Type of solution</th>
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</table>

Fig. 1. Results of numerical simulation of the problem on plasticity of bulk material: $a$ – total displacements; $b$ – equivalent plastic deformations by von Mises; $c$ – equivalent stresses by Drucker-Prager; $d$ – hydrostatic pressure
which uses bulk materials, and are a continuation of previous studies [3, 4, 18].

Further research may address the ways of solving the problems on plastic behavior of bulk materials in the CAP approximation and extended Drucker-Prager models employing hexagonal finite elements.

7. Conclusions

1. Based on the classic model of Drucker-Prager, we formulated a mathematical statement of the problem on elastic-plastic behavior of isotropic bulk material.

2. We have improved the procedure for solving numerically a problem on the mechanical state of bulk material using the return-mapping algorithm. The modified technique makes it possible to reduce the number of arithmetic operations by about $3n^2$ ($n$ is the dimensionality of SLAE) at each iteration step for each plastic finite element.

3. We have tested the programming code on the example of a model material, characterized by the associative law of current, at different values of the angle of native repose. It was found that the maximal value of error for such magnitudes as $\epsilon_{eqM}$, $\sigma_{eqM}$ and $\sigma_s$ is within 0.25–1.71 %, and for $\epsilon_{eqM}$ is 2.1–5.3 %.

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**IDENTIFICATION OF THE ADDITIONAL EXPOSURE ZONE FOR ENSURING A COMPLETE CONTACT OF THE TWO-LAYERED SYSTEM**

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The phenomenon of separation of interlayer bonds is observed in the operation of highways, airfield pavements, foundations of high-rise buildings with supports or isolation joints. The contact between the building sole and the base often leads to emergency situations. Such phenomena are modeled by contact problems with unilateral constraints.

A detailed research of the fundamental laws of contact interaction requires a comprehensive consideration of the geometric features and imperfections of the conjugate surfaces, physical and mechanical phenomena in the areas of their direct attachment (friction, slippage, adhesion, etc.). The numerical solution of contact problems is usually carried out on the basis of the finite element method. For the modeling of unilateral constraints, various physical models are used. The research on the stress-strain state of the model and the state of the contact zone (presence of a friction zone and a separation zone) can be defined as the solution of a direct problem.

Studies of contact problems of mechanics of deformable media are conducted in two directions. Within the framework of the first direction, problems of conjugation of media with sharply differing mechanical properties (layer-inhomogeneous elastic and elastoplastic bodies, conjugation of solids to liquid or gaseous media, etc.) are considered. In such problems, the boundaries of the contact area are, as a rule, specified and not changed in the course of deformation. To solve these problems, the methods of the function theory of a complex variable and the theory of potentials with integral