UDC 532.59
DOI: 10.15587/1729-4061.2017.111941

WAVE PROPAGATION IN A THREE-LAYER SEMI-INFINITE HYDRODYNAMIC SYSTEM WITH A RIGID LID

O. Avramenko
Doctor of Physical and Mathematical Sciences, Professor, Head of Department*
E-mail: oavramenko777@gmail.com
M. Lunyova
Postgraduate student*
E-mail: luneva.mariya@ukr.net
V. Naradovyi
PhD, Senior Lecturer*
E-mail: naradvova1986@gmail.com
*Department of Applied Mathematics, Statistics and Economics
Volodymyr Vynnychenko Central Ukrainian State Pedagogical University
Shevchenka str., 1, Kropyvnytskyi, Ukraine, 25006

Розглянуто слабконелінійну модель поширення та взаємодії хвиль вздовж поверхонь контакту у гідродинамічній системі «рідкий півпростір – шар – шар з твердою кришкою», наведено перші три її лінійні наближення, отримано умову поширення хвиль вздовж поверхонь контакту. Проаналізовано залежність відношення амплітуд хвиль на поверхнях контакту при різних геометричних та фізичних параметрах системи. Досліджено структуру хвильових рухів на поверхнях контакту. Результати дослідження можуть бути використані при розробці алгоритмів детектування хвильових рухів у різних рідких середовищах

Ключові слова: взаємодія хвиль, тришарова гідродинамічна система, амплітуди хвиль, відношення амплітуд

Рассмотрена слабонелинейная модель распространения и взаимодействия волн вдоль поверхностей контакта в гидродинамической системе «жидкое полупространство – слой – слой с твёрдой крышкой», приведены первые три её линейные приближения, получено условие распространения волн вдоль поверхностей контакта. Проанализирована зависимость отношения амплитуд волн на поверхностях контакта от различных геометрических и физических параметров системы. Исследована структура волновых движений на поверхностях контакта. Результаты исследования могут быть использованы при разработке алгоритмов детектирования волновых движений в различных жидких средах

Ключевые слова: взаимодействие волн, трехслойная гидродинамическая система, амплитуды волн, отношение амплитуд

1. Introduction

Exploration of wave phenomena in fluids of different types is one of the most difficult problems of modern science as it needs serious mathematical methods and construction of quite sophisticated mathematical models. Numerical and analytical research in such models requires creation of new approaches and application of modern CMS. A new impetus to the study of waves in the World Ocean was given by practical use of the energy component of propagation and interaction of waves. One of the problems with this is timely detection of internal wave motions.

2. Literature review and problem statement

Research into properties of propagation and interaction of different wave motions in hydrodynamic systems of various types is one of the relevant problems of modern theoretical and experimental hydrodynamics.

In paper [1], the problem of oscillations of the contact surface of two non-mixed viscous non-compressible liquids over the hard bottom in the gravitational field was explored. Correctness of the problem both with taking into account surface tension and without it was proved. The paper considered the case when heavier fluid is above. In this case, the Rayleigh-Taylor instability was found to be stabilized due to significant surface tension.

Article [2] examines high-frequency internal wave motions, observed in the Mediterranean Sea at a considerable depth. These wave motions are unusual for this sea. The causes of occurrence of such waves, associated with climate changes, were established.

Study [3] describes a new method of detection of parameters of internal waves by SAR-image. The new method was tested in the South China Sea, where the amplitudes and the phase velocities of internal waves were studied. Actual data were compared with the data, obtained using a new method of analysis of SAR images. Root mean square and relative error are 1 % and 7 %, respectively, which indicates applicability of the new method.

Paper [4] is devoted to studying scattering of linear waves in periodic multi-layer media. The Hemholtz equation is solved numerically using the method of perturbation of high order. The results of numerical modeling show spectral convergence of the proposed method.

Article [5] contains solution of the Korteweg-de-Vries equation in the assigned form of nodal waves and solitons for shallow water. Based on the obtained solution, numerical analysis for studying wave motions in the Baltic Sea in the framework of the three-layered model was carried out. An increase in the amplitude of a wave and a decrease in phase velocity in shallow waters were clearly demonstrated. Critical parameters of internal waves were determined.

In paper [6], wave motions in the South China Sea and related phenomena are explored. The topography of the double ridge, the Earth's rotation, changes in stratification have great impact on wave propagation. Assessment of these effects allows us with great precision to simulate internal tides in the sea. The specified effects were found to lead to a daily decrease in barotropic internal tides in the Luzon strait.

Research [7] is devoted to exploration of wave propagation in a three-layer system "layer of reinforced fiber – porous cross-elastic layer – viscoelastic half-space". Based on solution for velocities, obtained in a closed analytic form, a significant effect of parameters on phase velocity of wave propagation was revealed. The resulting model was tested for the system, which consists of the upper epoxy resin layer, reinforced by fiber, the cross-elastic layer of sandstone and the viscoelastic half-space.

In paper [8], internal waves, which propagate over the ridge in a two-layered liquid, were studied. The paper considered three types of interaction: weak, moderate or strong. It was found that different types of interaction between waves and the ridge are related to a modified blockage degree. It was found that the maximum speed of a wave, loss of energy and amplitude of waves have self-similar characteristic with the blockage degree.

In article [9], dispersion analysis of surface waves, which occur on the surface of the contact of liquid and homogeneous elastic body, was conducted. Analysis is performed by finding the roots of secular equation and research in physical content. Complete analysis of variance, with special focusing on the frequency range, in which phase velocity of shear waves is higher than velocity of fluid waves, was conducted.

Paper [10] addresses experimental study of formation of harmonic waves as a result of interaction of internal waves. It was found that at a collision of two non-resonant internal waves, harmonics are formed by the sum and difference of multiple frequencies of waves' collision. Phenomenon of transfer of relative kinetic energy from non-resonant waves to formation of harmonics after a collision was discovered experimentally

Paper [11] studied stability of gravity surface waves in a fluid of finite depth in the presence of surface tension. Solutions in the form of periodic waves that are formed under the influence of gravity and surface tension were found. Spectral stability of these waves was examined by the Hill method. It was revealed that taking into account surface tension does not lead to disappearance of phenomena of waves' instability, which was proved for waves of small amplitude.

Article [12] is dedicated to the study of periodic waves at the surface of the contact between two liquid layers of constant density, including waves with overturned crests. The study of these waves is carried out with assumption of continuity by physical parameters: the Bond and Atwood numbers, as well as by an average shear. Using the continuation method, different cases, which illustrate application of the criteria of the global bifurcation theorem of Ambrose, are considered. Bifurcation surfaces, which are both inverted and self-intersecting, are constructed. The relationship between the Stokes second harmonics of wave propagation and the form of bifurcation surfaces is outlined.

Study [13] presented a new non-hydrostatic model of the ocean with the coordinate system that is isopycnal by the vertical. The motivation of introduction of such a system lies in proper consideration of non-hydrostatic dispersion and studying formation of nonlinear internal single waves. Consideration of such a model in terms of calculation is the best because it allows us to use a smaller number of nodes of the computational grid and eliminates losses of amplitude of single waves. It was demonstrated that the specified model can describe nonlinear internal single waves for simplified the physically realistic problems of the ocean.

In paper [14], the authors received a new system of equations of the Boussinesq type for the study of interaction between long nonlinear waves in two-layered fluid of finite depth. Based on the derived equations, they developed an analytical model for the study of the evolution of the resonant triad, which consists of a surface wave and two sub-harmonious internal waves. The paper considers wave attenuation associated with weak viscosity of fluid. It was found that in viscous liquids, amplitude of surface waves should be larger than some critical value in order to overcome attenuation and to cause internal response waves. Dependence of critical amplitude, as well as the rate of increase and attenuation of internal waves on the depth, density and ratio of viscosity of liquid layers, amplitude and frequency of a surface wave, was studied.

Article [15] examines capillary-gravity wave motions in three-layer fluid in linear approximation. The authors obtained dispersion ratio and found analytical expressions for relationship of amplitudes of the waves that propagate on the contact surfaces. The latest ratios were analyzed depending on parameters of a hydrodynamic system.

Paper [16] proposes an analysis of waves' propagation in the two-layer hydrodynamic system "hard-bottomed layer – free surface layer". Using the method of large-scale expansions, the first three linear approximations of a nonlinear problem were obtained. Solutions of the first linear approximation were constructed and analyzed. Based on analysis of the ratio of amplitudes of the internal and surface waves, the interaction of wave motions in the studied system was analyzed. Further research of propagation and interaction of wave packets in the system "hard-bottomed layer – free surface layer" was carried out in papers [17–19], where, in particular, modular stability of wave packets was analyzed and conditions of origin and shape of waves were studied.

A large number of studies, related to propagation and interaction of wave motions in different types of systems are associated with mathematical complexity and lack of general approaches when constructing models. In fact, for each hydrodynamic system, you need to construct a new, and, often, more than one model, which makes it possible to fully explore wave processes in this system.

3. The aim and objectives of the study

The aim of present research is to analyze propagation and interaction of waves along the contact surfaces in a three-layer hydrodynamic system "liquid half-space – layer – layer with a lid". This will give the opportunity to assess more accurately and in detail the wave processes and wave interaction of three-layer systems.

To accomplish the set goal, the following problems were to be solved:

– to obtain the first three linear approximations of the weakly nonlinear problem that is explored;

– to determine the structure of wave motions in the studied system based on the ratio of amplitudes of the waves that propagate along different contact surfaces.

4. Statement and solution to a problem of waves' propagation in a three-layer hydrodynamic system

4. 1. Mathematical statement of the problem and research method

We explore the problem of propagation of three-dimensional wave packets of finite amplitude on the surface of a liquid layer

$$\Omega_1=\{(x,z):\ |x|<\infty,\ -\infty\le z<0\}$$

with density ρ_1, a middle liquid layer

$$\Omega_3=\{(x,z):\ |x|<\infty,\ h_2\le z<h_2+h_3\}$$

with density ρ_2 and the upper liquid layer

$$\Omega_3=\{(x,z):\ |x|<\infty,\ h_2\le z<h_2+h_3\}$$

with density ρ_3. Layers Ω_1 and Ω_2 are separated by the contact surface $z=\eta_1(x, t)$, and layers Ω_2 and Ω_3 are separated by the contact surface $z=h_2+\eta_2(x, t)$. When solving the problem, we take into account the forces of surface tension on the surfaces of contact. Gravity is directed perpendicularly to the contact surface in negative z-direction.

Mathematical statement of the problem has the following form:

$$\frac{\partial^2\varphi_j}{\partial x^2}+\frac{1}{\beta}\frac{\partial^2\varphi_j}{\partial z^2}=0 \text{ at } \Omega_j,\ j=1,2,3,$$

$$\frac{\partial\eta_1}{\partial t}-\frac{1}{\beta}\frac{\partial\varphi_j}{\partial z}=-\alpha\frac{\partial\eta_1}{\partial x}\frac{\partial\varphi_j}{\partial x} \text{ at } z=\alpha\eta_1(x,t),\ j=1,2,$$

$$\frac{\partial\eta_2}{\partial t}-\frac{1}{\beta}\frac{\partial\varphi_j}{\partial z}=-\alpha\frac{\partial\eta_2}{\partial x}\frac{\partial\varphi_j}{\partial x}$$

$$\text{at } z=h_2+\alpha\eta_2(x,t),\ j=2,3, \qquad (1)$$

$$\rho_1\frac{\partial\varphi_1}{\partial t}-\rho_2\frac{\partial\varphi_2}{\partial t}+(\rho_1-\rho_2)\eta_1+$$
$$+0.5\rho_1\alpha\left(\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_1}{\partial z}\right)^2\right)-$$
$$-0.5\rho_2\alpha\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-$$
$$-T_1\left[1+\left(\alpha\sqrt{\beta}\frac{\partial\eta_1}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_1}{\partial x^2}=0$$

at $z=\alpha\eta_1(x,t)$,

$$\rho_2\alpha\frac{\partial\varphi_2}{\partial t}-\rho_3\alpha\frac{\partial\varphi_3}{\partial t}+\alpha\eta_2(\rho_2-\rho_3)+$$
$$+h_2(\rho_2-\rho_3)+0.5\rho_2\alpha^2\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-$$
$$-0.5\rho_3\alpha^2\left(\left(\frac{\partial\varphi_3}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_3}{\partial z}\right)^2\right)-$$
$$-T_2\alpha\left[1+\left(\alpha\sqrt{\beta}\frac{\partial\eta_2}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_2}{\partial x^2}=0$$

at $z=h_2+\alpha\eta_2(x,t)$,

$$\frac{\partial\varphi_3}{\partial z}=0 \text{ at } z=h_2+h_3,$$

$$|\vec{\varphi_1}|\to 0 \text{ at } z\to-\infty,$$

where α and β are the coefficients of nonlinearity. Subsequently, we will consider the case when $\alpha<1$, and $\beta=1$.

To solve the set problem, we will use the method of multyscale expansions of the third order [20]. We will represent the desired functions of elevation of contact surfaces and potentials of velocities in the form

$$\eta_i(x,t)=\sum_{n=1}^{3}\alpha^{n-1}\eta_{in}(x_0,x_1,x_2,t_0,t_1,t_2)+O(\alpha^3),\ i=1,2,$$

$$\varphi_j(x,z,t)=\sum_{n=1}^{3}\alpha^{n-1}\varphi_{jn}(x_0,x_1,x_2,z,t_0,t_1,t_2)+O(\alpha^3),$$

$$j=1,2,3, \qquad (2)$$

where $x_k=\alpha^k x$ and $t_k=\alpha^k t$ ($k=0, 1, 2$).

Below, there are notations of linear approximations and analysis of the roots of dispersion equation.

4. 2. Statement of the first three approximations of the problem

Having applied the method of multyscale expansions and equating expressions at the same powers α, we will get three linear problems.

The problem on the first approximation (at α^0)

$$\varphi_{j1,x_0x_0}+\varphi_{j1,zz}=0 \text{ in } \Omega_j,\ j=1,2,3,$$

$$\eta_{11,t_0}+\varphi_{j1,z}=0 \text{ at } z=0,\ j=1,2,$$

$$\eta_{21,t_0}+\varphi_{j1,z}=0 \text{ at } z=h_2,\ j=2,3, \qquad (3)$$

$$\rho_1\varphi_{11,t_0}-\rho_2\varphi_{21,t_0}+(\rho_1-\rho_2)\eta_{11}-T_1\eta_{11,x_0x_0}=0 \text{ at } z=0,$$

$$\rho_2\varphi_{21,t_0}-\rho_3\varphi_{31,t_0}+(\rho_2-\rho_3)\eta_{21}-T_2\eta_{21,x_0x_0}=0 \text{ at } z=h_2,$$

$$\varphi_{31,z}=0 \text{ at } z=h_2+h_3,$$

$$\varphi_{11}\to 0 \text{ at } z\to-\infty.$$

The problem on the second approximation (at α^1)

$$\varphi_{j2,x_0x_0}+\varphi_{j2,zz}=-2\varphi_{j1,x_0x_1} \text{ in } \Omega_j,\ j=1,2,3,$$

$$\eta_{12,t_0}-\varphi_{j2,z}=-\eta_{11,t_1}-\eta_{11,x_0}\varphi_{j1,x_0}+\eta_{11}\varphi_{j1,zz} \text{ at } z=0,\ j=1,2,$$

$$\eta_{22,t_0}-\varphi_{j2,z}=-\eta_{21,t_1}-\eta_{21,x_0}\varphi_{j1,x_0}+\eta_{21}\varphi_{j1,zz}$$

at $z=h_2,\ j=2,3,$

$$\rho_1\varphi_{12,t_0}-\rho_2\varphi_{22,t_0}+(\rho_1-\rho_2)\eta_{12}-T_1\eta_{12,x_0x_0}=$$
$$=-\rho_1(\varphi_{11,t_1}+\eta_{11}\varphi_{11,t_0z})+\rho_2(\varphi_{21,t_1}+$$
$$+\eta_{11}\varphi_{21,t_0z})-0.5\rho_1(\varphi_{11,x_0}^2+\varphi_{11,z}^2)+$$
$$+0.5\rho_2(\varphi_{21,x_0}^2+\varphi_{21,z}^2)+2T_1\eta_{11,x_0x_1}$$

at $z=0,$

$$\rho_2\varphi_{22,t_0}-\rho_3\varphi_{32,t_0}+(\rho_2-\rho_3)\eta_{22}-T_2\eta_{22,x_0x_0}=$$
$$=-\rho_2(\varphi_{21,t_1}+\eta_{21}\varphi_{21,t_0z})+\rho_3(\varphi_{31,t_1}+$$
$$+\eta_{21}\varphi_{31,t_0z})-0.5\rho_2(\varphi_{21,x_0}^2+\varphi_{21,z}^2)+$$
$$+0.5\rho_3(\varphi_{31,x_0}^2+\varphi_{31,z}^2)+2T_2\eta_{21,x_0x_1}$$

at $z=h_2,$

$$\varphi_{32,z}=0 \text{ at } z=h_2+h_3,$$

$$\varphi_{12}\to 0 \text{ at } z\to-\infty.$$

The problem on the third approximation (at α^2)

$$\varphi_{j3,x_0x_0}+\varphi_{j3,zz}=-\varphi_{j1,x_1x_1}-2\varphi_{j1,x_0x_1}-2\varphi_{j1,x_0x_2}$$

in $\Omega_j,\ j=1,2,3,$

$$\eta_{13,t_0}-\varphi_{j3,z}=-\eta_{11,t_2}-\eta_{12,t_1}-\eta_{11,x_0}\varphi_{j1,x_1}-$$
$$-\eta_{11,x_1}\varphi_{j1,x_0}-\eta_{12,x_0}\varphi_{j1,x_0}-\eta_{11,x_0}\varphi_{j2,x_0}+$$
$$+\eta_{11}\varphi_{j2,zz}+\eta_{12}\varphi_{j1,zz}+0.5\eta_{11}^2\varphi_{j1,zzz}-\eta_{11}\eta_{11,x_0}\varphi_{j1,x_0z}$$

at $z=0,\ j=1,2,$

$$\eta_{23,t_0}-\varphi_{j3,z}=-\eta_{21,t_2}-\eta_{22,t_1}-\eta_{21,x_0}\varphi_{j1,x_1}-$$
$$-\eta_{21,x_1}\varphi_{j1,x_0}-\eta_{22,x_0}\varphi_{j1,x_0}-\eta_{21,x_0}\varphi_{j2,x_0}+$$
$$+\eta_{21}\varphi_{j2,zz}+\eta_{22}\varphi_{j1,zz}+0.5\eta_{21}^2\varphi_{j1,zzz}-\eta_{21}\eta_{21,x_0}\varphi_{j1,x_0z}$$

at $z=h_2,\ j=2,3,$

$$\rho_1\varphi_{13,t_0}-\rho_2\varphi_{23,t_0}+(\rho_1-\rho_2)\eta_{13}-T_1\eta_{13,x_0x_0}=$$
$$=-\rho_1(\varphi_{11,t_2}+\varphi_{12,t_1}+\eta_{12}\varphi_{11,t_0z}+$$
$$+\eta_{11}\varphi_{11,t_1z}+0.5\eta_{12}^2\varphi_{11,t_0zz})+$$
$$+\rho_2(\varphi_{21,t_2}+\varphi_{22,t_1}+\eta_{12}\varphi_{21,t_0z}+\eta_{11}\varphi_{21,t_1z}+$$
$$+0.5\eta_{12}^2\varphi_{21,t_0zz})-\rho_1(\eta_{11}\varphi_{12,t_0z}+\varphi_{11,x_0}\varphi_{11,x_1}+$$
$$+\varphi_{11,x_0}\varphi_{12,x_1}+\eta_{11}\varphi_{11,x_0}\varphi_{11,x_0z}+$$
$$+\varphi_{11,z}\varphi_{11,z}+\eta_{11}\varphi_{11,z}\varphi_{11,zz})+\rho_2(\eta_{11}\varphi_{22,t_0z}+$$
$$+\varphi_{21,x_0}\varphi_{21,x_1}+\varphi_{21,x_0}\varphi_{22,x_1}+$$
$$+\eta_{11}\varphi_{21,x_0}\varphi_{21,x_0z}+\varphi_{21,z}\varphi_{21,z}+\eta_{11}\varphi_{21,z}\varphi_{21,zz})+$$
$$+T_1\eta_{11,x_1x_1}+2T_1\eta_{12,x_0x_1}+2T_1\eta_{12,x_0x_2}-$$
$$-1.5T_1(\eta_{11,x_0})^2\eta_{11,x_0x_0}$$

at $z=0,$

$$\rho_2\varphi_{23,t_0}-\rho_3\varphi_{33,t_0}+(\rho_2-\rho_3)\eta_{23}-T_2\eta_{23,x_0x_0}=$$
$$=-\rho_2(\varphi_{21,t_2}+\varphi_{22,t_1}+\eta_{22}\varphi_{21,t_0z}+$$
$$+\eta_{21}\varphi_{21,t_1z}+0.5\eta_{22}^2\varphi_{21,t_0zz})+$$
$$+\rho_3(\varphi_{31,t_2}+\varphi_{32,t_1}+\eta_{22}\varphi_{31,t_0z}+\eta_{21}\varphi_{31,t_1z}+0.5\eta_{22}^2\varphi_{31,t_0zz})-$$
$$-\rho_2(\eta_{21}\varphi_{22,t_0z}+\varphi_{21,x_0}\varphi_{21,x_1}+\varphi_{21,x_0}\varphi_{22,x_1}+$$
$$+\eta_{21}\varphi_{21,x_0}\varphi_{21,x_0z}+\varphi_{21,z}\varphi_{21,z}+\eta_{21}\varphi_{21,z}\varphi_{21,zz})+$$
$$+\rho_3(\eta_{21}\varphi_{32,t_0z}+\varphi_{31,x_0}\varphi_{31,x_1}+\varphi_{31,x_0}\varphi_{32,x_1}+\eta_{21}\varphi_{31,x_0}\varphi_{31,x_0z}+$$
$$+\varphi_{31,z}\varphi_{31,z}+\eta_{21}\varphi_{31,z}\varphi_{31,zz})+T_2\eta_{21,x_1x_1}+$$
$$+2T_2\eta_{22,x_0x_1}+2T_2\eta_{22,x_0x_2}-1.5T_2(\eta_{21,x_0})^2\eta_{21,x_0x_0}$$

at $z=h_2,$

$$\varphi_{33,z}=0 \text{ at } z=h_2+h_3,$$

$$\varphi_{13}\to 0 \text{ at } z\to-\infty.$$

5. Dispersion equation and solutions of the first approximation

The problem of propagation of waves in a three-layer hydrodynamic system in the first approximation is a linear boundary value problem of the second order, the solutions of which were sought for in the form of progressive waves. As a result, the condition of waves' origin in the studied three-layer system in the form of the dispersion equation was detected (4).

$$\rho_3\omega^2\mathrm{cth}(kh_3)+$$
$$+\frac{\rho_2\omega^2(-\rho_2\omega^2+(-\rho_1\omega^2+k(\rho_1-\rho_2)+T_1k^3)\mathrm{cth}(kh_2))}{-\rho_2\omega^2\mathrm{cth}(kh_2)+(-\rho_1\omega^2+k(\rho_1-\rho_2)+T_1k^3)}=$$
$$=k(\rho_2-\rho_3)+T_2k^3. \tag{4}$$

Dispersion equation (4) is a biquadratic equation relative to frequency ω

$$\left[-\rho_2\rho_3\mathrm{cth}(kh_2)\mathrm{cth}(kh_3)-\rho_1\rho_3\mathrm{cth}(kh_3)-\right.$$
$$\left.-\rho_2^2-\rho_1\rho_2\mathrm{cth}(kh_2)\right]\omega^4+$$
$$+\left[\rho_3k(\rho_1-\rho_2)\mathrm{cth}(kh_3)+T_1k^3\rho_3\mathrm{cth}(kh_3)+\right.$$
$$+\rho_2k(\rho_1-\rho_2)\mathrm{cth}(kh_2)+T_2k^3\rho_2\mathrm{cth}(kh_2)+$$
$$\left.+T_2k^3\rho_2\right]\omega^2-k(\rho_2-\rho_3)(\rho_1-\rho_2)+$$
$$+T_1k^4(\rho_2-\rho_3)-T_2k^4(\rho_1-\rho_2)-T_1k^6T_2. \tag{5}$$

Equation (5) has two pairs of roots – frequencies of centers of the wave package

$$\omega_1^2=\frac{-b-\sqrt{b^2-4ac}}{2a},\quad \omega_2^2=\frac{-b+\sqrt{b^2-4ac}}{2a}, \tag{6}$$

where

$$a=\rho_2\rho_3\mathrm{cth}(kh_2)\mathrm{cth}(kh_3)+\rho_1\rho_3\mathrm{cth}(kh_3)+\rho_2^2+\rho_1\rho_2\mathrm{cth}(kh_2),$$

$$b=-k(\rho_1-\rho_2)(\rho_2\mathrm{cth}(kh_2)+\rho_3\mathrm{cth}(kh_3))-T_1k^3(\rho_2\mathrm{cth}(kh_2)+$$
$$+\rho_3\mathrm{cth}(kh_3))-(\rho_1+\rho_2\mathrm{cth}(kh_2))(k(\rho_2-\rho_3)+T_2k^3),$$

$$c=(T_1k^3+k(\rho_1-\rho_2))(k(\rho_2-\rho_3)+T_2k^3).$$

Physical reliability of the results is proved by two limited cases, in which a three-layer system degenerates into a two-layer system "layer with a lid – half-space" [21].

The first limited case occurs under conditions of equality of two layers' density $\rho_2=\rho_3=0.9$. In this case, we have a layer of liquid, limited from the top by a rigid lid and liquid half-space under it with the contact surface contact η_1. The second limited case occurs under conditions of equality of two lower layers' density $\rho_1=\rho_2=1$. In this case, we have a layer of liquid, bounded from the top by the lid and half-space with contact surface η_1. Comparative analysis of frequencies from the thickness of the top layer showed coincidences of the indicated dependencies with shear by thickness of the middle layer $\rho_2=1$, which also proves physical reliability.

According to (6), there are two pairs of independent solutions for a linear problem (3),

– for ω_1:

$$\eta_{11}^{(1)}=A\cos(kx-\omega_1t),$$

$$\eta_{21}^{(1)}=\frac{\rho_2\omega_1^2\text{ch}(kh_2)+(\rho_1\omega_1^2-k\rho_1+k\rho_2-T_1k^3)\text{sh}(kh_2)}{\rho_2\omega_1^2}\times$$
$$\times A\cos(kx-\omega_1t),$$

$$\varphi_{11}^{(1)}=\frac{\omega_1}{k}\exp(kz)A\sin(kx-\omega_1t),$$

$$\varphi_{21}^{(1)}=\left[\frac{(-\rho_1\omega_1^2+k(\rho_1-\rho_2)+T_1k^3)\text{ch}(k(h_2-z))}{\rho_2\omega_1k\text{ch}(kh_2)}+\right.$$
$$\left.+\frac{(-\rho_2\omega_1^2\text{ch}(kh_2)+(-\rho_1\omega_1^2+k(\rho_1-\rho_2)+T_1k^3)\text{sh}(kh_2))\text{sh}(kz)}{\rho_2\omega_1k\text{ch}(kh_2)}\right]\times$$
$$\times A\sin(kx-\omega_1t),$$

$$\varphi_{31}^{(1)}=\frac{(-\rho_2\omega_1^2\text{ch}(kh_2)+(-\rho_1\omega_1^2+k(\rho_1-\rho_2)+T_1k^3)\text{sh}(kh_2))\text{ch}(k(h_2+h_3-z))}{\rho_2\omega_1k\text{sh}(kh_3)}\times$$
$$\times A\sin(kx-\omega_1t);$$

– for ω_2:

$$\eta_{11}^{(2)}=\frac{\rho_2\omega_2^2}{\rho_2\omega_2^2\text{ch}(kh_2)+(\rho_1\omega_2^2-k\rho_1+k\rho_2-T_1k^3)\text{sh}(kh_2)}\times$$
$$\times B\cos(kx-\omega_2t),$$

$$\eta_{21}^{(2)}=B\cos(kx-\omega_2t),$$

$$\varphi_{11}^{(2)}=\frac{\omega_2}{k}\exp(kz)\times$$
$$\times\frac{\rho_2\omega_2^2}{\rho_2\omega_2^2\text{ch}(kh_2)+(\rho_1\omega_2^2-k\rho_1+k\rho_2-T_1k^3)\text{sh}(kh_2)}\times$$
$$\times B\sin(kx-\omega_2t),$$

$$\varphi_{21}^{(2)}=\left[\frac{\omega_2(-\rho_1\omega_2^2+k(\rho_1-\rho_2)+T_1k^3)\text{ch}(k(h_2-z))}{k(\rho_2\omega_2^2\text{ch}(kh_2)+(\rho_1\omega_2^2-k\rho_1+k\rho_2-T_1k^3)\text{sh}(kh_2))\text{ch}(kh_2)}-\right.$$
$$\left.-\frac{\omega_2\text{sh}(kz)}{k\text{ch}(kh_2)}\right]B\sin(kx-\omega_2t),$$

$$\varphi_{31}^{(2)}=\frac{\omega_2\text{ch}(k(h_2+h_3-z))}{k\text{sh}(kh_3)}B\sin(kx-\omega_2t), \qquad (7)$$

where $\eta_{21}^{(1)}$ is the wave-response to wave $\eta_{11}^{(1)}$ with frequency ω_1 and amplitude A, which propagates on the lower contact surface. $\eta_{11}^{(2)}$ is the wave-response to wave $\eta_{21}^{(2)}$ with frequency ω_2 and amplitude B, which propagates on the upper contact surface.

6. Analysis of solutions of the problem of propagation of wave processes in the examined system

6. 1. Dependence of ratios of amplitudes of elevations of contact surfaces on geometrical parameters

In the first approximation of the problem, we will carry out analysis of the ratio of amplitudes of elevations of the lower surface of contact $\eta_1(x,t)$ and upper surface of contact $\eta_2(x,t)$, which correspond to ω_1^2 and ω_2^2. We will designate these ratios as a_1 and a_2, respectively

$$a_1=\frac{\rho_2\omega_1^2\text{ch}(kh_2)+(\rho_1\omega_1^2-k\rho_1+k\rho_2-T_1k^3)\text{sh}(kh_2)}{\rho_2\omega_1^2}, \qquad (8)$$

$$a_2=\frac{\rho_2\omega_2^2}{\rho_2\omega_2^2\text{ch}(kh_2)+(\rho_1\omega_2^2-k\rho_1+k\rho_2-T_1k^3)\text{sh}(kh_2)},$$

where a_1 characterizes contribution of the wave with frequency ω_1 to wave motion on the surface of contact of liquid media Ω_2 and Ω_3, and value a_2 is the contribution of the wave with frequency ω_2 to wave motion on the surface of contact of liquid media Ω_1 and Ω_2.

Fig. 1, *a, b* shows dependences of a_1 and a_2 on the thickness of the upper layer h_3 at different wave numbers k=0.1, 0.3, 0.5, 0.7, 1 in the case when $h_2=1$, $T=T_0=0$, $\rho_1=1$, $\rho_2=0.9$, $\rho_3=0.8$. At an increase in the thickness of the upper layer for each fixed value of k, there are limited values, to which values a_1 and a_2 tend, in this case coefficient a_1 acquires positive values, while coefficient a_2 acquires negative values.

Fig. 1, *c, d* shows dependences of a_1 and a_2 on the thickness of the upper layer h_3 at different wave numbers k=0.1, 0.3, 0.5, 0.7, 1 in the case when $h_2=1$, $T=T_0=0$, $\rho_1=1$, $\rho_2=0.95$, $\rho_3=0.8$. As in the previous graph, for each fixed value of k, there are limited values, to which values a_1 and a_2 tend.

We will note that at equal density jump on the contact surfaces $\rho_1-\rho_2=\rho_2-\rho_3=0.1$, coefficients a_1 and a_2 are equal by module (Fig. 1, *a, b*). If density jump on the lower contact surface $\rho_1-\rho_2=0.05$, and density jump on the upper contact surface $\rho_2-\rho_3=0.15$, then $|a_1|>|a_2|$. Change in density ρ_2 significantly influences both a_1 for waves with frequency ω_1, and a_2 for waves with frequency ω_2. An increase in value of density of the middle layer ρ_2 from 0.9 to 0.95 leads to an increase in values of a_1 and in increase in absolute value of a_2, in addition, a change in density ρ_2 has much more considerable impact on a_2 than on a_1.

We will note that when plotting the charts, the region of determining of a_1 and a_2 was taken in the interval from $h_3=1$, because for small values of thickness of the upper layer, the linear model is not acceptable.

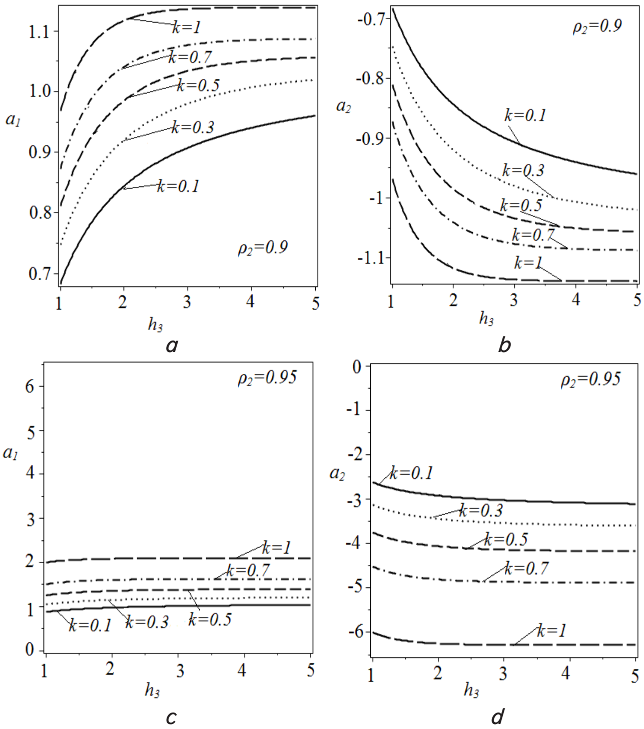

Fig. 1. Dependence of a_1 and a_2 on thickness of the upper layer h_3 at k=0.1, 0.3, 0.5, 0.7, 1, h_2=1, T=T_0=0, ρ_1=1, ρ_3=0.8; *a, b* – ρ_2=0.9; *c, d* – ρ_2=0.95

6. 3. Analysis of dependences of ratios of amplitudes of contact surfaces' elevation on physical parameters

Analysis of the ratio of amplitudes of the waves that propagate along the contact surfaces, depending on a change in ratio of densities at different wave numbers, was performed (Fig. 2).

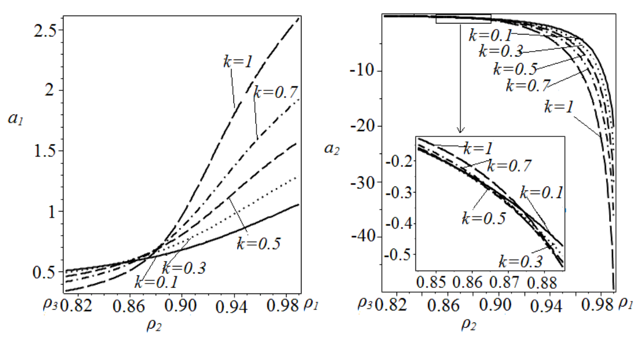

Fig. 2. Dependences of a_1 and a_2 on density of medium layer ρ_2 at k=0.1, 0.3, 0.5, 0.7, 1; h_2=1, h_3=1, T=T_0=0, ρ_1=1, ρ_3=0.8

Fig. 2 shows the graph of dependence of a_1 and a_2 on density of the middle layer ρ_2 at different values of wave number k=0.1, 0.3, 0.5, 0.7, and in the case when h_2=1, h_3, T=T_0=0, ρ_1=1, ρ_3=0.8.

It can be seen that at an increase in ρ_2, absolute values of coefficients a_1 and a_2 increase, in addition, for smaller values of the wave number, an increase is slower than for the case where the wave number is larger. It was also discovered that under certain conditions, coefficients a_1 and a_2 coincide for different values of wave numbers. At an increase in wave number k and at values of density of the middle layer ρ_2, close to density of the upper layer ρ_3, absolute values of coefficients a_1 and a_2 decrease. Otherwise, when values of density of the middle layer are close to the value of density of the lower layer of density ρ_1, absolute values of coefficients a_1 and a_2 increase

The results, obtained for ω_1 indicate that a decrease in density jump on the lower contact surface of the two liquid media and a simultaneous increase in density jump on the upper contact surface lead to formation of large elevation on the upper surface of the contact. At the same time, at the actual absence of density jump on the lower contact surface, another solution of dispersion equation (frequency ω_2) tends to zero, which causes the need for a detailed study of this limited case. Fig. 3, *a, b* shows the graph of dependence of a_1 and a_2 on wave number k at different values of thickness of the upper layer h_3 when h_2=1, T=T_0=0, ρ_1=1, ρ_2=0.9, ρ_3=0.8.

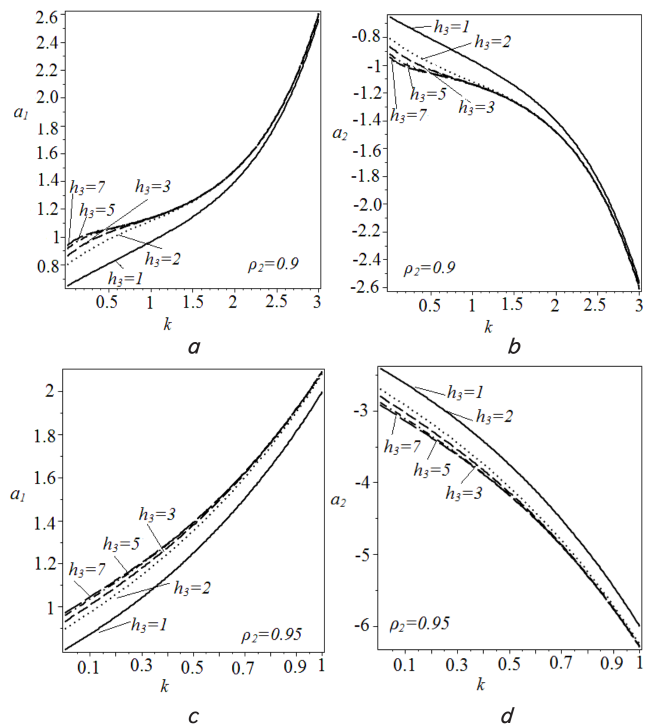

Fig. 3. Dependences of a_1 and a_2 on wave number k at h_3=1, 2, 3, 5, 7; h_2=1, T=T_0=0, ρ_1=1, ρ_3=0.8: *a, b* – ρ_2=0.9; *c, d* – ρ_2=0.95

At values of thickness of the upper layer h_3=2, 3, 5, 7 graphs of dependences of a_1 and a_2 on wave number k are located not far from each other and very quickly converge to a common limited value. At the same time, the graphs of dependences of a_1 and a_2 on wave number k at value of thickness of the upper layer h_3=1 have a somewhat separated character and not so quickly converge to the same limited value. Similar to dependences of a_1 and a_2 on h_3, there is equality of absolute values of $|a_1|=|a_2|$ if density jump on the contact surface are equal to $\rho_1-\rho_2=\rho_2-\rho_3=0.1$.

Fig. 3, *c, d* shows the graph of dependence of a_1 and a_2 on wave number k at different values of thickness of the upper layer h_3 in case when h_2=1, T=T_0=0, ρ_1=1, ρ_2=0.95, ρ_3=0.8.

At value of density of the medium layer ρ_2=0.95, density jump on the lower contact surface $\rho_1-\rho_2$=0.05, and on the upper contact surface $\rho_2-\rho_3$=0.15, in this case, the ratio $|a_1|>|a_2|$ is met, unlike dependences, considered above for ρ_2=0.9 (Fig. 3, *a, b*). The ratio between absolute values of $|a_1|$ and $|a_2|$ of dependence on k for ρ_2=0.9 and ρ_2=0.95 agree with similar results of dependences on h_3, represented in 6.1.

We will note that for all cases, shown in Fig. 3, absolute values of a_1 and a_2 for gravity elevations are smaller than for capillary ones

6. 4. Propagation of waves along surfaces of contact at different moments of time

Relationship between amplitudes $A^{'}$ and $A^{'}$ of waves of response to the waves on the lower and upper surfaces of contact A and B are accordingly determined from ratio

$$A^{'}=\frac{\rho_2\omega_1^2\mathrm{ch}(kh_2)+(\rho_1\omega_1^2-k\rho_1+k\rho_2-T_1k^3)\mathrm{sh}(kh_2)}{\rho_2\omega_1^2}\times$$
$$\times A=a_1A,$$

$$B^{'}=\frac{\rho_2\omega_2^2}{\rho_2\omega_2^2\mathrm{cth}(kh_2)+(\rho_1\omega_2^2-k\rho_1+k\rho_2-T_1k^3)\mathrm{sh}(kh_2)}\times$$
$$\times B=a_2B. \quad (9)$$

Elevation of contact surfaces in the first approximation consists of the sum of two harmonics

$$\eta_1=A\cos(kx-\omega_1t)+$$
$$+a_2B\cos(kx-\omega_2t), \quad (10)$$

$$\eta_2=a_1A\cos(kx-\omega_1t)+$$
$$+B\cos(kx-\omega_2t).$$

Here A and B are constants, which determine the amplitudes of harmonics with frequencies ω_1 and ω_2.

Fig. 4, *a–c* shows the pictures of wave motion on the lower surface of contact $\eta_1(x, t)$ and upper surface of contact $\eta_2(x, t)$ for the following values of parameters k=1, ρ_1=1, ρ_2=0.9, ρ_3=0.8, h_3=1, h_2=1 at different values of waves' amplitudes on the surface of the contact A and B: a – A=0.1, B=0.1; b – A=0.1, B=0.05; c – A=0.05, B=0.1 and at different moment of time t.

We can see the structure of wave processes on the lower and upper contact surfaces, in particular, contribution of main waves and waves-responses. Fig. 5, *a*, *b* shows the pictures of wave motion on the lower inner surface of contact $\eta_1(x, t)$ and on the upper inner surface of contact $\eta_2(x, t)$ for the following values of parameters k=0.1, ρ_1=1, ρ_3=0.8, h_3=1,3, h_2=1, A=0.1 B=0.1 at different values of density of the middle layer ρ_2=0.85, 0.9, 0.95, and at different moment of time t. As Fig. 5, *a*, *b* shows, under equal conditions, a change in density of the middle layer has a significant impact on amplitude of elevation of both the upper and the lower surfaces of contact, in this case, at approximation of value of density of the middle layer ρ_2 to value ρ_1=1, this influence is more pronounced on the lower surface of the contact.

For pre-set parameters of the hydrodynamic system for both cases of thickness of the upper layer h_3=1 and h_3=3, a change in density of the middle layer ρ_2=0.85, 0.9, 0.95 has a significant impact on the amplitude of the lower surface of contact. When it comes to the upper surface of contact, its elevation is influenced by thickness of the upper layer h_3. In particular, at h_3=1 and at density ρ_2=0.85 at certain moment of time, amplitude of elevation is significantly larger than amplitude at ρ_2=0.9 and ρ_2=0.95 (Fig. 5, *a*), which is not observed at bigger thickness of the upper layer h_3=3.

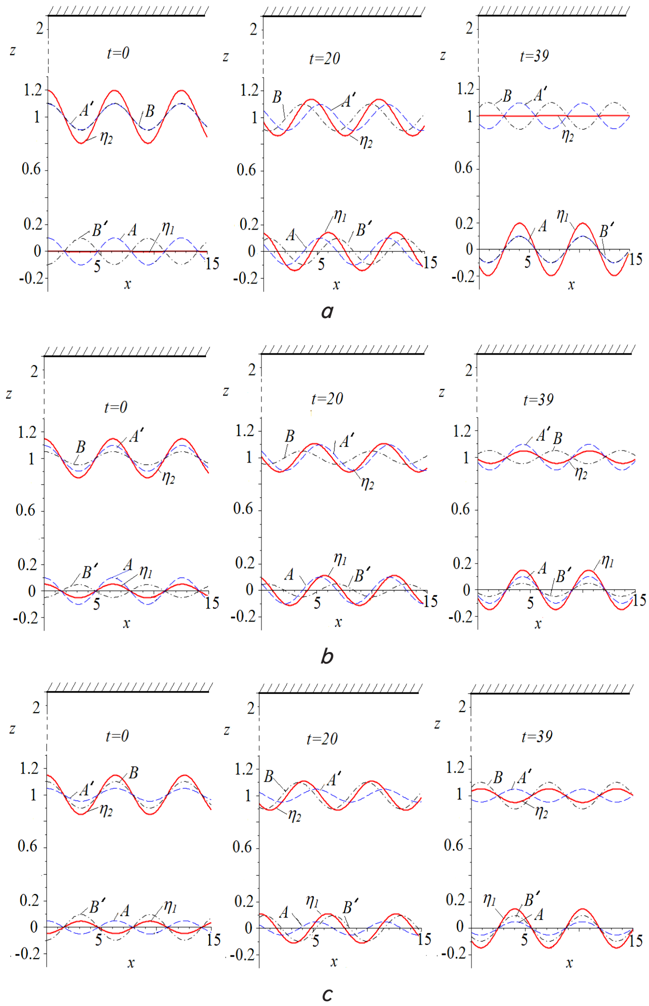

Fig. 4. Elevation of surfaces of contact $\eta_1(x, t)$ and $\eta_2(x, t)$ at different moment of time t for values of amplitudes: *a* – A=0.1, B=0.1; *b* – A=0.1, B=0.05; *c* – A=0.05, B=0.1

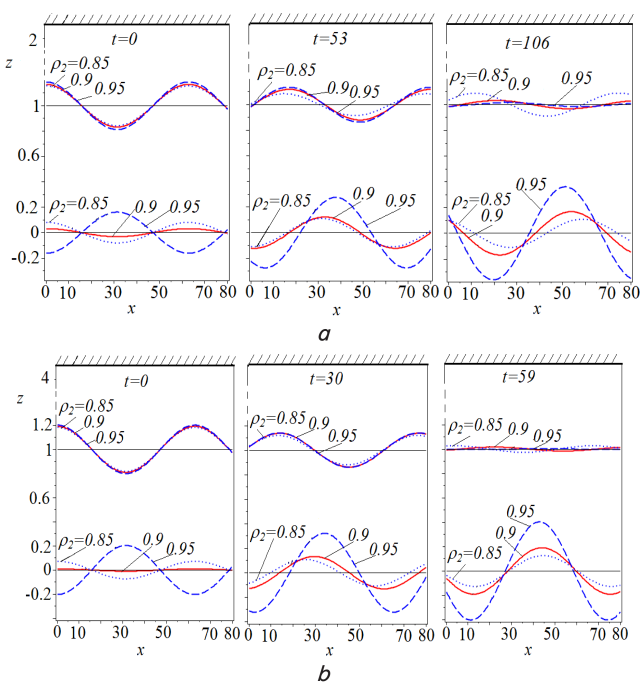

Fig. 5. Elevations of surfaces of contact $\eta_1(x, t)$ and $\eta_2(x, t)$ at different moments of time t at k=0.1, ρ_1=1, ρ_3=0.8, h_2=1, ρ_2=0.85, 0.9, 0.95, A=0.1, B=0.1: *a* – h_3=1; *b* – h_3=3

7. Discussion of results of research into wave processes in a three-layer hydrodynamic system

We will note that the study was conducted in the framework of a weakly nonlinear model. Existence of a large number of physical and geometrical parameters of the system that are interrelated leads to the need of detailed analysis and interpretation of obtained results. All results are related, they create a complete picture of the wave process. We will underline that in the process of research, the authors revealed the case, in which a three-layer system degenerates in a two-layer system, and one of the frequencies of waves that can propagate in the system tends to zero, which causes the need for additional research. Previously obtained results for similar two-layer systems " half-space – a layer with a rigid lid" and "solid-bottomed layer – layer with rigid lid" were used for testing and verification of physical validity of the new results. Results of the research are fully applicable to studying wave processes in the ocean with the ice lid with the laminated structure of fluid that occurs near the mouth of rivers, as well as in the open ocean in the ice melt period. Advantages of this research in propagation and interaction of waves in the hydrodynamic system "half-space – layer – layer with a rigid lid" refers to identification of dependences of ratios of amplitudes of waves on one of the contact surfaces and response to them on the other surface on different parameters of the system.

Completed study describes the phenomenon of waves' propagation in linear approximation, though problem statement (1) is nonlinear. In future, it is planned to obtain solutions of higher approximations, which will make it possible to obtain evolution equations of bypassed wave packets on contact surfaces, as well as to explore the phenomenon of modulation stability of waves.

Conducted analysis of the ratio of amplitudes is important in subsequent study of energy processes in the studied system. At propagation of waves in multi-layered hydrodynamic systems, the phenomenon of waves' energy pumping occurs. In the future it is planned to use the results, obtained in present research, during the study of this phenomenon.

8. Conclusions

1. By using the method of large-scale expansions, we obtained the first three linear approximations of a weakly nonlinear problem of propagation and interaction of waves along the contact surfaces in a three-layer hydrodynamic system "liquid half-space – layer – layer with a rigid lid".

2. For each frequency, dependences of amplitudes of waves that run along contact surfaces at various geometrical and physical parameters of the hydrodynamic system were constructed. At an increase of thickness of the upper layer, amplitudes of waves-responses begin to tend to the limited value. At equal density jumps on the contact surfaces, the impact of propagation of a wave on one of the contact surfaces on propagation of a wave on the other contact surface coincides, no matter whether a wave propagates on the upper or on the lower surface. If density jump on the lower contact surface is smaller than density jump on the upper contact surface, wave-responses that appear on the lower surface are larger than wave-responses on the upper surface. A change in density of the inner layer from value of density of the lower layer to value of density of the upper layer leads to changes of amplitudes of waves-responses, in this case, for smaller values of wave number, an increase is slower than in the case when the wave number is large. The cases of coincidence of amplitudes of waves-responses for different values of wave numbers were identified. A change in the value of wave number leads to the fact that values of amplitudes of waves-responses very quickly converge to common limited value at values of thickness of the upper layer of more than two. Analysis of the ratio between amplitudes of waves on contact surfaces and amplitudes of responses was conducted, which revealed equality at the same density jumps on contact surfaces. We identified the need for a detailed study of the limited case when there is no actual density jump, in which one of the solutions of dispersion equation tends to zero. A change in density of the middle layer has a significant effect on amplitude of elevation of both the upper and the lower surfaces of contact. At approximation of value of density of the middle layer to the value of density of the lower layer, this effect is more pronounced on the lower surface of the contact. A change in density of the middle layer has a significant impact on the amplitude of the lower surface of contact; in this case amplitude of the upper contact surface undergoes significant changes only under certain properties of the system (at a change in geometrical parameters).

References

1. Wang, Y. The Viscous Surface-Internal Wave Problem: Global Well-Posedness and Decay [Text] / Y. Wang, I. Tice, C. Kim // Archive for Rational Mechanics and Analysis. – 2013. – Vol. 212, Issue 1. – P. 1–92. doi: 10.1007/s00205-013-0700-2
2. Van Haren, H. High-frequency internal wave motions at the ANTARES site in the deep Western Mediterranean [Text] / H. van Haren // Ocean Dynamics. – 2014. – Vol. 64, Issue 4. – P. 507–517. doi: 10.1007/s10236-014-0702-0
3. Fan, K. Internal wave parameters retrieval from space-borne SAR image [Text] / K. Fan, B. Fu, Y. Gu, X. Yu, T. Liu, A. Shi et. al. // Frontiers of Earth Science. – 2015. – Vol. 9, Issue 4. – P. 700–708. doi: 10.1007/s11707-015-0506-7
4. Hong, Y. A high-order perturbation of surfaces method for scattering of linear waves by periodic multiply layered gratings in two and three dimensions [Text] / Y. Hong, D. P. Nicholls // Journal of Computational Physics. – 2017. – Vol. 345. – P. 162–188. doi: 10.1016/j.jcp.2017.05.017
5. Massel, S. R. On the nonlinear internal waves propagating in an inhomogeneous shallow sea [Text] / S. R. Massel // Oceanologia. – 2016. – Vol. 58, Issue 2. – P. 59–70. doi: 10.1016/j.oceano.2016.01.005
6. Li, Q. Numerical assessment of factors affecting nonlinear internal waves in the South China Sea [Text] / Q. Li // Progress in Oceanography. – 2014. – Vol. 121. – P. 24–43. doi: 10.1016/j.pocean.2013.03.006
7. Singh, A. K. Effect of loosely bonded undulated boundary surfaces of doubly layered half-space on the propagation of torsional wave [Text] / A. K. Singh, A. Lakshman // Mechanics Research Communications. – 2016. – Vol. 73. – P. 91–106. doi: 10.1016/j.mechrescom.2016.02.007

8. Zhu, H. Numerical simulation of interaction between internal solitary waves and submerged ridges [Text] / H. Zhu, L. Wang, E. J. Avital, H. Tang, J. J. R. Williams // Applied Ocean Research. – 2016. – Vol. 58. – P. 118–134. doi: 10.1016/j.apor.2016.03.017
9. Rosi, G. Surface waves at the interface between an inviscid fluid and a dipolar gradient solid [Text] / G. Rosi, V.-H. Nguyen, S. Naili // Wave Motion. – 2015. – Vol. 53. – P. 51–65. doi: 10.1016/j.wavemoti.2014.11.004
10. Smith, S. Experiments on nonlinear harmonic wave generation from colliding internal wave beams [Text] / S. Smith, J. Crockett // Experimental Thermal and Fluid Science. – 2014. – Vol. 54. – P. 93–101. doi: 10.1016/j.expthermflusci.2014.01.012
11. Deconinck, B. Stability of periodic gravity waves in the presence of surface tension [Text] / B. Deconinck, O. Trichtchenko // European Journal of Mechanics – B/Fluids. – 2014. – Vol. 46. – P. 97–108. doi: 10.1016/j.euromechflu.2014.02.010
12. Akers, B. F. Overturned internal capillary-gravity waves [Text] / B. F. Akers, D. M. Ambrose, K. Pond, J. D. Wright // European Journal of Mechanics – B/Fluids. – 2016. – Vol. 57. – P. 143–151. doi: 10.1016/j.euromechflu.2015.12.006
13. Vitousek, S. A nonhydrostatic, isopycnal-coordinate ocean model for internal waves [Text] / S. Vitousek, O. B. Fringer // Ocean Modelling. – 2014. – Vol. 83. – P. 118–144. doi: 10.1016/j.ocemod.2014.08.008
14. Tahvildari, N. Generation of long subharmonic internal waves by surface waves [Text] / N. Tahvildari, J. M. Kaihatu, W. S. Saric // Ocean Modelling. – 2016. – Vol. 106. – P. 12–26. doi: 10.1016/j.ocemod.2016.07.004
15. Shiryaeva, S. O. On the surface and internal gravitational waves in a three-layer immiscible liquid [Text] / S. O. Shiryaeva, A. I. Grigor'ev, L. S. Yakovleva // Technical Physics. – 2015. – Vol. 60, Issue 2. – P. 1772–1777. doi: 10.1134/s1063784215120208
16. Selezov, I. T. Nonlinear interaction of internal and surface gravity waves in a two-layer fluid with free surface [Text] / I. T. Selezov, O. V. Avramenko, Yu. V. Gurtovyi, V. V. Naradovyi // Journal of Mathematical Sciences. – 2010. – Vol. 168, Issue 4. – P. 590–602. doi: 10.1007/s10958-010-0010-2
17. Avramenko, O. V. Conditions of Wave Propagation in a Two-Layer Liquid with Free Surface [Text] / O. V. Avramenko, V. V. Naradovyi, I. T. Selezov // Journal of Mathematical Sciences. – 2015. – Vol. 212, Issue 2. – P. 131–141. doi: 10.1007/s10958-015-2654-4
18. Avramenko, O. V. Analysis of propagation of weakly nonlinear waves in a two-layer fluid with free surface [Text] / O. V. Avramenko, V. V. Naradovyi // Eastern-European Journal of Enterprise Technologies. – 2015. – Vol. 4, Issue 7 (76). – P. 39–44. doi: 10.15587/1729-4061.2015.48282
19. Avramenko, O. V. Enerhiya vnutrishnikh i poverkhnevykh khvylovykh rukhiv u dvosharoviy hidrodynamichniy systemi [Text] / O. V. Avramenko, V. V. Naradovyi, I. T. Selezov // Matematychni metody ta fizyko-mekhanichni polia. – 2016. – Vol. 59, Issue 1. – P. 111–120.
20. Nayfeh, A. H. Nonlinear Propagation of Wave-Packets on Fluid Interfaces [Text] / A. H. Nayfeh // Journal of Applied Mechanics. – 1976. – Vol. 43, Issue 4. – P. 584–588. doi: 10.1115/1.3423936
21. Selezov, I. T. Evolyuciya nelineynyh volnovyh paketov v gidrodinamicheskoy sisteme "sloy-poluprostranstvo" s uchetom poverhnostnogo natyazheniya [Text] / I. T. Selezov, O. V. Avramenko // Matematychni metody ta fizyko-mekhanichni polia. – 2001. – Vol. 44, Issue 2. – P. 113–122.