Доведено аналог класичної теореми Бореля для цілих гармонійних в $\mathbb{R}^n$, $n\geq 3$, функцій, який встановлює зв'язок між максимумом модуля гармонійної функції скінченного порядку та максимальним членом степеневого ряду деякої цілої в площині функції, коефіцієнти якої певним чином зв'язані з коефіцієнтами розкладу в ряд за сферичними функціями Лапласа даної гармонійної функції. Також отримано вирази для узагальненого та нижнього узагальненого порядків цілої гармонійної в $\mathbb{R}^n$ функції через рівномірну норму сферичних функцій Лапласа в її розкладі в ряд

Ключові слова: ціла гармонійна функція, сферична функція Лапласа, узагальнений порядок, нижній узагальнений порядок

Доказано аналог классической теоремы Бореля для целых гармонических в $\mathbb{R}^n$, $n\geq 3$, функций, который устанавливает связь между максимумом модуля гармонической функции конечного порядка и максимальным членом степенного ряда некоторой целой в плоскости функции, коэффициенты которой некоторым образом связаны с коэффициентами разложения в ряд по сферическим функциям Лапласса данной гармонической функции. Также получены выражения для обобщенного и нижнего обобщенного порядков целой гармонической в $\mathbb{R}^n$ функции через равномерную норму сферических функций Лапласса в ее разложении в ряд

Ключевые слова: целая гармоническая функция, сферическая функция Лапласса, обобщенный порядок, нижний обобщенный порядок

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ANALOG OF THE CLASSICAL BOREL THEOREM FOR ENTIRE HARMONIC FUNCTIONS IN $\mathbb{R}^n$ AND GENERALIZED ORDERS

O. Veselovska
PhD, Associate Professor
Department of Mathematics
Lviv Polytechnic National University
S. Bandery str.,12, Lviv, Ukraine, 79013
E-mail: veselovskaov@gmail.com

1. Introduction

Harmonic functions, being a natural generalization of linear functions of one variable, are in some sense the simplest functions of several variables. At the same time, the range of such functions is very rich and varied.

Harmonic functions are closely related to such fundamental sections of analysis as the theory of analytic functions, the theory of potential, and differential equations with partial derivatives of the elliptic type [1]. Moreover, these functions are widely used in mathematical physics [2]. This is due to the fact that the potentials of the most important vector fields considered in physics are harmonic functions and any harmonic function can be considered as the potential of a certain field.

Often, harmonic functions in space $\mathbb{R}^n$ are given by the series in Laplace spherical functions, and in the three-dimensional space, in addition, by the series of adjointed Legendre polynomials. This requires a study of the relationship between the growth of these functions and the behaviour of the coefficients of such expansions. The need to take into account the growth of functions arises in the theory of series, differential equations, and approximation.

Therefore, it is important to express the generalized growth characteristics of harmonic functions in the space $\mathbb{R}^n$ in terms of the uniform norm of spherical Laplace functions that are included in the expansions of such functions in series as well as to establish an analog of the classical Borel theorem for harmonic functions of the n-dimensional space.

2. Literature review and problem statement

The first study of the relationship between the growth of harmonic functions of the three-dimensional space and the behaviour of the coefficients of the expansion of these functions in series by adjointed Legendre polynomials is presented in [3]. By means of the integral Bergman representation of a harmonic function in $\mathbb{R}^3$ by an entire function of a complex variable and a real parameter the order of the harmonic function and the type of the axisymmetric harmonic function in $\mathbb{R}^3$ are expressed in terms of expansion coefficients of these functions by the adjointed Legendre polynomials. This result is refined and generalized in [4, 5]. In [6], formulae are obtained for generalized and lower generalized orders of a harmonic function in $\mathbb{R}^n$ in terms of the coefficients of the expansion of this function in the Fourier series. The formulae for the order and type and also for generalized and lower generalized orders of harmonic functions in $\mathbb{R}^n$ through the norm of the gradient at the origin are given in [7, 8], respectively. Paper [9] is devoted to researching the growth of a J-universal harmonic function in the space.

In [10], the necessary and sufficient conditions under which a harmonic function in the ball of a three-dimensional space continues to the entire harmonic one are established. Also, the order and type of an entire harmonic function in $\mathbb{R}^3$ are expressed in terms of an approximation error of the continued function by harmonic polynomials. A similar issue is studied in [11] for the (p, q) orders and types of harmonic function in $\mathbb{R}^3$. The results obtained in [10, 11] are generalized in [12] onto the n-dimensional space. To characterize the growth of entire harmonic function in $\mathbb{R}^n$, generalized characteristics introduced in [13] are used.

Article [14] is devoted to the study of uniform approximation of generalized axisymmetric potentials by polynomials. The order and type of potentials are expressed by the error of approximation. In [15], approximation of entire solutions of the Helmholtz equation by Chebyshev polynomials is studied, and some estimates are given on the growth parameters of these solutions in terms of coefficients and the approximation error with respect to the sup norm. In [16], research continues on the basis of [15]. Paper [17] is devoted to the study of generalized and lower generalized q-types of solutions of the usual elliptic differential equation with partial derivatives. Approximation of entire solutions of the Helmholtz equation in Banach spaces $B(p, q, m)$ by Chebyshev polynomials is considered in [18]. Expressions for the order and type of solutions of some linear differential equations with partial derivatives in terms of the error of axisymmetric harmonic polynomial approximation and Lagrange interpolation are obtained in [19]. In [20], slow growth and approximation of pseudoanalytic functions on a disk are studied.

Consequently, the characteristics of the growth of harmonic functions of the three-dimensional or n-dimensional spaces have been expressed in terms of the errors of approximation or interpolation of these functions by different polynomials in relation to different norms, as well as through the expansion coefficients of harmonic functions into series by the adjointed Legendre and Chebyshov polynomials or the norm of the gradient at the origin. The issue that is still open for studying is the description of the growth of the harmonic function of the n-dimensional space by applying the uniform norm of Laplace spherical functions in the expansion of this harmonic function in a series.

In studying entire functions of one complex variable, the important question is the connection between the maximum modulus of such functions and the maximum term of their power series; in particular, the Borel theorem is known for entire functions of finite order in the plane [21]. An analog of the classical Borel theorem for harmonic functions of the three-dimensional space, which are decomposed in series by adjointed Legendre polynomials, is obtained in [5]. There is no analog for harmonic functions in the case of the n-dimensional space, which are decomposed into series by Laplace spherical functions.

3. The aim and objectives of the study

The aim of the work is to establish an analog of the Borel classical theorem for entire harmonic finite-order functions in space $\mathbb{R}^n$ and obtain formulae for the most general characteristics of the growth of entire harmonic functions in the n-dimensional space in terms of the uniform norm of Laplace spherical functions in the expansion of these functions in series. This will help investigate the growth of the harmonic function of the n-dimensional space directly by the coefficients of the expansion of this function in a series.

To achieve this aim, the following tasks are solved:

– to determine the relation between the maximum terms of entire finite-order functions in the plane, the coefficients of which satisfy certain conditions;

– to estimate from above the maximum modulus of the entire harmonic function in $\mathbb{R}^n$ by a maximum modulus of some entire function in the plane whose coefficients of the power series are expressed in terms of the uniform norm of the Laplace spherical functions;

– to estimate from below the maximum modulus of the entire harmonic function in the space $\mathbb{R}^n$ by the maximum term of the power series of some entire function in the plane whose coefficients of the power series are expressed in terms of the uniform norm of the Laplace spherical functions.

4. The relation between the maximum terms of entire functions in the plane

Let the entire function f in the plane be given by a power series

$$f(z)=\sum_{k=0}^{\infty} c_k z^k. \qquad (1)$$

Let us denote through

$$\mu(r,\mathrm{f})=\max_{\mathrm{k}\ge 0}|c_k|r^k$$

the maximum term of series (1) in the circle $\{z\in\mathbb{C}:|z|=r\}$, and through $\nu(r,f)$ the largest number of the maximum term of this series, which is called the central index.

Theorem 1. Let

$$f_1(z)=\sum_{k=0}^{\infty} b_k z^k,\quad f_2(z)=\sum_{k=0}^{\infty} d_k b_k z^k$$

be, the entire functions in the plane of the finite orders ρ_1 and ρ_2, respectively, with $b_k>0$,

$$\frac{1}{h(k)}\le d_k\le h(k),$$

where h is a non-decreasing positive function. Then there exists a constant $K>0$ such that

$$\left|\ln\mu(r,f_2)-\ln\mu(r,f_1)\right|\le \ln\mathrm{h}\left(Kr^{\rho}\right)$$

for all $r>0$ and $\rho=\max\{\rho_1,\rho_2\}$.

Proof. Suppose $\nu_1(r)$, $\nu_2(r)$ are the central indexes of the power series of the functions $f_1(z)$, $f_2(z)$, respectively. Then

$$b_{\nu_1(r)}r^{\nu_1(r)}\ge b_{\nu_2(r)}r^{\nu_2(r)},$$

$$d_{\nu_2(r)}b_{\nu_2(r)}r^{\nu_2(r)}\ge d_{\nu_1(r)}b_{\nu_1(r)}r^{\nu_1(r)}.$$

Let us multiply the first inequality by $d_{\nu_2(r)}$. Taking into account the conditions imposed on the coefficients d_k, we obtain

$$\frac{\mu(r,\mathrm{f}_1)}{h(\nu_1(r))}\le\mu(r,\mathrm{f}_2)\le h(\nu_2(r))\mu(r,\mathrm{f}_1). \qquad (2)$$

Since the central index is a positive and non-decreasing function, on the basis of the known relation [22],

$$\ln\mu(r,\mathrm{f})-\ln|c_0|=\int_0^r\frac{\nu(t,f)}{t}\mathrm{d}t$$

that takes place for the function f given by (1), we have

$$\ln\left(\frac{1}{|c_0|}\mu(er,\mathrm{f})\right)\geq\int_r^{er}\frac{\nu(t,f)}{t}\mathrm{d}t\geq$$

$$\geq\nu(r,f)\int_r^{er}\frac{\mathrm{d}t}{t}=\nu(r,f).$$

Consequently, for the functions f_1 and f_2, we get

$$\nu_i(r)\leq\ln\left(K_i\,\mu(er,\mathrm{f}_i)\right),$$

where i=1, 2,

$$K_1=\frac{1}{b_0},\quad K_2=\frac{1}{d_0b_0}.$$

Taking into account the obtained inequalities and also the fact that the function h is non-decreasing, we find from inequalities (2) that

$$\frac{\mu(r,\mathrm{f}_1)}{h\left(\ln\left(K_1\,\mu(er,\mathrm{f}_1)\right)\right)}\leq\mu(r,\mathrm{f}_2)\leq$$

$$\leq h\left(\ln\left(K_2\,\mu(er,\mathrm{f}_2)\right)\right)\mu(r,\mathrm{f}_1). \quad (3)$$

From the finiteness of the orders $\rho_1,\ \rho_2$ of functions f_1 and f_2, respectively, it follows that there exists a constant K>0 such that

$$\ln\left(K_i\,\mu(er,\mathrm{f}_i)\right)\leq Kr^{\rho},$$

where $i=1,2,\ \ \rho=\max\{\rho_1,\rho_2\}$. Hence using (3), we get

$$\frac{\mu(r,\mathrm{f}_1)}{h(\mathrm{K}r^{\rho})}\leq\mu(r,\mathrm{f}_2)\leq h(\mathrm{K}r^{\rho})\mu(r,\mathrm{f}_1).$$

With the logarithm of the resulting inequality, we arrive at the assertion of Theorem 1.

Consequence 1. If $h(x)=o(e^x),\ \ x\to\infty,$ then

$$\ln\mu(r,\mathrm{f}_2)=\ln\mu(r,\mathrm{f}_1)+o(r^{\rho}),\quad r\to\infty.$$

Consequence 2. If $h(x)=x^{\alpha}$, $\alpha>0$, then

$$\ln\mu(r,\mathrm{f}_2)=\ln\mu(r,\mathrm{f}_1)+O(\ln r),\quad r\to\infty.$$

5. The relationship between the maximum modulus of an entire harmonic function in $\mathbb{R}^n$, $n\geq3$, and the maximum term of a series of some entire function in the plane

Let $S^n=\{x\in\mathbb{R}^n:|x|=1\}$ be a unit sphere in $\mathbb{R}^n$ centered at the origin, and

$$\omega_n=\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

is its surface area, where Γ denotes a gamma function.

A spherical harmonic or a Laplace spherical function of degree k,

$$k\in Z_+=\{0,1,2,\ldots\},$$

denoted by $Y^{(k)}$, is called a restriction of a homogeneous harmonic polynomial of degree k on the unit sphere S^n $n\geq2$. [23].

A set of spherical harmonics of degree k can be considered as a sub-space of the space $L^2(S^n)$ of real-valued functions with the scalar product

$$(f,g)=\frac{1}{\omega_n}\int_{S^n}f(x)g(x)\mathrm{d}S,$$

where dS is the element of the surface area on the sphere S^n. If

$$\left\{Y_1^{(k)},\ldots,Y_{\gamma_k}^{(k)}\right\}$$

is an orthonormal base in this subspace, then

$$\bigcup_{k=0}^{\infty}\left\{Y_1^{(k)},\ldots,Y_{\gamma_k}^{(k)}\right\}$$

will be an orthonormal base in the space $L^2(S^n)$. Here

$$\gamma_k=\frac{(2k+n-2)(k+n-3)!}{k!(n-2)!}$$

is the quantity of linearly independent spherical harmonics of degree k.

Let u be an entire harmonic function in $\mathbb{R}^n$, that is, the harmonic function over the whole space $\mathbb{R}^n$. Then it expands into a Fourier-Laplace series [1]

$$u(rx)=\sum_{k=0}^{\infty}Y^{(k)}(x;u)\,r^k,\quad x\in S^n, \quad (4)$$

where

$$Y^{(k)}(x;u)=a_1^{(k)}Y_1^{(k)}(x)+a_2^{(k)}Y_2^{(k)}(x)+\ldots+a_{\gamma_k}^{(k)}Y_{\gamma_k}^{(k)}(x),$$

$$a_j^{(k)}=\left(u,Y_j^{(k)}\right),\quad j=\overline{1,\gamma_k},$$

$\left(u,Y_j^{(k)}\right)$ is the scalar product in $L^2(S^n)$.

For n=2, the spherical harmonics are reduced to ordinary trigonometric functions of an angle. For $n\geq3$, they have a more complicated structure and are expressed in terms of polynomials of a special form.

Let us assume that d_2=1 and d_n=n–2 at n>2 and

$$\nu=\frac{n-2}{2}.$$

Then

$$Y^{(k)}(x;u)r^k=\frac{2(k+\nu)}{d_n\omega_n}\int_{S^n}C_k^{\nu}\left[(x,y)\right]u(ry)\mathrm{d}S(y),$$

where $k\in Z_+,\ x\in S^n,\ (\cdot,\cdot)$ is the scalar product in $\mathbb{R}^n$, and C_k^{ν} denotes the Gegenbauer polynomial of degree k and order ν [23], which are determined from the relation

$$\frac{1-\tau^2}{\left(1-2\tau t+\tau^2\right)^{\nu+1}}=1+2\sum_{k=1}^{\infty}\frac{k+\nu}{d_n}C_k^{\nu}(t)\,\tau^k,$$

where $|t|\leq1,\ \ 0\leq\tau<1.$

Let us consider the function

$$M(r,\mathrm{u})=\max_{x\in S^m}|u(rx)|,\quad r>0,$$

with the help of which we shall measure the growth of an entire harmonic function u in the space $\mathbb{R}^n$, $n\geq 3$.

The most commonly used characteristics of the growth of the function u are the order $\rho(u)$ and the lower order $\lambda(u)$, which are determined, respectively, by the relations

$$\rho(u)=\overline{\lim_{r\to\infty}}\frac{\ln(\ln M(r,\mathrm{u}))}{\ln \mathrm{r}}$$

and

$$\lambda(u)=\underline{\lim_{r\to\infty}}\frac{\ln(\ln M(r,\mathrm{u}))}{\ln \mathrm{r}},$$

but in the case of $0<\rho<\infty$, the type is $\sigma(u)$:

$$\sigma(u)=\overline{\lim_{r\to\infty}}\frac{\ln M(r,\mathrm{u})}{\mathrm{r}^{\rho}}.$$

The generalization of the order and type produces the proximate order $\rho(r)$ [24] and the type $\sigma^*(u)$ of the relatively proximate order $\rho(r)$, which is determined by the relation

$$\sigma^*(u)=\overline{\lim_{r\to\infty}}\frac{\ln M(r,\mathrm{u})}{\mathrm{r}^{\rho(r)}}.$$

In the case of entire harmonic functions in $\mathbb{R}^n$ of infinite and zero orders, we shall use more general scale of growth introduced in [13].

Let the function γ be defined and differentiable on the interval $[a;+\infty)$ at some $a\geq 0$ strictly monotonically increasing, with $\to\infty$ going to ∞. According to [13], it belongs to the class L^0, if for any real function ψ such that $\psi(t)\to 0$ as $t\to\infty$, the following equality holds

$$\lim_{t\to\infty}\frac{\gamma[(1+\psi(t))t]}{\gamma(t)}=1,$$

and it belongs to the class Λ if for all c, $0<c<\infty$,

$$\lim_{t\to\infty}\frac{\gamma(ct)}{\gamma(t)}=1.$$

We note that $\Lambda\subset L^0$, but according to the example of the function t^m, $m>0$, $\Lambda\neq L^0$.

Using the functions α and β of the classes L^0 and Λ, by analogy with [13], we introduce the generalized and lower generalized orders of the entire harmonic function u in $\mathbb{R}^n$ by the equalities

$$\rho_{\alpha\beta}(u)=\overline{\lim_{r\to\infty}}\frac{\alpha(\ln M(r,\mathrm{u}))}{\beta(r)}$$

and

$$\lambda_{\alpha\beta}(\mathrm{u})=\underline{\lim_{r\to\infty}}\frac{\alpha(\ln M(r,\mathrm{u}))}{\beta(r)}.$$

We note that from these growth characteristics, with the appropriate choice of the functions α and β, we can obtain all the above-mentioned characteristics of the growth of the entire harmonic function in $\mathbb{R}^n$.

We put

$$B_k=\sqrt{\frac{(2\nu)!}{2}}\frac{1}{(k+2\nu)^{\nu}}\max_{x\in S^n}\left|Y^{(k)}(x;u)\right|. \tag{5}$$

Theorem 2. If u is an entire harmonic function in the space $\mathbb{R}^n$, $n\geq 3$, given by series (4), then the function

$$g(z)=\sum_{k=0}^{\infty}B_k z^k, \tag{6}$$

is entire and in the case of finiteness of the order of the function u, the true equality is

$$\ln M(r,u)=\ln\mu(r,\mathrm{g})+O(\ln r),\quad r\to\infty. \tag{7}$$

Theorem 2 is an analog of the classical Borel theorem [21], which establishes a connection between the maximum modulus of the entire finite-order function in the plane and the maximum term of its power series.

To prove Theorem 2, we shall use the following lemma.

Lemma 1. For the entire harmonic function u in $\mathbb{R}^n$, $n\geq 3$, given by series (4), the following inequalities holds

$$B_k\leq M(r,u)r^{-k} \tag{8}$$

for all $k\in Z_+$ and $r>0$.

The proof of this lemma is given in [5]. We note that inequalities (8) are analogous to Cauchy's inequalities for entire functions of one complex variable.

Proof of Theorem 2. The fact that the function g defined by (6) is entire follows directly from Lemma 1. Indeed, choosing $r_0>\frac{2}{\varepsilon}$, we obtain

$$\sqrt[k]{B_k}\leq\frac{\sqrt[k]{M(r_0,u)}}{\frac{2}{\varepsilon}}<\frac{2}{\frac{2}{\varepsilon}}=\varepsilon$$

for all $k\geq k_0$, that is

$$\lim_{k\to\infty}\sqrt[k]{B_k}=0,$$

which means that the function g is entire.

Let us prove relation (7). From the decomposition of (4) and the definition of the numbers B_k, it follows that

$$M(r,u)\leq\sum_{k=0}^{\infty}\max_{\xi\in S^m}\left|Y^{(k)}(\xi;u)\right|r^k=\sum_{k=0}^{\infty}\sqrt{\frac{2}{(2\nu)!}}(k+2\nu)^{\nu}B_k r^k. \tag{9}$$

Let us consider the function

$$g_1(z)=\sum_{k=0}^{\infty}\sqrt{\frac{2}{(2\nu)!}}(k+2\nu)^{\nu}B_k z^k. \tag{10}$$

Obviously, function g_1 is entire and has a finite order. Indeed, it follows from Lemma 1 that

$$\ln\mu(r,\mathrm{g})\leq\ln M(r,u), \tag{11}$$

where we obtain $\rho(\mathrm{g})\leq\rho(\mathrm{u})$. The orders of the functions g and g_1 are equal on the basis of the Hadamard formula [24] for finding the order of the entire function of one complex variable in terms of the coefficients of its power series. Therefore, $\rho(g_1)\le\rho(u)$, and since the order of the function u is finite, the order of function g_1 is finite, too.

Then from inequality (9) and the classical Borel theorem [21], it follows that

$$\ln M(r,u)\le\ln M(r,g_1)=\ln\mu(r,g_1)+O(\ln r),\quad r\to\infty.$$

Taking into account the consequence of 2, we find

$$\ln M(r,u)\le\ln\mu(r,g)+O(\ln r),\quad r\to\infty. \tag{12}$$

By combining inequalities (11) and (12), we arrive at the statement of Theorem 2.

Consequence 3. If the entire harmonic function in $\mathbb{R}^n$ has the order ρ, $0<\rho<\infty$, then its type σ^* with respect to the proximate order $\rho(r)$ is determined by the equality

$$\left(\sigma^*\rho e\right)^{1/\rho}=\overline{\lim_{k\to\infty}}\,\psi(k)\sqrt[k]{B_k}, \tag{13}$$

where $r=\psi(t)$ denotes a function inverse to $t=r^{\rho(r)}$.

Proof. It follows from Theorem 2 and the classical Borel theorem [21] that

$$\rho(u)=\overline{\lim_{r\to\infty}}\frac{\ln(\ln M(r,u))}{\ln r}=$$
$$=\overline{\lim_{r\to\infty}}\frac{\ln(\ln\mu(r,g))}{\ln r}=\overline{\lim_{r\to\infty}}\frac{\ln(\ln M(r,g))}{\ln r}=\rho(g).$$

By analogy, we have

$$\overline{\lim_{r\to\infty}}\frac{\ln M(r,u)}{r^{\rho(r)}}=\overline{\lim_{r\to\infty}}\frac{\ln M(r,g)}{r^{\rho(r)}}.$$

From this, we obtain that the proximate order $\rho(r)$ of the function u is a proximate order of the entire function g in plane and vice versa. It is also obvious that the types of the relatively well-defined order $\rho(r)$ of the functions u and g are the same. Therefore, for the function g, using the formula [24] to determine the type of the relatively proximate order through the coefficients of its expansion into a power series, we obtain equality (13).

If $\psi(t)=t^{\frac{1}{\rho}}$, then relation (13) produces

$$(\sigma\rho e)^{1/\rho}=\overline{\lim_{k\to\infty}}\,k^{\frac{1}{\rho}}\sqrt[k]{B_k}. \tag{14}$$

Formula (14) defines the type σ of the entire harmonic function u in $\mathbb{R}^n$.

6. Generalized and lower generalized orders of the entire harmonic function in $\mathbb{R}^n$

Let us assume that

$$F(t,c)=\beta^{-1}(c\alpha(t)), \tag{15}$$

where β^{-1} is a function inverse to β.

Theorem 3. Let u be a harmonic function in $\mathbb{R}^n$, $n\geq3$, with B_k defined by relation (5). If for all c, $0<c<\infty$, one of the conditions is satisfied:

a) $\alpha,\beta\in\Lambda,\dfrac{d\ln F(t,c)}{d\ln t}=O(1),\quad t\to\infty;$

b) $\alpha,\beta\in L^0,\lim\limits_{t\to\infty}\dfrac{d\ln F(t,c)}{d\ln t}=p,\quad 0<p<\infty,$

where the function $F(t, c)$ is determined by relation (15), then the generalized order $\rho_{\alpha\beta}(u)$ of the entire harmonic function u in $\mathbb{R}^n$ is determined by the equation

$$\rho_{\alpha\beta}(u)=\overline{\lim_{k\to\infty}}\frac{\alpha(pk)}{\beta\left(e^pB_k^{-1/k}\right)},$$

and, in the case, condition (a) is satisfied, the number p is considered to be an arbitrary positive one.

Proof of Theorem 3. Let the entire functions g and g_1 of the complex variable z be given respectively by relations (6) and (10). Then on the basis of inequalities (8) and (9), we obtain

$$\mu(r,g)\le M(r;u)\le M(r,g_1). \tag{16}$$

From here,

$$\rho_{\alpha\beta}(g)\le\rho_{\alpha\beta}(u)\le\rho_{\alpha\beta}(g_1). \tag{17}$$

Using the known formula [13] that expresses the generalized order of the entire function of one complex variable in terms of the coefficients of its power series, we have

$$\rho_{\alpha\beta}(g)=\overline{\lim_{k\to\infty}}\frac{\alpha(pk)}{\beta\left(e^pB_k^{-1/k}\right)},$$

$$\rho_{\alpha\beta}(g_1)=\overline{\lim_{k\to\infty}}\frac{\alpha(pk)}{\beta\left((1+o(1))e^pB_k^{-1/k}\right)}.$$

Since, under the condition of the theorem 3, the function β belongs to the class L^0 or Λ, then

$$\rho_{\alpha\beta}(g)=\rho_{\alpha\beta}(g_1),$$

which together with (17) completes the proof of Theorem 3.

We note that from Theorem 3 for the entire harmonic function u in $\mathbb{R}^n$ we can obtain the following:

1) at $\alpha(t)=\beta(t)=\ln t$, the formula for the order $\rho(u)$:

$$\rho(u)=\overline{\lim_{k\to\infty}}\frac{k\ln k}{\ln B_k^{-1}};$$

2) at $\alpha(t)=t,\ \beta(t)=t^\rho,\ p=\dfrac{1}{\rho}$, where ρ is the order of the function u, formula (14) for the type $\sigma(u)$;

3) at $\alpha(t)=t,\ \beta(t)=t^{\rho(t)}$, where $\rho(t)$ is the proximate order of the function u, formula (13) for the type $\sigma^*(u)$ relatively to the proximate order $\rho(t)$.

Theorem 3 is complemented by the following theorem.

Theorem 4. Let u be an entire harmonic function in $\mathbb{R}^n$, $n\geq3$, with B_k, $F(t, c)$ determined by relations (5) and (15), respectively. If $\beta\in L^0$, and α is such that $\alpha(e^t)\in L^0$, and for all c, $0<c<\infty$, the following condition is satisfied

$$\ln\left(\frac{d\ln F(t,c)}{d\ln t}\right)=o(\ln t),\quad t\to\infty,$$

then the generalized order $\rho_{\alpha\beta}(u)$ of the entire harmonic function u in $\mathbb{R}^n$ is determined by the equality

$$\rho_{\alpha\beta}(u)=\overline{\lim_{k\to\infty}}\frac{\alpha(k)}{\beta\left(B_k^{-1/k}\right)}.$$

We note that if $\alpha\left(e^t\right)\in L^0$, then $\alpha(t)\in\Lambda$. However, as is shown by the example of the function $\alpha(t)=e^{\sqrt{\ln t}}$, the inverse statement is not valid.

In the case of entire harmonic functions of zero order, a more precise growth characteristic is given by the following theorem.

Theorem 5. Let u be an entire harmonic function in $\mathbb{R}^n$, $n\geq3$, with B_k defined by relation (2), $\alpha\in\Lambda$,

$$\Phi(t,c)=\alpha^{-1}\left(c\alpha(t)\right)$$

and for all c, $0<c<\infty$, at sufficiently large t, the true inequality is

$$0\leq\frac{d\Phi(t,c)}{dt}\leq A_1 e^{A_2\Phi(t,c)},$$

where A_1 and A_2 are such constants that $0<A_1<\infty$ and $0<A_2<\infty$. Then

$$\overline{\lim_{k\to\infty}}\frac{\alpha\left(\ln\ln M(r,u)\right)}{\alpha(\ln\ln r)}=\max\left\{1,\overline{\lim_{k\to\infty}}\frac{\alpha(\ln k)}{\alpha\left[\ln\left(\frac{1}{k}\ln B_k^{-1}\right)\right]}\right\}.$$

By the function α satisfying the conditions of Theorem 5, it is possible to choose

$$\alpha(x)=\ln_j x,$$

where $j\geq1$, a $\ln_1 x=\ln x$, $\ln_j x=\ln\left(\ln_{j-1}x\right)$ means a j-th iteration of the logarithm.

Theorems 4 and 5 directly follow from inequality (16) and similar results for the entire functions of one complex variable [13].

Theorem 6. Let u be an entire harmonic function in $\mathbb{R}^n$, $n\geq3$, with B_k defined by relations (5) while α and β are functions of the classes L^0, Λ, 0<p<∞. Then

$$\lambda_{\alpha\beta}(u)\geq\underline{\lim_{k\to\infty}}\frac{\alpha(pk)}{\beta\left(e^p B_k^{-1/k}\right)}. \qquad (18)$$

If, moreover, the ratio $\frac{B_k}{B_{k+1}}$ is a non-decreasing function of k and one of the conditions – (a) or (b) – of Theorem 3 is satisfied, then inequality (18) transforms into the equality.

The proof of this theorem is similar to the proof of Theorem 3.

Consequence 4. Let u be an entire harmonic function in $\mathbb{R}^n$, $n\geq3$. Then

$$\lambda(u)\geq\underline{\lim_{k\to\infty}}\frac{k\ln k}{\ln B_k^{-1}}.$$

The inequality becomes an equality when the ratio $\frac{B_k}{B_{k+1}}$ is a non-decreasing function of k.

Consequence 4 is derived from Theorem 6 if to choose that $\alpha(t)=\beta(t)=\ln t$.

7. Discussion of the results of studying the growth of harmonic functions in the space $\mathbb{R}^n$

The study has determined a relation between the maximum terms of entire functions of finite order in the plane given by power series

$$\sum_{k=0}^{\infty}b_k z^k,\ \ \sum_{k=0}^{\infty}d_k b_k z^k,$$

the coefficients of which satisfy the conditions

$$b_k>0,\ \ \frac{1}{h(k)}\leq d_k\leq h(k),$$

where h is a non-decreasing positive function.

The result is the estimation of the maximum modulus of the entire harmonic function of several variables through the maximum modulus of some entire function of a complex variable in which coefficients of the power series are somewhat connected with the coefficients of the expansion of the harmonic function in a series by Laplace spherical functions. This finding has made it possible to obtain an analog of the classical Borel theorem for harmonic functions of finite order in the space $\mathbb{R}^n$.

Besides, the research has helped determine the most general characteristics of the growth of the harmonic function in $\mathbb{R}^n$ in terms of the uniform norm of Laplace spherical functions in the expansion of this function in a series. This allows estimating the growth of a harmonic function directly by the behaviour of the coefficients of its expansion in a series, which is important in the theory of series, differential equations, and approximation.

The results obtained in the case of n=3 can be used in geodesy, where it is natural to have Laplace series, in particular for describing the gravitational field of Earth, the form of Earth, relief or other values that are given in the form of a map on spherical surfaces.

Other areas in which further research can be carried out are the use of norms other than uniform, the establishment of formulae for the generalized characteristics of the growth of harmonic functions for the case when the space is exhausted by some complete regions, and also the improvement of the condition of finiteness of order in the analog of the Borel theorem for harmonic functions in an n- dimensional space.

7. Conclusions

1. We obtain a relation between the maximum terms of entire finite-order functions in the plane given by power series whose coefficients are somewhat connected. This has made it possible to determine how the logarithms of the maximum terms of entire functions in the plane differ, depending on the coefficients of the expansion of these functions in the power series; the finding was used to prove the analog of the classical Borel theorem for entire harmonic finite-order functions in $\mathbb{R}^n$.

2. An estimate for the maximum modulus of an entire harmonic function of several variables through the maximum modulus of some entire function of one complex variable has been obtained. This has made it possible to prove the analog of the classical Borel theorem for entire harmonic

functions of finite order in $\mathbb{R}^n$ and express the generalized characteristics of the growth of harmonic functions in the space in terms of the uniform norm of Laplace spherical functions in the expansion of harmonic functions in series.

3. An estimate has been made for the maximum modulus of the entire harmonic function of several variables through the maximum term of some entire function of one complex variable. This made it possible to express the generalized characteristics of the growth of harmonic functions in the space through the uniform norm of Laplace spherical functions in the expansion of harmonic functions in series.

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