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STABILITY OF STRUCTURAL ELEMENTS OF SPECIAL LIFTING MECHANISMS IN THE FORM OF CIRCULAR ARCHES

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Викладений алгоритм вирішення задач стійкості плоскої форми згину елементів спеціальних вантажопідйомних механізмів у вигляді кругових арок з перерізами, що мають дві вісі симетрії. Проінтегрована система диференціальних рівнянь стійкості елементів у вигляді кругових арок. Побудовані варіанти систем фундаментальних функцій диференціальних рівнянь стійкості арок з постійними коефіцієнтами. Задачі стійкості запропоновано вирішувати МГЕ

Ключові слова: стійкість, система диференціальних рівнянь зі змінними коефіцієнтами, фундаментальні функції, МГЕ

Изложен алгоритм решения краевых задач устойчивости плоской формы изгиба элементов специальных грузоподъемных механизмов в виде круговых арок с сечениями, имеющими две оси симметрии. Проинтегрирована система дифференциальных уравнений устойчивости элементов в виде круговых арок. Построены варианты систем фундаментальных функций дифференциальных уравнений устойчивости арок с постоянными коэффициентами. Задачи устойчивости предложено решать МГЭ

Ключевые слова: устойчивость, система дифференциальных уравнений с переменными коэффициентами, фундаментальные функции, МГЭ

1. Introduction

Booms of special lifting machines have the form of circular arches. The use of circular arches is due to the advantages over rectilinear rods in strength and rigidity. In this regard, arch elements of crane structures very often have a large ratio of the axial moments of inertia of cross-sections. In this case, the design meets the requirements of strength and rigidity, but at the same time, there is a risk of lateral-torsional buckling. After buckling, the rod experiences two bends and torsion. Significant cross-section displacements often lead to various accidents.

The phenomenon of buckling can be prevented by calculation. However, this requires appropriate, sufficiently accurate and reliable mathematical models of buckling processes. At present, theoretical developments of stability of the simple bending of the circular arch are rudimentary and do not allow solving important practical problems in the needed amount. Thus, the problem of creating computational models of stability problems of circular arches is relevant and necessary for practice [1–7].

2. Literature review and problem statement

The problem of stability of the simple bending of rectilinear beams with sections in the form of a narrow strip has been posed as early as the 19th century. Much later, the theory of spatial stability of plane and spatial rods and rod systems has been generalized [1].

The constructed theory could not be used for a long time because the corresponding differential equations had variable coefficients and integration encountered serious mathematical difficulties [2]. There are known solutions to various problems of calculating the curves of rods in the form of circular arches taking into account only bending deformation [3].

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This problem has found the effective resolution only with the advent of a numerical-analytic version of the boundary element method (BEM). This method allows mathematically rigorous and exact solution of boundary value problems for the linear homogeneous and inhomogeneous differential equations with variable coefficients [4, 5].

Various solutions of differential stability equations are accumulated for rectilinear rods, while for circular arches there are no fundamental solution functions for Cauchy problems of stability of the simple bending. The problems of stability of the simple bending of circular arches can be solved by means of professional packages of the finite element method (FEM) such as Ansys, Solid Works, Abaqus, etc. At this time, the FEM is the most common numerical method, has a rather simple algorithm logic and a large number of arithmetic operations [6]. However, the lack of an exact stiffness matrix of the problems of stability of the simple bending of structural elements in the form of circular arches does not allow obtaining accurate and reliable results with an arbitrarily large sampling of the structure. The application of the BEM algorithm compares favorably [7]. It uses an exact system of differential equations of the problem, a mathematically rigorous procedure for constructing its solution, and a very logically simple process of forming a resolving system of linear algebraic equations of the boundary value stability problem [8]. In addition, as shown in [9], the BEM allows obtaining exact values of the problem parameters (forces, displacements, stresses, natural vibration frequencies [10, 11], buckling forces) both at the boundary and within the region. Moreover, the BEM has the simplest algorithm logic among other numerical methods, good convergence of the solution, high stability of arithmetic operations, and a very small accumulation of rounding errors in numerical operations [12]. At the same time, the method is characterized by the simplicity of the algorithm logic [10–12], good convergence of the minimum error of the solution results and high stability.

In this regard, the literature review logically leads to the following formulation of the aim and objectives of the study.

3. The aim and objectives of the study

The aim of this paper is to construct a system of fundamental orthonormal functions for problems of stability of the simple bending of circular arches with sections with two or more axes of symmetry.

To achieve the aim, the following objectives were set:

– to simplify the general differential equations of stability of circular arches with allowance for the symmetry of their sections;

– to obtain the resolving ordinary differential equation of the problems under consideration;

– to construct the systems of fundamental orthonormal functions of the differential equation for the two most important cases of the roots of the characteristic equation;

– to present practical recommendations on the application of the resulting calculated ratios of boundary value problems of stability of arches.

4. Development of software

The system of equations of stability of the simple bending of a circular rod, after taking into account the symmetry of the section, is reduced to the following form [1]:

$$\begin{cases} EI_y w^{IV}(\alpha)+\dfrac{EI_\varpi}{R}\theta^{IV}(\alpha)+\left[M_Z(\alpha)-\dfrac{GI_d}{R}\right]\theta^{II}(\alpha)=0; \\ EI_\omega\theta^{IV}(\alpha)-GI_d\theta^{II}(\alpha)+\left[M_Z(\alpha)-\dfrac{EI_y}{R}\right]w^{II}(\alpha)=0, \end{cases} \quad (1)$$

where EI_y – rigidity of the section in the horizontal xOz plane; $w(\alpha)$ – flexural motion of the rod axis along the Oz axis (Fig. 1); EI_{w} – sectorial rigidity of the section under the constrained torsion; R – radius of the axis of the circular rod; $\theta(\alpha)$ – angle of torsion of the section around the Ox axis; $M_Z(\alpha)$ – bending moment in the section caused by a given transverse load; GI_d – rigidity of the section under torsion; α – angular coordinate of the current section.

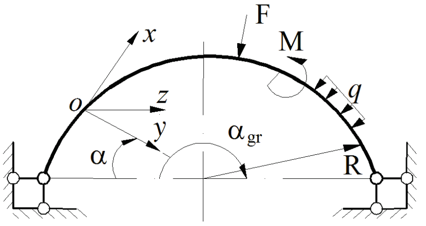

Fig. 1. Design scheme of the problem of stability of a circular rod

It can be seen that the system (1) has variable coefficients in the form of the bending moment $M_Z(\alpha)$. Considering that this is generally a set of simple functions, the difficulties that will be encountered in the integration of this system become obvious.

The problem can be substantially simplified if we use the numerical-analytical version of the BEM [2–5].

In this method, it is necessary to have a solution of the Cauchy problem for equations (1), but with constant coefficients. We now describe the procedure for integrating the simplified system of equations. The initial parameters of the constrained torsion and bending in the horizontal plane are as follows:

$GI_d\theta(0)$ – torsion angle, kNm$_2$;

$GI_d\theta^I(0)$ – derivative of the torsion angle, kNm;

$B_\varpi(0)=-\dfrac{GI_d}{k^2}\theta^{II}(0)$ – bimoment, kNm^2;

$k=\sqrt{\dfrac{GI_d}{EI_\omega}}$ – flexural-torsional characteristic of the section, $\dfrac{1}{\text{m}}$;

$M_\varpi(0)=-\dfrac{GI_d}{k^2}\theta^{III}(0)$ – flexural-torsional moment, kNm;

$EI_y w(0)$ – motion of the section towards the Oz axis, kNm^3;

$EI_y w'(0)=EI_y\phi(0)$ – angle of rotation of the section, kNm^2;

$EI_y w''(0)=-M_y(0)$ – bending moment in the horizontal plane, kNm;

$EI_y w'''(0)=-Q_z(0)$ – transverse force in the horizontal plane, kN.

These initial parameters and the system of equations (1) form the Cauchy problem of stability of the plane of the bending shape of the circular rod. To form fundamental solutions of the Cauchy problem, we perform a number of transformations.

From the second equation of the system (1), it follows that (M_Z=const):

$$w''(\alpha)=\frac{1}{\left(M_Z-\frac{EI_y}{R}\right)}\left[-EI_\omega\theta^{IV}(\alpha)+GI_d\theta^{II}(\alpha)\right]. \qquad (2)$$

By double integration of this expression, we obtain a connection between the flexural motion $w(\alpha)$ and the torsion angle $\theta(\alpha)$:

$$w(\alpha)=\frac{1}{\left(M_Z-\frac{EI_y}{R}\right)}\left[-EI_\omega\theta^{II}(\alpha)+GI_d\theta(\alpha)\right]+$$
$$+(A\cdot\alpha+B)\frac{1}{\left(M_Z-\frac{EI_y}{R}\right)}, \qquad (3)$$

where the integration constants are equal to

$$B=\left(M_Z-\frac{EI_y}{R}\right)w_{(0)}+EI_\omega\theta^{II}(0)-GI_d\theta(0);$$
$$A=\left(M_Z-\frac{EI_y}{R}\right)w'_{(0)}+EI_\omega\theta^{III}(0)-GI_d\theta'(0). \qquad (4)$$

If we substitute $w''(\alpha)$ from (2) into the first equation of the system (1), we obtain the resolving differential equation of stability of the simple bending the circular rod:

$$-z_1\theta^{VI}_{(\alpha)}+z_2\theta^{IV}_{(\alpha)}+z_3\theta^{II}_{(\alpha)}=0, \qquad (5)$$

where

$$z_1=\frac{EI_y\cdot EI_\omega}{\left(M_Z-\frac{EI_y}{R}\right)};\quad z_2=\frac{EI_y\cdot GI_d}{\left(M_Z-\frac{EI_y}{R}\right)}+\frac{EI_\omega}{R};$$
$$z_3=M_Z-\frac{GI_d}{R}. \qquad (6)$$

The equation (5) is classified as the sixth-order linear homogeneous differential equation with constant coefficients. Its solution can be obtained according to the standard scheme. The characteristic equation for (5) has the form:

$$(-z_1)t^6+z_2t^4+z_1t^2=0. \qquad (7)$$

Its roots are of various kinds. Consider the two most important combinations of the roots.

First case.

$$t_{1,2}=0$$

– valid multiples;

$$t_{3,4}=\pm\sqrt{\frac{-z_2+\sqrt{z_2^2+4z_1z_3}}{-2z_1}} \qquad (8)$$

– two valid roots;

$$t_{5,6}=\pm i\sqrt{\frac{z_2+\sqrt{z_2^2+4z_1z_3}}{2z_1}}$$

– two imaginary roots.

The general solution of the equation (5) can be written in the form:

$$\theta(\alpha)=C_1+C_2\cdot\alpha+C_3\operatorname{ch}a\alpha+$$
$$+C_4\operatorname{sh}a\alpha+C_5\cdot\cos b\alpha+C_6\sin b\alpha, \qquad (9)$$

where

$$a=\sqrt{\frac{-z_2+\sqrt{z_2^2+4z_1z_3}}{-2z_1}};\quad b=\sqrt{\frac{z_2+\sqrt{z_2^2+4z_1z_3}}{2z_1}}. \qquad (10)$$

By five-time differentiation of the expression (9), taking into account the ratios between the initial parameters and expression (3), we can form a system of linear algebraic equations for the integration constants C_1–C_2:

$$\begin{pmatrix} 1 & & 1 & & 1 & \\ & 1 & & a & & b \\ & & a^2 & & -b^2 & \\ & & & a^3 & & -b^3 \\ & & A_{53} & & A_{55} & \\ & & & A_{64} & & A_{66} \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix}=\begin{pmatrix} \theta(0) \\ \theta'(0) \\ -\frac{B_\omega(0)k^2}{GI_d} \\ -\frac{M_\omega(0)k^2}{GI_d} \\ -\frac{M_y(0)}{GI_y} \\ -\frac{Q_z(0)}{EI_y} \end{pmatrix}, \qquad (11)$$

where elements of the coefficient matrix of the equation (11) have the form:

$$A_{53}=\frac{a^2\left(-EI_\omega a^2+GI_d\right)}{M_Z-\frac{EI_y}{R}};\quad A_{55}=\frac{-b^2\left(EI_\omega b^2+GI_d\right)}{M_Z-\frac{EI_y}{R}};$$
$$A_{64}=\frac{a^3\left(-EI_\omega a^2+GI_d\right)}{M_Z-\frac{EI_y}{R}};\quad A_{66}=\frac{-b^3\left(EI_\omega b^2+GI_d\right)}{M_Z-\frac{EI_y}{R}}. \qquad (12)$$

The integration constants after solving the system of the equations (11) are written in the form:

$$C_1=\theta_{(0)}-\frac{a^2+b^2}{x_1b^2-x_2a^2}\left[-\frac{M_{y(0)}}{EI_y}\right]+\frac{x_1+x_2}{x_1b^2-x_2a^2}\left[-\frac{B_{\omega(0)k^2}}{GI_d}\right];$$
$$C_2=\theta^I_{(0)}-\frac{a^2+b^2}{x_1b^2-x_2a^2}\left[-\frac{Q_{z(0)}}{EI_y}\right]+\frac{x_1+x_2}{x_1b^2-x_2a^2}\left[-\frac{M_{\omega(0)k^2}}{GI_d}\right];$$
$$C_3=\frac{b^2}{x_1b^2-x_2a^2}\left[-\frac{M_{y(0)}}{EI_y}\right]-\frac{x_2}{x_1b^2-x_2a^2}\left[-\frac{B_{\omega(0)k^2}}{GI_d}\right];$$
$$C_4=\frac{b^2}{a\left(x_1b^2-x_2a^2\right)}\left[-\frac{Q_{z(0)}}{EI_y}\right]-\frac{x_2}{a\left(x_1b^2-x_2a^2\right)}\left[-\frac{M_{\omega(0)k^2}}{GI_d}\right];$$
$$C_5=\frac{a^2}{x_1b^2-x_2a^2}\left[-\frac{M_{y(0)}}{EI_y}\right]-\frac{x_1}{x_1b^2-x_2a^2}\left[-\frac{B_{\omega(0)k^2}}{GI_d}\right]; \qquad (13)$$

where the following are denoted:

$$x_1 = \frac{a^2\left(-EI_\omega a^2 + GI_d\right)}{M_Z - \frac{EI_y}{R}};$$

$$x_2 = \frac{b^2\left(EI_\omega b^2 + GI_d\right)}{M_Z - \frac{EI_y}{R}}. \qquad (14)$$

The constants C_1–C_6 are substituted into the expression for the torsion angle $\theta(\alpha)$ (9) and then four bending parameters (using the expression (3)) and four parameters of the constrained torsion relative to the corresponding initial parameters can be formed. After rationing of the fundamental functions, it is convenient to present these expressions in the matrix form as follows:

$$\begin{pmatrix} EI_y w(\alpha) \\ EI_y \varphi(0) \\ M_y(\alpha) \\ Q_z(\alpha) \\ GI_d\theta(\alpha) \\ GI_d\theta'(\alpha) \\ B_\omega(\alpha) \\ M_\omega(\alpha) \end{pmatrix} = \\ = \begin{pmatrix} 1 & \alpha & -A_{13} & -A_{14} & & & -A_{17} & -A_{18} \\ & 1 & -A_{23} & -A_{24} & & & -A_{27} & -A_{28} \\ & & A_{33} & A_{34} & & & A_{37} & A_{38} \\ & & A_{43} & A_{44} & & & A_{47} & A_{48} \\ & & -A_{53} & -A_{54} & 1 & \alpha & -A_{57} & -A_{58} \\ & & -A_{63} & -A_{64} & & 1 & -A_{67} & -A_{68} \\ & & A_{73} & A_{74} & & & A_{77} & A_{78} \\ & & A_{83} & A_{84} & & & A_{87} & A_{88} \end{pmatrix} \times \\ \times \begin{pmatrix} EI_y w(0) \\ EI_y \varphi(0) \\ M_y(0) \\ Q_z(0) \\ GI_d\theta(0) \\ GI_d\theta'(0) \\ B_\omega(0) \\ M_\omega(0) \end{pmatrix}. \qquad (15)$$

From this expression, it follows that when solving the problems of stability of circular arches by the BEM, it is necessary to solve only eight equations, with an error of less than 1 % [11]. According to the FEM, as the experiment shows [12], it will be required to derive a thousand equations, with an error of 5 % or more.

The fundamental orthonormal functions of the equation (15) take the form:

$$A_{13} = \frac{-\left(a^2+b^2\right)c + b^2\frac{x_1}{a^2}cha\alpha + a^2\frac{x_2}{a^2}\cos b\alpha}{x_1b^2 - x_2a^2}; \quad C = \frac{GI_d}{M_z - \frac{EI_y}{R}};$$

$$A_{14} = \frac{-ab\left(a^2+b^2\right)c\alpha + b^3\frac{x_1}{a^2}sha\alpha + a^3\frac{x_2}{a^2}\sin b\alpha}{ab\left(x_1b^2 - x_2a^2\right)};$$

$$A_{17} = \frac{k^2\left(x_1+x_2\right)c - k^2x_2\frac{x_1}{a^2}cha\alpha + k^2x_1\frac{x_2}{b^2}\cos b\alpha + \left(x_1b^2 - x_2a^2\right)c}{x_1b^2 - x_2a^2}\cdot\frac{EI_y}{GI_d};$$

$$A_{18} = \frac{k^2ab\left(x_1+x_2\right)c\alpha - k^2\cdot x_2b\frac{x_1}{a^2}sha\alpha - k^2ax_1\frac{x_2}{b^2}\sin b\alpha + ab\left(x_1b^2 - x_2a^2\right)c\alpha}{ab\left(x_1b^2 - x_2a^2\right)}\cdot\frac{EI_y}{GI_d};$$

$$A_{23} = \frac{x_1b^3\frac{x_1}{a^2}sha\alpha - x_2a^3\sin b\alpha}{ab\left(x_1b^2 - x_2a^2\right)}; \quad A_{24} = A_{13}; \quad A_{34} = A_{23};$$

$$A_{27} = \frac{-k^2x_1x_2bsha\alpha + k^2x_1x_2a\sin b\alpha}{ab\left(x_1b^2 - x_2a^2\right)}\cdot\frac{EI_y}{GI_d}; \quad A_{44} = A_{33};$$

$$A_{33} = \frac{x_1b^2cha\alpha - x_2a^2\cos b\alpha}{x_1b^2 - x_2a^2}; \quad A_{37} = \frac{[x_1x_2b^2\left(cha\alpha - \cos b\alpha\right)]k^2}{x_1b^2 - x_2a^2}\cdot\frac{EI_y}{GI_d};$$

$$A_{43} = \frac{x_1ab^2sha\alpha + x_2a^2\sin b\alpha}{x_1b^2 - x_2a^2}; \quad A_{47} = \frac{-x_1x_2b^2\left(asha\alpha + b\sin b\alpha\right)k^2}{x_1b^2 - x_2a^2}\cdot\frac{EI_y}{GI_d};$$

$$A_{28} = A_{17}; \quad A_{38} = A_{27}; \quad A_{48} = A_{37}; \quad A_{53} = \frac{-b^2\left(1 - cha\alpha\right) - a^2\left(1 - \cos b\alpha\right)}{x_1b^2 - x_2a^2}\cdot\frac{GI_d}{EI_y};$$

$$A_{54} = \frac{-b^3\left(a\alpha - sha\alpha\right) - a^3\left(b\alpha - \sin b\alpha\right)}{a\alpha\left(x_1b^2 - x_2a^2\right)}\cdot\frac{GI_d}{EI_y};$$

$$A_{57} = \frac{\left[x_2\left(1 - cha\alpha\right) + x_1\left(1 - \cos b\alpha\right)\right]k^2}{x_1b^2 - x_2a^2};$$

$$A_{58} = \frac{\left[bx_2\left(a\alpha - sha\alpha\right) + ax_1\left(1 - \cos b\alpha\right)\right]k^2}{ab\left(x_1b^2 - x_2a^2\right)}; \quad A_{63} = \frac{ab^2sha\alpha - a^2b\sin b\alpha}{x_1b^2 - x_2a^2}\cdot\frac{GI_d}{EI_y};$$

$$A_{64} = A_{53}; \quad A_{67} = \frac{\left(-x_2asha\alpha + x_1b\sin b\alpha\right)k^2}{x_1b^2 - x_2a^2};$$

$$A_{68} = A_{57}; \quad A_{73} = \frac{a^2b^2\left(cha\alpha - \cos b\alpha\right)k^2}{x_1b^2 - x_2a^2}\cdot\frac{GI_d}{k^2EI_y};$$

$$A_{74} = \frac{A_{63}}{k^2}; \quad A_{77} = \frac{-x_2a^2cha\alpha + x_1b^2\cos b\alpha}{x_1b^2 - x_2a^2};$$

$$A_{78} = \frac{A_{67}}{k^2}; \quad A_{83} = \frac{a^3b^2sha\alpha + a^2b^3\sin b\alpha}{x_1b^2 - x_2a^2}\cdot\frac{GI_d}{k^2EI_y};$$

$$A_{84} = A_{73}; \quad A_{87} = \frac{-x_2a^3sha\alpha - x_1b^3\sin b\alpha}{x_1b^2 - x_2a^2}; \quad A_{88} = A_{77}. \qquad (16)$$

The expression (15) is the resolving equation of the BEM for solving boundary value problems of stability of the simple bending of structures in the form of individual arches, rings, ring systems, and combined arch systems.

Second case.

The roots are valid multiple and imaginary:

$$r^4+s^4>0;\ \ s^4<0;\ \ r^4<0.$$

$$b_1=\sqrt{-r^2-\sqrt{r^4+s^4}};\ \ b_2=\sqrt{-r^2+\sqrt{r^4+s^4}};$$

$$r^2=\frac{z_2}{2z_1};\ \ s^4=\frac{z_3}{z_1};$$

$$z_1=\frac{EI_yEI_\omega}{M_z-\dfrac{EI_y}{R}};\ \ z_2=\frac{EI_yGI_d}{\left(M_z-\dfrac{EI_y}{R}\right)}+\frac{EI_\omega}{R};$$

$$z_3=\left(M_z-\frac{GI_d}{R}\right). \tag{17}$$

The general solution of the equation (5) takes the form:

$$\theta(\alpha)=C_1+C_2\alpha+C_3\cdot\cos b_1\alpha+$$
$$+C_4\sin b_1\alpha+C_5\cos b_2\alpha+C_6\sin b_2\alpha. \tag{18}$$

The integration constants, expressed through the initial parameters of the equation (11) for this case, have the form:

$$C_1=\theta_{(0)}+\frac{b_2^2-b_1^2}{x_1b_2^2-x_2b_1^2}\left[-\frac{M_{y(0)}}{EI_y}\right]-\frac{x_2-x_1}{x_1b_2^2-x_2b_1^2}\left[-\frac{B_{\omega(0)}k^2}{GI_d}\right];$$

$$C_2=\theta_{(0)}+\frac{b_1b_2\left(b_2^2-b_1^2\right)}{x_3b_2^3-x_4b_1^3}\left[-\frac{Q_{z(0)}}{EI_y}\right]-\frac{b_1x_1-b_2x_3}{x_3b_2^3-x_4b_1^3}\left[-\frac{M_{\omega(0)}k^2}{GI_d}\right];$$

$$C_3=-\frac{b_2^2}{x_1b_2^2-x_2b_1^2}\left[-\frac{M_{y(0)}}{EI_y}\right]-\frac{x_2}{x_1b_2^2-x_2b_1^2}\left[-\frac{B_{\omega(0)}k^2}{GI_d}\right];$$

$$C_4=-\frac{b_2^3}{x_3b_2^3-x_4b_1^3}\left[-\frac{Q_{z(0)}}{EI_y}\right]-\frac{x_4}{x_3b_2^3-x_4b_1^3}\left[-\frac{M_{\omega(0)}k^2}{GI_d}\right];$$

$$C_5=\frac{b_1^2}{x_1b_2^2-x_2b_1^2}\left[-\frac{M_{y(0)}}{EI_y}\right]-\frac{x_1}{x_1b_2^2-x_2b_1^2}\left[-\frac{B_{\omega(0)}k^2}{GI_d}\right];$$

$$C_6=\frac{b_1^3}{x_3b_2^3-x_4b_1^3}\left[-\frac{Q_{z(0)}}{EI_y}\right]-\frac{x_3}{x_3b_2^3-x_4b_1^3}\left[-\frac{M_{\omega(0)}k^2}{GI_d}\right];$$

$$x_1=\frac{b_1^2\left(EI_\omega b_1^2+GI_d\right)}{\left(M_z-\dfrac{EI_y}{R}\right)};\ \ x_2=\frac{b_2^2\left(EI_\omega b_2^2+GI_d\right)}{\left(M_z-\dfrac{EI_y}{R}\right)};$$

$$x_3=b_1x_1;\ \ x_4=b_2\cdot x_2. \tag{19}$$

The fundamental orthonormal functions of the equation (15) after all transformations are written in the form:

$$A_{13}=\frac{\left(b_2^2-b_1^2\right)c-b_2^2\dfrac{x_1}{b_1^2}\cos b_1\alpha-b_1^2\dfrac{x_2}{b_2^2}\cos b_2\alpha}{x_1b_2^2-x_2b_1^2};$$

$$A_{14}=\frac{b_1b_2\left(b_2^2-b_1^2\right)c\alpha-b_2^3\dfrac{x_3}{b_1^3}\sin b_1\alpha+b_1^3\dfrac{x_4}{b_2^3}\sin b_2\alpha}{x_3b_2^3-x_4b_1^3};$$

$$A_{17}=\frac{\begin{bmatrix}-\left(x_2-x_1\right)c+x_2\dfrac{x_1}{b_1^2}\cos b_1\alpha-\\-\ x_1\dfrac{x_2}{b_2^2}\cos b_2\alpha\end{bmatrix}k^2+\left(x_1b_2^2-x_2b_1^2\right)c}{x_1b_2^2-x_2b_1^2}\cdot\frac{EI_y}{GI_d};$$

$$A_{18}=\frac{\begin{bmatrix}-\left(b_1x_4-b_2x_3\right)c\alpha+\\+x_4\dfrac{x_3}{b_1^3}\sin b_1\alpha-x_3\dfrac{x_4}{b_2^3}\sin b_2\alpha\end{bmatrix}k^2+\left(x_3b_2^3-x_4b_1^3\right)c}{x_1b_2^3-x_2b_1^3}\cdot\frac{EI_y}{GI_d};$$

$$A_{23}=\frac{b_2^2\dfrac{x_1}{b_1}\sin b_1\alpha-b_1^2\dfrac{x_2}{b_2}\sin b_2\alpha}{x_1b_2^2-x_2b_1^2};$$

$$A_{24}=\frac{b_1b_2\left(b_2^2-b_1^2\right)c-b_2^3\dfrac{x_3}{b_1^2}\cos b_1\alpha+b_1^3\dfrac{x_4}{b_2^2}\cos b_2\alpha}{x_3b_2^3-x_4b_1^3};$$

$$A_{27}=\frac{\left[-x_2\dfrac{x_1}{b_1}\sin b_1\alpha+x_1\dfrac{x_2}{b_2}\sin b_2\alpha\right]k^2}{x_1b_2^2-x_2b_1^2}\cdot\frac{EI_y}{GI_d};$$

$$A_{28}=\frac{\left[-\left(b_1x_4-b_2x_3\right)c+x_4\dfrac{x_3}{b_1^2}\cos b_1\alpha-x_3\dfrac{x_4}{b_2^2}\cos b_2\alpha\right]}{x_3b_2^3-x_4b_1^3}\cdot\frac{EI_y}{GI_d};$$

$$A_{33}=\frac{b_2^2x_1\cos b_1\alpha-b_1^2x_2\cos b_2\alpha}{x_1b_2^2-x_2b_1^2};$$

$$A_{34}=\frac{b_2^3\dfrac{x_3}{b_1}\sin b_1\alpha-b_1^3\dfrac{x_4}{b_2}\sin b_2\alpha}{x_3b_2^3-x_4b_1^3};$$

$$A_{37}=\frac{\left[-x_1x_2\cos b_1\alpha+x_1x_2\cos b_2\alpha\right]k^2}{x_1b_2^2-x_2b_1^2}\cdot\frac{EI_y}{GI_d};$$

$$A_{38}=\frac{\left[-x_4\dfrac{x_3}{b_1}\sin b_1\alpha+x_3\dfrac{x_4}{b_2}\sin b_2\alpha\right]k^2}{x_3b_2^3-x_4b_1^3}\cdot\frac{EI_y}{GI_d};$$

$$A_{43}=\frac{-b_1b_2^2x_1\sin b_1\alpha+b_1^2b_2x_2\sin b_2\alpha}{x_1b_2^2-x_2b_1^2};$$

$$A_{44}=\frac{b_2^3x_3\cos b_1\alpha-b_1^3x_4\cos b_2\alpha}{x_3b_2^3-x_4b_1^3};$$

$$A_{47}=\frac{\left[x_1x_2b_1\sin b_1\alpha-x_1x_2b_2\sin b_2\alpha\right]k^2}{x_1b_2^2-x_2b_1^2}\cdot\frac{EI_y}{GI_d};$$

$$A_{48}=\frac{\left[-x_3x_4\frac{x_3}{b_1}\cos b_1\alpha+x_3x_4\frac{x_4}{b_2}\cos b_2\alpha\right]k^2}{x_3b_2^3-x_4b_1^3}\cdot\frac{EI_y}{GI_d};$$

$$A_{53}=\frac{b_2^2\left(1-\cos b_1\alpha\right)-b_1^2\left(1-\cos b_2\alpha\right)}{x_1b_2^2-x_2b_1^2}\cdot\frac{GI_d}{EI_y};$$

$$A_{54}=\frac{b_2^3\left(b_1\alpha-\sin b_1\alpha\right)-b_1^3\left(b_2\alpha-\sin b_2\alpha\right)}{x_3b_2^3-x_4b_1^3}\cdot\frac{GI_d}{EI_y};$$

$$A_{57}=\frac{\left[-x_2\left(1-\cos b_1\alpha\right)+x_1\left(1-\cos b_2\alpha\right)\right]k^2}{x_1b_2^2-x_2b_1^2};$$

$$A_{58}=\frac{\left[-x_4\left(b_1\alpha-\sin b_1\alpha\right)+x_3\left(b_2\alpha-\sin b_2\alpha\right)\right]k^2}{x_3b_2^3-x_4b_1^3};$$

$$A_{63}=\frac{b_1b_2^2\sin b_1\alpha-b_1^2b_2\sin b_2\alpha}{x_1b_2^2-x_2b_1^2}\cdot\frac{GI_d}{EI_y};$$

$$A_{64}=\frac{b_1b_2^3\left(1-\cos b_1\alpha\right)-b_1^3b_2\left(1-\cos b_2\alpha\right)}{x_3b_2^3-x_4b_1^3}\cdot\frac{GI_d}{EI_y};$$

$$A_{67}=\frac{\left[-x_2b_1\sin b_1\alpha+x_1b_2\sin b_2\alpha\right]k^2}{x_1b_2^2-x_2b_1^2};$$

$$A_{68}=\frac{\left[-x_4b_1\left(1-\cos b_1\alpha\right)+x_3b_2\left(1-\cos b_2\alpha\right)\right]k^2}{x_3b_2^3-x_4b_1^3};$$

$$A_{73}=\frac{b_1^2b_2^2\cos b_1\alpha-b_1^2b_2^2\cos b_2\alpha}{\left(x_1b_2^2-x_2b_1^2\right)k^2}\cdot\frac{GI_d}{EI_y};$$

$$A_{74}=\frac{b_1^2b_2^3\sin b_1\alpha-b_1^3b_2^2\sin b_2\alpha}{k^2\left(x_3b_2^3-x_4b_1^3\right)}\cdot\frac{GI_d}{EI_y};$$

$$A_{77}=\frac{-x_2b_1^2\cos b_1\alpha+x_1b_2^2\cos b_2\alpha}{x_1b_2^2-x_2b_1^2};$$

$$A_{78}=\frac{-x_4b_1^2\sin b_1\alpha+x_3b_2^2\sin b_2\alpha}{x_3b_2^3-x_4b_1^3};$$

$$A_{83}=\frac{-b_1^3b_2^3\sin b_1\alpha+b_1^2b_2^3\sin b_2\alpha}{k^2\left(x_1b_2^2-x_2b_1^2\right)}\cdot\frac{GI_d}{EI_y};$$

$$A_{84}=\frac{b_1^3b_2^3\cos b_1\alpha-b_1^3b_2^3\cos b_2\alpha}{k^2\left(x_3b_2^3-x_4b_1^3\right)}\cdot\frac{GI_d}{EI_y};$$

$$A_{87}=\frac{x_2b_1^3\sin b_1\alpha-x_1b_2^3\sin b_2\alpha}{\left(x_3b_2^2-x_2b_1^2\right)};$$

$$A_{88}=\frac{-x_4b_1^3\cos b_1\alpha+x_3b_2^3\sin b_2\alpha}{x_3b_2^3-x_4b_1^3};$$

$$C=\frac{GI_d}{\left(M_z-\frac{EI_y}{R}\right)}. \tag{20}$$

These fundamental functions, as well as the expressions (16), serve as the initial mathematical model of stability problems of circular arches.

5. Discussion of the proposed approach to solving stability problems

5. 1. The case when M_Z=const

This case for circular arches is very rare and is possible only with the hinge support and loading by concentrated equal bending moments. In this case, equation (15) can be used directly for the entire structure using the BEM algorithm [2–6].

5. 2. The case when M_Z is some function of the angular coordinate α

This is the most common case for arch structures. Here it is necessary to have an analytical expression for the $M_Z(\alpha)$ function. This function can be constructed most simply by the BEM algorithm [5, 6], where the procedure for calculating the $M_Z(\alpha)$ function from the existing loads is described exhaustively. Then the arch is broken into n parts [7–9]. In each part, the values of the bending moment M_Z are calculated from the known expression so that the area of the step figure M_Z is equal to the area of the valid plot M_Z. If this condition is met, then for $n\geq30$ almost exact results of critical loads M_{cr}, F_{cr}, q_{cr} are obtained [10–12].

It should be noted that the conducted studies have removed the problems of mathematical modeling of very complex problems of stability of structural elements of lifting machines.

6. Conclusions

1. When solving the problems of stability of the simple bending of the arch by the FEM, it is necessary to solve about 1,000 linear algebraic equations. The error of the solution will be about 5 %. To solve the problems of stability of arches by the BEM, it will be required to solve only eight equations and the error of the results will be less than 1 %.
2. The simplified system of differential equations of problems of stability of the simple bending of rods in the form of circular arches with variable coefficients is presented. Horizontal motions and angles of torsion of the axis of circular arches serve as unknowns.
3. The sixth-order ordinary differential equation with constant coefficients for the considered stability problems and use of the BEM technology is derived. The resulting equation allows constructing an exact analytical solution of the problems of stability of circular arches according to the known theory.
4. The matrix equation of boundary value problems of stability of the simple bending of circular arches by the BEM is formed. This equation makes it possible to substantially simplify the logic of solving stability problems and obtain exact values of critical loads.

The analysis of the presented material shows that in the framework of the algorithm of the numerical-analytical version of the BEM it is possible to construct the resolving equation of stability problems of the simple bending of circular rods. This equation can be applied to the solution of very complex problems of stability of various structures containing rods, outlined along the circle arch.

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