Вивчаються енергетичні характеристики поширення хвиль вздовж поверхонь контакту у гідродинамічній системі «рідкий півпростір – шар – шар з твердою кришкою». На основі розв'язків першого наближення слабконелінійної моделі отримано співвідношення для енергії хвильового руху у кожному шарі та для сумарної енергії системи. Проаналізовано залежність енергії хвильового руху при різних геометричних та фізичних параметрах системи

Ключові слова: слабконелінійна модель, тришарова гідродинамічна система, внутрішні хвилі, енергія хвильового руху

Изучаются энергетические характеристики распространения волн вдоль поверхностей контакта в гидродинамической системе «жидкое полупространство – слой – слой с твердой крышкой». На основе решений первого приближения слабонелинейной модели получено соотношение для энергии волнового движения в каждом слое и для суммарной энергии системы. Проанализирована зависимость энергии волнового движения при различных геометрических и физических параметрах системы

Ключевые слова: слабонелинейная модель, трехслойная гидродинамическая система, внутренние волны, энергия волнового движения

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ANALYSIS OF ENERGY OF INTERNAL WAVES IN A THREE-LAYER SEMI-INFINITE HYDRODYNAMIC SYSTEM

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1. Introduction

At the present stage of development of continuum mechanics, more intensive research is carried out into internal and surface waves in fluids of different types. Similar studies are among the most complex ones in modern science as they require construction of complex mathematical models. Practical use of the energy component of waves' propagation and interaction was the impetus to study wave processes. The relevance of qualitative and quantitative analysis of energy characteristics of the internal wave motion in the areas where the ocean has a layered structure is determined by the substantial influence of wave energy distribution between the fluid layers on the navigation safety in such areas. Currently, there is a need to develop the theoretical basis for designing new methods for cancellation of internal waves.

2. Literature analysis and problem statement

Numerous scientific developments address solving the problem on the generation and propagation of wave-packets, as well as studying the energy of wave processes in heterogeneous liquid media of different types.

In paper [1], asymptotic models of propagation of internal waves on the surface of the contact between two layers of immiscible fluids of different density with a rigid lid and a flat bottom were derived. Different asymptotic models were obtained by expanding non-local operators relative to correspondent small parameters, which in different ways depend on the relation of the amplitude, wavelength and the relation of the depth of two layers. In article [2], the problem of fluctuations of the contact surface of two immiscible viscous non-compressed liquids over a hard bottom in the gravitational field is explored. The correctness of the problem, both taking into account the surface tension and without it, is proven and the case when a heavier fluid is above is considered. It was established that due to the significant surface tension, the non-stability of Rayleigh-Taylor stabilizes. Papers [1, 2] contain significant mathematical results regarding construction of the new models and research into conditions of stability of hydrodynamic systems, the natural continuation of which would be application of the theoretical information to detection of new physical effects and patterns of wave motion in layered fluids.

In article [3], the theoretical research into inter-phase waves in a two-layered system with the free surface was conducted. It was shown that a great density difference or a relative wave height leads to greater drift velocity. In addition, trajectories and velocities of fluid particles were constructed with the help of the package of symbolic calculations. However, the results need development in the context of evaluation of energy characteristics of the given two-layer system and other more complex layered systems.

In paper [4], the dynamics of internal single waves in shallow water in an essentially non-linear three-layer system was explored. It was shown that the instability mechanism is manifested as instabilities of the Kelvin-Helmholtz type, similar to the MCC mode (Miyata, Choi, and Camassa) with two layers. An essentially non-linear three-layer system is

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examined in the framework of the model of shallow water that, generally speaking, in this case is justified, but there occurs the need to replace the model with a more complex one at changing the relation of geometric parameters.

Papers [5, 6] focus on the study of phenomena of energy loss and transfer in hydrodynamic systems. In particular, paper [5] examines internal waves, which propagate over the ridge in two-layered fluid, and considers three types of interaction. It was found that different types of waves and ridge interaction are associated with the modified blocking degree. It was established that the maximum wave velocity, loss of wave energy and amplitude have a self-similar characteristic with the blocking degree. Work [6] is devoted to experimental study of formation of harmonic waves as a result of interaction of internal waves. It was found that harmonics by the sum and difference of multiple frequencies of waves' collision are formed at collision of two non-resonance internal waves. The phenomenon of the transfer of relative kinetic energy from non-resonance waves to harmonics formed after the collision was experimentally found. It is desirable to conduct similar research into three-layer systems for more complete coverage of the phenomena of energy loss and transfer.

The concept of maintaining the energy flow for internal waves that propagate in non-homogeneous water in shallow waters was considered in article [7]. The author made an emphasis on application of the Korteweg-de Vries (KdV) equation in the assigned form of cnoidal and single waves. An increase in the wave height and a decrease in phase velocity under condition of an insignificant value of the water depth were demonstrated. The features of dynamics of internal waves under field conditions were established, kinematic criteria of a break were explored in detail and critical heights of internal waves were determined. Numerical modeling was applied to the shallow waters of the southern Baltic Sea, which is an important foundation for subsequent study of energy characteristics of the wave process in hydrodynamic layered systems of the finite or non-finite depth.

Research into propagation and interaction of wave-packets in the system " layer with a bottom – free surface layer" was conducted in article [8]. Specifically, modulation stability of wave packets was analyzed and conditions of passage and the shape of waves were explored, as well as dependences of motion energy of internal and surface waves on geometrical and physical parameters of the system were analyzed. Contribution of the second approximation to the total energy of the system was assessed. This article studies a two-layer system, characteristics of which are useful but insufficient to explore fully the complex processes that occur in the depths of the ocean on the contact of layers of water of different density. Energy characteristics of three-layer hydrodynamic systems are of special interest.

In paper [9], a weakly non-linear model of waves propagation and interaction along contact surfaces in hydrodynamic system "liquid half-space – layer –layer with a rigid lid" was considered, the first three linear approximations were presented. The condition of waves' propagation along the contact surfaces was obtained. The dependence of the relation of waves' amplitudes on the contact surfaces at different geometrical and physical parameters of the system was analyzed. The structure of wave movements on the contact surfaces was studied. Energy characteristics were not analyzed in this paper, dependences of energy of the motion of internal and surface waves on geometrical and physical parameters of the system were not analyzed either.

Thus, based on an analysis of the literary sources, the unresolved part of the problem of waves' propagation in hydrodynamic layered structures now is a qualitative and quantitative analysis of energy characteristics of the hydrodynamic system "liquid half-space – layer –layer with a rigid lid". Estimation of the contribution of the wave motion of every layer to the total energy of the system for different geometric and physical properties of the system is of a particular interest. Such evaluation may reveal existence of the phenomenon of energy pumping between the layers of the system and other effects and patterns.

3. The aim and objectives of the study

The aim of present study is qualitative and quantitative analysis of energy of internal waves that propagate along contact surfaces in a three-layer hydrodynamic system "liquid half-space – layer – layer with a rigid lid". This will make it possible to analyze in more detail the energy component of waves' propagation and interaction in three-layer systems.

To accomplish the aim, the following tasks have been set:

– to obtain the relations for estimation of wave motion energy in each of the layers of a three-layer hydrodynamic system "liquid half-space – layer –layer with a rigid lid";

– to identify quantitative and qualitative features of dependence of total energy of the system and energy of wave motions in the layers on the wave number;

– to identify and analyze dependences of total wave motion energy on thickness of the top layer at different amplitudes of progressive waves on contact surfaces.

4. Statement and methods for studying the problem on waves propagation in a three-layer hydrodynamic system

We explore the problem on propagation of wave-packets of finite amplitude on the surface of a liquid layer

$$\Omega_1 = \{(x,z):\ |x| < \infty,\ -\infty \le z < 0\}$$

with density ρ_1 and

$$\Omega_2 = \{(x,z):\ |x| < \infty,\ 0 \le z < h_2\}$$

with density ρ_2, which are separated by the contact surface

$$z = \eta_1(\mathrm{x,t}) \text{ and } \Omega_3 = \{(x,z):\ |x| < \infty,\ h_2 \le z < h_2 + h_3\}$$

with density ρ_3, which is separated from liquid layer Ω_2 by contact surface

$$z = h_2 + \eta_2(\mathrm{x,t}).$$

When solving the problem, forces of surface tension on contact surfaces are taken into consideration. Gravity is directed perpendicular to the contact surface in the negative z-direction, fluids are considered non-compressed (Fig. 1).

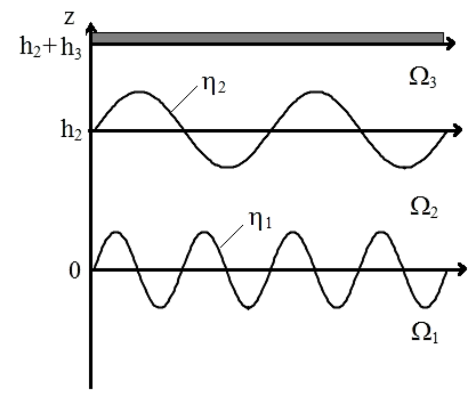

Fig. 1. Diagram of the tree-layer hydrodynamic system “liquid half-space – layer –layer with a rigid lid”

The mathematical model in dimensionless values takes the following form

$$\frac{\partial^2\varphi_j}{\partial x^2}+\frac{1}{\beta}\frac{\partial^2\varphi_j}{\partial z^2}=0 \text{ in } \Omega_j,\ j=1,2,3,$$

$$\frac{\partial\eta_1}{\partial t}-\frac{1}{\beta}\frac{\partial\varphi_j}{\partial z}=-\alpha\frac{\partial\eta_1}{\partial x}\frac{\partial\varphi_j}{\partial x} \text{ at } z=\alpha\eta_1(x,t),\ j=1,2,$$

$$\frac{\partial\eta_2}{\partial t}-\frac{1}{\beta}\frac{\partial\varphi_j}{\partial z}=-\alpha\frac{\partial\eta_2}{\partial x}\frac{\partial\varphi_j}{\partial x}$$

at $z=h_2+\alpha\eta_2(x,t),\ j=2,3,$ (1)

$$\rho_1\frac{\partial\varphi_1}{\partial t}-\rho_2\frac{\partial\varphi_2}{\partial t}+(\rho_1-\rho_2)\eta_1+$$
$$+0.5\rho_1\alpha\left(\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_1}{\partial z}\right)^2\right)-$$
$$-0.5\rho_2\alpha\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-$$
$$-T_1\left[1+\left(\alpha\sqrt{\beta}\frac{\partial\eta_1}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_1}{\partial x^2}=0$$

at $z=\alpha\eta_1(x,t),$

$$\rho_2\alpha\frac{\partial\varphi_2}{\partial t}-\rho_3\alpha\frac{\partial\varphi_3}{\partial t}+\alpha\eta_2(\rho_2-\rho_3)+$$
$$+h_2(\rho_2-\rho_3)+0.5\rho_2\alpha^2\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-$$
$$-0.5\rho_3\alpha^2\left(\left(\frac{\partial\varphi_3}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_3}{\partial z}\right)^2\right)-$$
$$-T_2\alpha\left[1+\left(\alpha\sqrt{\beta}\frac{\partial\eta_2}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_2}{\partial x^2}=0$$

at $z=h_2+\alpha\eta_2(x,t),$

$$\frac{\partial\varphi_3}{\partial z}=0 \text{ at } z=h_2+h_3,$$

$$\left|\overrightarrow{\varphi_1}\right|\to 0 \text{ at } z\to-\infty,$$

where φ_j (j=1, 2, 3) are the potentials of velocities in Ω_j, η_1 and η_2 are the contact surfaces of two liquid layers Ω_1 and Ω_2, Ω_2 and Ω_3, respectively.

Mathematical statement of problem (1) was obtained by means of introduction of dimensionless values with the help of characteristic length H, which is equal to thickness of internal layer h_2, characteristic length of wave L, maximal elevation a of the contact surface between layers Ω_2 and Ω_3, characteristic $\frac{L}{\sqrt{gH}}$, density of lower layer ρ_1. We will proceed to dimensionless values, which will be designated with an asterisk,

$$x=Lx^*,\quad z=Hz^*,\quad \eta_j=a\eta_j^*\ (j=1,2),$$

$$t=\frac{L}{\sqrt{gH}}t^*,\quad \varphi=\frac{gLa}{\sqrt{gH}}\varphi^*,$$

$$T_{1,2}=L^2\rho_1 gT_{1,2}^*,\quad \rho_{1,2,3}=\rho_1\rho_{1,2,3}^*. \tag{2}$$

Values

$$\alpha=\frac{a}{H} \text{ and } \beta=\frac{H^2}{L^2}$$

are non-linearity coefficients and subsequently we will consider the case when α<1, and β=1. The asterisks will be subsequently omitted.

To solve the stated problem, we will use the method of multi-scale expansions to the third order [10]. We will represent the sought-for functions of elevation of contact surfaces and potentials of velocities in the form of

$$\eta_i(x,t)=\sum_{n=1}^{3}\alpha^{n-1}\eta_{in}(x_0,x_1,x_2,t_0,t_1,t_2)+O(\alpha^3),\quad i=1,2,$$

$$\varphi_j(x,z,t)=$$
$$\sum_{n=1}^{3}\alpha^{n-1}\varphi_{jn}(x_0,x_1,x_2,z,t_0,t_1,t_2)+O(\alpha^3),\ j=1,2,3, \tag{3}$$

where $x_k=\alpha_k x$ and $t_k=\alpha_k t$ (k=0, 1, 2). From formulae (3), we derived approximation of the studied system to the third order [9] and performed detailed analysis of solutions of the first approximation.

5. Estimation of wave motion energy of the system based on solutions of the first approximation

While studying the first linear problem [9], dispersion relation was derived

$$\rho_3\omega^2\text{cth}(kh_3)+$$
$$+\frac{\rho_2\omega^2(-\rho_2\omega^2+(-\rho_1\omega^2+k(\rho_1-\rho_2)+T_1k^3)\text{cth}(kh_2))}{-\rho_2\omega^2\text{cth}(kh_2)+(-\rho_1\omega^2+k(\rho_1-\rho_2)+T_1k^3)}=$$
$$=k(\rho_2-\rho_3)+T_2k^3, \tag{4}$$

which has two pairs of roots ω_1 and ω_2. As it was found earlier [9], the existence of two pairs of roots of dispersion relation offers the possibility to obtain two pairs of independent solutions of the first approximation

– for ω_1:

$$\eta_{11}^{(1)} = A\cos(kx - \omega_1 t),$$

$$\eta_{21}^{(1)} = \frac{\rho_2\omega_1^2 \mathrm{ch}(kh_2) + (\rho_1\omega_1^2 - k\rho_1 + k\rho_2 - T_1k^3)\mathrm{sh}(kh_2)}{\rho_2\omega_1^2} A\cos(kx - \omega_1 t),$$

$$\varphi_{11}^{(1)} = \frac{\omega_1}{k}\exp(kz) A\sin(kx - \omega_1 t),$$

$$\varphi_{21}^{(1)} = \left[\frac{(-\rho_1\omega_1^2 + k(\rho_1 - \rho_2) + T_1k^3)\mathrm{ch}(k(h_2 - z))}{\rho_2\omega_1 k\,\mathrm{ch}(kh_2)} + \right.$$
$$\left. + \frac{(-\rho_2\omega_1^2 \mathrm{ch}(kh_2) + (-\rho_1\omega_1^2 + k(\rho_1 - \rho_2) + T_1k^3)\mathrm{sh}(kh_2))\mathrm{sh}(kz)}{\rho_2\omega_1 k\,\mathrm{ch}(kh_2)}\right] A\sin(kx - \omega_1 t), \quad (5)$$

$$\varphi_{31}^{(1)} = \frac{(-\rho_2\omega_1^2 \mathrm{ch}(kh_2) + (-\rho_1\omega_1^2 + k(\rho_1 - \rho_2) + T_1k^3)\mathrm{sh}(kh_2))\mathrm{ch}(k(h_2 + h_3 - z))}{\rho_2\omega_1 k\,\mathrm{sh}(kh_3)} \times$$
$$\times A\sin(kx - \omega_1 t);$$

– for ω_2:

$$\eta_{11}^{(2)} = \frac{\rho_2\omega_2^2}{\rho_2\omega_2^2 \mathrm{ch}(kh_2) + (\rho_1\omega_2^2 - k\rho_1 + k\rho_2 - T_1k^3)\mathrm{sh}(kh_2)} B\cos(kx - \omega_2 t),$$

$$\eta_{21}^{(2)} = B\cos(kx - \omega_2 t),$$

$$\varphi_{11}^{(2)} = \frac{\omega_2}{k}\exp(kz) \times$$
$$\times \frac{\rho_2\omega_2^2}{\rho_2\omega_2^2 \mathrm{ch}(kh_2) + (\rho_1\omega_2^2 - k\rho_1 + k\rho_2 - T_1k^3)\mathrm{sh}(kh_2)} B\sin(kx - \omega_2 t), \quad (6)$$

$$\varphi_{21}^{(2)} = \left[\frac{\omega_2(-\rho_1\omega_2^2 + k(\rho_1 - \rho_2) + T_1k^3)\mathrm{ch}(k(h_2 - z))}{k(\rho_2\omega_2^2 \mathrm{ch}(kh_2) + (\rho_1\omega_2^2 - k\rho_1 + k\rho_2 - T_1k^3)\mathrm{sh}(kh_2))\mathrm{ch}(kh_2)} - \frac{\omega_2 \mathrm{sh}(kz)}{k\,\mathrm{ch}(kh_2)}\right] \times$$
$$\times B\sin(kx - \omega_2 t),$$

$$\varphi_{31}^{(2)} = \frac{\omega_2 \mathrm{ch}(k(h_2 + h_3 - z))}{k\,\mathrm{sh}(kh_3)} B\sin(kx - \omega_2 t),$$

where $\eta_{21}^{(1)}$ is the wave-response to wave $\eta_{11}^{(1)}$ with frequency ω_1 and amplitude A that propagates on the lower contact surface; $\eta_{11}^{(2)}$ is the wave-response to wave $\eta_{21}^{(2)}$ with frequency ω_2 and amplitude B that propagates on the upper contact surface.

Energy, transferred on the upper and lower contact surfaces by internal waves, will be studied according to [11]. We will consider energy, transferred within period τ by two-dimensional progressive waves, assigned by solutions (5) and (6), through the plane x=const. Then the formulae are used

$$E_1 = -\frac{\rho_1}{\tau}\int_t^{t+\tau} \mathrm{d}t \int_{-\infty}^{0} \frac{\partial(\varphi_{11}^{(1)} + \varphi_{11}^{(2)})}{\partial t}\frac{\partial(\varphi_{11}^{(1)} + \varphi_{11}^{(2)})}{\partial x}\mathrm{d}z, \quad (7)$$

$$E_2 = -\frac{\rho_2}{\tau}\int_t^{t+\tau} \mathrm{d}t \int_{0}^{h_2} \frac{\partial(\varphi_{21}^{(1)} + \varphi_{21}^{(2)})}{\partial t}\frac{\partial(\varphi_{21}^{(1)} + \varphi_{21}^{(2)})}{\partial x}\mathrm{d}z, \quad (8)$$

$$E_3 = -\frac{\rho_3}{\tau}\int_t^{t+\tau} \mathrm{d}t \int_{h_2}^{h_2+h_3} \frac{\partial(\varphi_{31}^{(1)} + \varphi_{31}^{(2)})}{\partial t}\frac{\partial(\varphi_{31}^{(1)} + \varphi_{31}^{(2)})}{\partial x}\mathrm{d}z, \quad (9)$$

$$E_s = E_1 + E_2 + E_3, \quad (10)$$

where E_1, E_2, E_3 are the wave motion energy in half-space, middle and upper layers, respectively; E_s is the total energy of the three-layer system. Wave motion energy is found by substitution of expressions of potentials (5)–(6) in formulae (7)–(9), all subsequent calculations are performed in the package of symbolic calculations due to awkwardness of sub-integral expressions. In the course of calculations we take into account that potentials $\varphi_{j1}^{(1)}$ and $\varphi_{j1}^{(2)}$ do not have a common period, since ω_1 and ω_2 are not, generally speaking, rational numbers, sufficiently large values of the length of the integration section τ are used when calculating integral by t.

The formulae (7)–(10) give an estimate of the total energy and its components based on solutions of the first approximation; their analysis is represented in chapter 6.

6. Analysis of wave motion energy based on solutions of the first approximation

6. 1. Wave motion energy in limit cases of the studied system

We will start the study of energy characteristics of the system “liquid half-space – layer –layer with a rigid lid” with consideration of two limit cases, in which a three-layer system degenerates in a two-layer system “liquid half-space – layer with a rigid lid”. The obtained results will be compared with calculation of energy, transferred by waves in the hydrodynamic system “liquid half-space –layer with a rigid lid”, which was performed in [12]. Solutions to this problem in a linear statement take the following form

$$\zeta = 2C\cos(kx - \omega t),$$

$$\varphi_1 = \frac{2\omega C\sin(kx - \omega t)e^{kz}}{k}, \quad (11)$$

$$\varphi_2 = -\frac{2\omega C\sin(kx - \omega t)\cosh(k(h - z))}{k\sinh(kh)}.$$

Energy, transferred by the waves in the mentioned two-layer system, is calculated as follows

$$E_{1c} = -\frac{1}{\tau}\int_t^{t+\tau} \mathrm{d}t \int_{-\infty}^{0} \frac{\partial(\varphi_1)}{\partial t}\frac{\partial(\varphi_1)}{\partial x}\mathrm{d}z = \frac{2\pi\omega^2 A^2}{k^2}, \quad (12)$$

$$E_{2c} = -\frac{\rho_2}{\tau}\int_t^{t+\tau} \mathrm{d}t \int_{0}^{h} \frac{\partial(\varphi_2)}{\partial t}\frac{\partial(\varphi_2)}{\partial x}\mathrm{d}z =$$
$$= \frac{2\pi\omega^2(\cosh(kh_2)\sinh(kh_2) + kh_2)A^2}{k^2\sinh^2(kh_2)}, \quad (13)$$

$$E_c = E_{1c} + E_{2c}. \quad (14)$$

The first limit case of degeneration of a three-layer system "liquid half-space – layer – layer with a rigid lid" in a two-layer system occurs on condition of equality of two layers $\rho_2=\rho_3=0.9$, in this case the amplitude of the upper internal progressive wave $B=0$. In this case, we have the layer of fluid, bounded from above by the rigid lid and liquid half-space below it with the contact surface η_1. Fig. 2, *a* shows dependence E_1, E_2, E_3, E_s and E_c on thickness of the upper layer for the following parameters: $k=0.1$, $T_1=T_2=0$, $h_2=1$, amplitude of the lower internal progressive wave $A=0.1$.

As it is noticeable in Fig. 2, at an increase in thickness of the upper layer, energy E_1, which is transferred on the half-space, increases. At the same time, energy of the middle layer E_2 falls to some level. Energy of the upper layer E_3 increases to its maximum value with an increase in thickness of the upper layer h_3, and at further increase falls to its limit value. We will note that case $h_3 \in [-1, 0]$ is of a purely theoretical nature, where components of energy gain conditional values. As it can be seen from the diagram, total energy of system E_s coincides with the energy of the corresponding two-layer system "liquid half-space – layer with a rigid lid" E_c with thickness of the upper layer $h=h_2+h_3$. Similar coincidence of the diagrams in the final case proves physical accuracy of the obtained results.

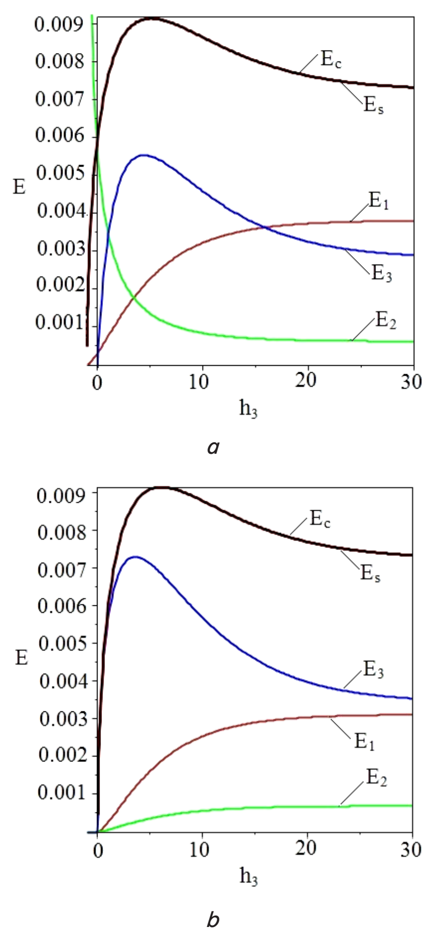

Fig. 2. Diagram of dependence E_1, E_2, E_3, E_s and E_c on thickness of the upper layer h_3: *a* – first degenerated case; *b* – second degenerated case

The second limit case of degeneration of a three-layer system "liquid half-space – layer –layer with a rigid lid" in a two-layer system occurs on condition of equality of density of two lower layers $\rho_1=\rho_2=1$. In this case, we have a liquid layer of thickness h_3, bounded from above by the lid, and half-space, between which there is the contact surface η_2. Fig. 2, *b* shows dependence E_1, E_2, E_3, E_s and E_c on thickness of the upper layer for the following parameters: $k=0.1$, $T_1=T_2=0$, $h_2=1$, $A=0$, $B=0.1$. Similarly to the first limit case, wave motion energy of half-space E_1 increases at an increase in thickness of the upper layer. Energy of the upper layer E_3 also is ascending in character for small values of thickness h_3, but at an increase it starts falling to a certain limit value. A significant difference between the two limit cases lies in dependence of energy of the middle layer E_2 on thickness of the upper layer. In the second case, energy of the middle layer does not decrease, but rather increases to the level close to that observed in the first case. We will note that in the second limit case, total energy of system E_s coincides with energy of system E_c, which proves that physical accuracy of results for a three-layer system.

6. 2. Analysis of dependence of wave energy on wave number

Fig. 3 shows dependence of E_1, E_2, E_3, the sum E_s on wave number k at different values of thickness of the upper layer $h_3=10$, $h_3=3$, $h_3=1$ for the following parameters of the system

$$\rho_1=1,\ \rho_2=0.9,\ \rho_3=0.8,\ T_1=T_2=0,\ h_2=1,\ A=0.1,\ B=0.05.$$

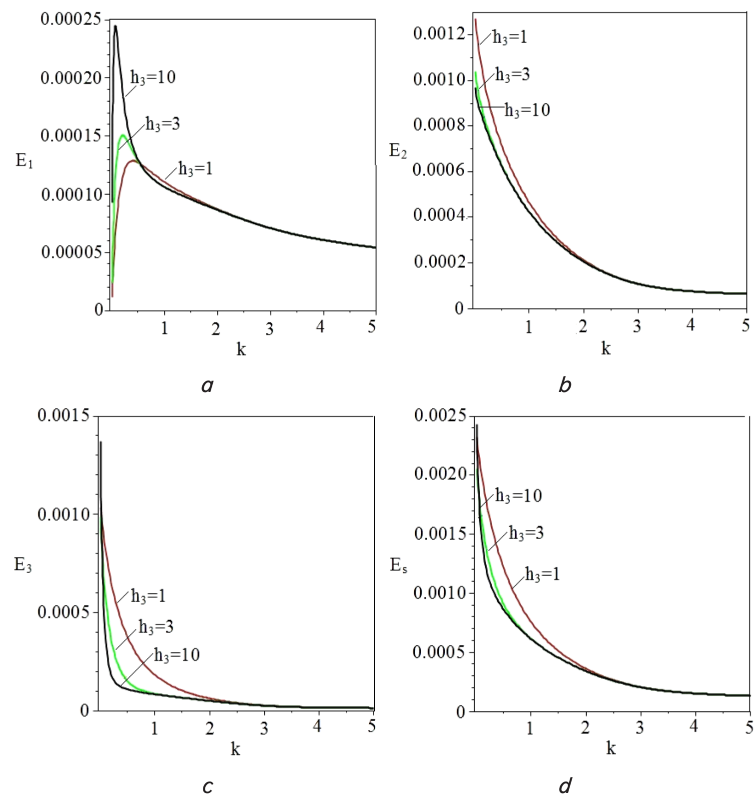

Fig. 3. Diagram of dependence of wave motion energy on wave number *k* at different values of thickness of the upper layer: *a* – wave motion energy in half-space E_1; *b* – energy of the middle layer E_2; *c* – energy of the upper layer E_3; *d* – total energy of system E_s

Fig. 3, *a* shows that at an increase in the value of wave number, the value of wave motion energy in half-space E_1 first increases to the maximum value and after reaching it quickly decreases to a certain limit value. Such dependence is observed in the region of small wave numbers (for long waves) for each thickness of the upper layer h_3, in this case, the higher h_3, the higher the maximum value E_1. We will note that for large wave numbers (for short waves), half-space energy hardly depends on thickness of the upper layer.

Fig. 3, *b* shows the diagram of dependence of energy of the middle layer E_2 on wave number k at different values of thickness of the upper layer h_3. Fig. 3 shows that at an increase in the value of wave number k, the value of energy of the middle layer is descending in character and quickly enough converge to a specific limit value. That is, for short waves, energy of the middle layer almost does not depend on thickness of the upper layer. Fig. 3 shows that dependence of energy of the upper layer E_3 on wave number k at different values of thickness of this layer h_3 also has a descending character and on certain intervals of the diagram does not depend on the value of its thickness.

The total energy of the three-layer system E_s (Fig. 3, *d*) decreases at an increase in the value of wave number k, and, as it can be seen from the diagram, for large values, k does not depend on the value of thickness of the upper layer h_3.

6. 3. Analysis of dependence of energy of waves of different length on thickness of the upper layer

In this chapter, we will consider three cases of dependence of energy of waves of different length on thickness of the upper layer at different amplitudes A and B of progressive waves on the lower and upper contact surfaces, respectively:

1) $A \neq 0$, $B=0$;

2) $A=0$, $B \neq 0$;

3) $A \neq 0$, $B \neq 0$.

1) We will consider a three-layer system on condition of passage of a progressive wave with amplitude $A \neq 0$ on the lower contact surface in the absence of a progressive wave on the upper contact surface.

Fig. 4 shows dependence of energy E_s on thickness of the upper layer h_3 at different values of wave number $k=0.01$, $k=0.1$, $k=1$, $k=2$ for the following parameters of the system: $\rho_1=1$, $\rho_2=0.9$, $\rho_3=0.8$, $T_1=T_2=0$, $h_2=1$, wave amplitude on the lower contact surface $A=0.1$ and on the upper one $B=0$.

As the diagram shows, the total energy of system E_s (Fig. 4) is descending in character at an increase in thickness of the upper layer and for long waves has a larger value compared with short waves.

2) We will consider a three-layer system on conditions of passage of a progressive wave with amplitude $B \neq 0$ on the upper contact surface, provided there is no progressive wave on the lower contact surface, i. e. $A=0$.

Fig. 5 shows dependence of total energy E_s on thickness of the upper layer h_3 at different values of wave number $k=0.01$, $k=0.1$, $k=1$, $k=2$ for the following parameters of the system: $\rho_1=1$, $\rho_2=0.9$, $\rho_3=0.8$, $T_1=T_2=0$, $h_2=1$, amplitude of the upper internal progressive wave $B=0.05$. Total energy of the system for a small value of wave number ($k=0.01$) increases, reaches the maximum value and starts falling to the limit value. At other values of wave number ($k=0.1$, $k=1$, $k=2$), dependence of energy on thickness of the upper layer has a weakly descending character and converges to its limit value quite quickly.

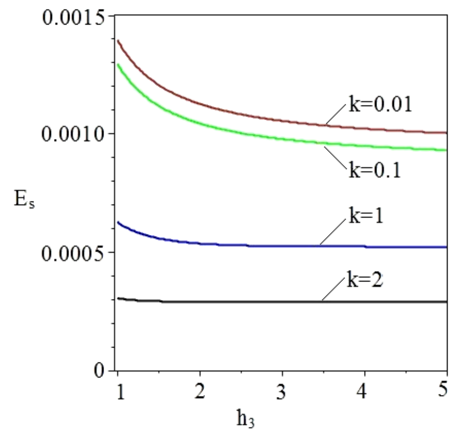

Fig. 4. Diagram of dependence of total energy of system E_s on thickness of the upper layer h_3 at different values of wave number k

3) We will consider a three-layer system on condition of simultaneous passage of a progressive wave with amplitude $B \neq 0$ on the upper contact surface and passage of a progressive wave with amplitude $A \neq 0$ on the lower contact surface.

Consider dependence E_s (Fig. 6) on thickness of the upper layer for different wave numbers $k=0.01$, $k=0.1$, $k=1$, $k=2$ and the following parameters of the system: $\rho_1=1$, $\rho_2=0.9$, $\rho_3=0.8$, $T_1=T_2=0$, $h_2=1$, $A=0.1$, $B=0.05$.

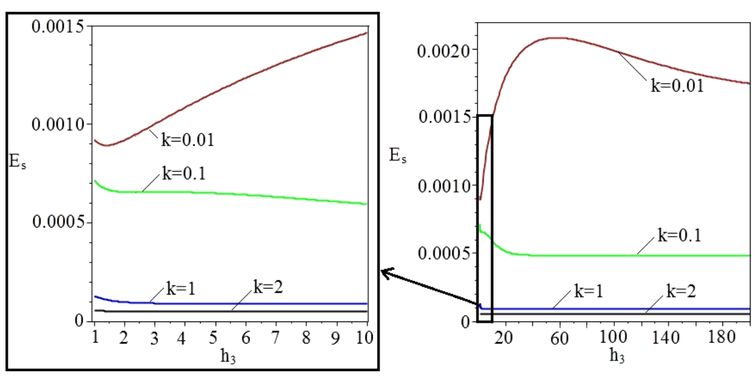

Fig. 5. Dependence of energy of system E_s on thickness of the upper layer h_3 at different values of wave number k

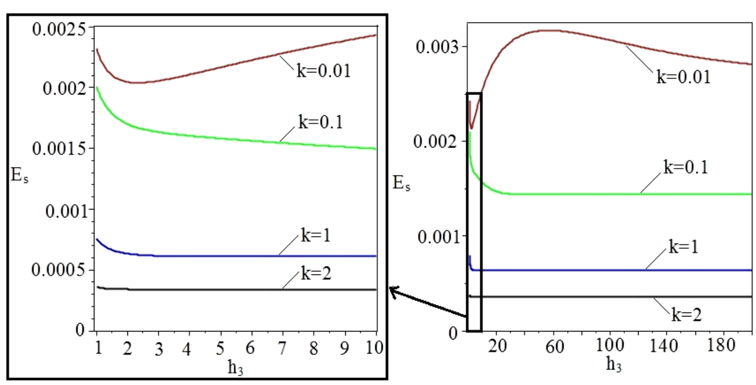

Fig. 6. Diagram of dependence of energy E_s on thickness of the upper layer h_3 at different values of wave number k

Fig. 6 shows the total energy of the system, which increases at an increase in thickness h_3 for a small wave number (k=0.01), reaches maximum value and begins falling to the limit value. At the same time, for larger values of wave number, the total energy of the system rather quickly falls to its limit value.

We should note that in this specific case of propagation of progressive waves along both contact surfaces, total energy of the system is close to the sum of energies of the system in cases, in which a wave does not propagate on one of the contact surfaces.

7. Discussion of results of the study of wave processes in a three-layer hydrodynamic system

The study of energy characteristics of a three-layer hydrodynamic system "liquid half-space – layer –layer with a rigid lid" was conducted within the framework of a weakly non-linear model, specifically, in its first approximation.

The main attention was paid to the study of the influence of different physical and geometrical parameters both on the total energy of the system and the energy of separate layers. Research into reciprocal influence of changes of physical and geometrical parameters of the system on energy characteristics allows us to assess qualitatively and quantitatively energy processes in each layer separately and in the system in general. The obtained results essentially complement and enrich the picture of the wave process, which was presented in detail in the previous studies. To prove physical reliability and to test the computer program in the package of symbolic computation, two systems, degenerated to a two-layer system "liquid half-space – layer –layer with a rigid lid", were explored. In both cases the obtained results coincided with the results of the test problems. Practical application of the obtained results involves studying energy characteristics of wave processes in the ocean with the layer structure in the presence of the ice cap. As it is known, such phenomena occur at mouths of rivers, as well as in the open ocean in areas adjacent to a thermal wedge. The advantage of this study is the consistent study of energy characteristics for dispersion relations between physical and geometrical parameters of the hydrodynamic system "liquid half-space – layer –layer with a rigid lid". In particular, it is important in the cases of absence of progressive waves on one of the contact surfaces and simultaneous passage of waves on both contact surfaces, since it makes possible to reveal the peculiarities of propagation of energy of progressive waves both taken separately and in interaction. A natural extension of this research should be obtaining and studying higher approximations of a weakly non-linear problem with subsequent deriving evolutionary equations of envelopes of wave packets on the contact surfaces and plotting the diagrams of modulation stability. In terms of continuation of the study of energy characteristics of a three-layer "liquid half-space – layer – layer with a rigid lid", it is necessary to obtain estimation of the energy contribution of higher harmonics. The obtained results can be compared with the known data for a two-layer system and energy pumping in a wave packet can be explored.

8. Conclusions

1. We obtained the relations for quantitative and qualitative evaluation of energy of three layers of fluid and total energy of a hydrodynamic system "liquid half-space – layer – layer with a rigid lid" in the form of integrals for time and vertical space variable over product of derivatives of potentials of velocities by time and horizontal space variable. Symbolic transformations and related calculations were performed in the software package.

2. It was found that energy of the middle and upper layer of a three-layer hydrodynamic system is descending in character at an increase in the value of wave number k and converges fast enough to a certain limit value, which for short waves does not depend on thickness of the upper layer. At the same time, energy of the lower layer increases to a certain value, and after reaching the maximum quite quickly falls to its limit value. In this case, the energy of the system, which is the sum of energies of three layers, also is descending in character and fast enough converges to its limit value.

3. An analysis of the influence of geometrical parameters of the system on energy characteristics of the system showed:

– if the amplitude of a progressive wave on the upper contact surface is equal to zero and the amplitude of a progressive wave on the lower contact surface is different from zero, energy of the system is descending in character at an increase in thickness of the upper layer and has a higher value for long waves in comparison with short waves;

– if the amplitude of a progressive wave on the lower contact surface is equal to zero and the amplitude of a progressive wave on the upper contact surface is different from zero, the total energy reaches maximum values for some values of thickness of the upper layer and tends to its limit value at an increase in thickness of the upper layer;

– if both amplitudes of progressive waves on the upper and the lower contact surfaces are different from zero, the total energy of the system from thickness of the upper layer also has maximum values, and at an increase in thickness of the upper layer falls and tends to the limit value. At the same time, for short waves, the total energy of the system is descending in character and rather quickly falls to its limit value. We will note that in the case of non-zero amplitudes of waves on both contact surfaces, the total energy of the system is close to the sum of energies of the system in the two above mentioned cases.

References

1. Bona J. L., Lannes D., Saut J.-C. Asymptotic models for internal waves // Journal de Mathématiques Pures et Appliquées. 2008. Vol. 89, Issue 6. P. 538–566. doi: 10.1016/j.matpur.2008.02.003
2. Wang Y., Tice I., Kim C. The Viscous Surface-Internal Wave Problem: Global Well-Posedness and Decay // Archive for Rational Mechanics and Analysis. 2013. Vol. 212, Issue 1. P. 1–92. doi: 10.1007/s00205-013-0700-2
3. Hsu H.-C., Tsai C.-C. Lagrangian approach to interfacial water waves with free surface // Applied Ocean Research. 2016. Vol. 59. P. 616–637. doi: 10.1016/j.apor.2016.08.001
4. Jo T.-C., Choi Y.-K. Dynamics of strongly nonlinear internal long waves in a three-layer fluid system // Ocean Science Journal. 2014. Vol. 49, Issue 4. P. 357–366. doi: 10.1007/s12601-014-0033-6

5. Numerical simulation of interaction between internal solitary waves and submerged ridges / Zhu H., Wang L., Avital E. J., Tang H., Williams J. J. R. // Applied Ocean Research. 2016. Vol. 58. P. 118–134. doi: 10.1016/j.apor.2016.03.017
6. Smith S., Crockett J. Experiments on nonlinear harmonic wave generation from colliding internal wave beams // Experimental Thermal and Fluid Science. 2014. Vol. 54. P. 93–101. doi: 10.1016/j.expthermflusci.2014.01.012
7. Massel S. R. On the nonlinear internal waves propagating in an inhomogeneous shallow sea // Oceanologia. 2016. Vol. 58, Issue 2. P. 59–70. doi: 10.1016/j.oceano.2016.01.005
8. Avramenko O. V., Naradovyi V. V., Selezov I. T. Energy of Motion of Internal and Surface Waves in a Two-Layer Hydrodynamic System // Journal of Mathematical Sciences. 2018. Vol. 229, Issue 3. P. 241–252. doi: 10.1007/s10958-018-3674-7
9. Avramenko O., Lunyova M., Naradovyi V. Wave propagation in a three-layer semi-infinite hydrodynamic system with a rigid lid // Eastern-European Journal of Enterprise Technologies. 2017. Vol. 5, Issue 5 (89). P. 58–66. doi: 10.15587/1729-4061.2017.111941
10. Nayfeh A. H. Nonlinear Propagation of Wave-Packets on Fluid Interfaces // Journal of Applied Mechanics. 1976. Vol. 43, Issue 4. P. 584. doi: 10.1115/1.3423936
11. Tarapov I. E. Continuum Mechanics. Vol. 3. Mechanics of Inviscid Liquid. Kharkiv: Zolotye Stranitsy, 2005.
12. Avramenko O. V., Hurtovyi Yu. V., Naradovyi V. V. Analiz enerhiyi khvylovoho rukhu v dvosharovykh hidrodynamichnykh systemakh // Naukovi zapysky. Seriya: Matematychni nauky. 2014. Issue 73. P. 3–8.

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STUDY OF THE METHOD FOR ASSESSING ATMOSPHERIC TURBULENCE BY THE ENVELOPE OF SODAR SIGNALS

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Досліджено метод оцінки атмосферної турбулентності за статистичними характеристиками обвідної сигналів содара. Показано, що обвідна ехо-сигналів розподілена по закону Райса, параметр закону розподілу пов'язаний з інтенсивністю турбулентності. Отримані значення параметра закону розподілу exoсигналів для турбулентності певних класів. Використання дослідженого методу додатково до вже застосованих дозволить збільшити точність і часове розрізнення содарів

Ключові слова: акустичне зондування, содар, турбулентність, ехо-сигнал, обвідна, авіаційна метеорологія, вітрова енергетика

Исследован метод оценки атмосферной турбулентности по статистическим характеристикам огибающей сигналов содара. Показано, что огибающая эхо-сигналов распределена по закону Райса, параметр закона распределения связан с интенсивностью турбулентности. Получены значения параметра закона распределения эхо-сигналов для турбулентности определённых классов. Использование исследованного метода дополнительно к уже применяемым позволит увеличить точность и временное разрешение содаров

Ключевые слова: акустическое зондирование, содар, турбулентность, эхо-сигнал, огибающая, авиационная метеорология, ветровая энергетика

1. Introduction

Acoustic locators (sodars) are important sources of information about velocity, wind direction and the degree of turbulence of air masses at altitudes of up to 1 km. Information of sodars is widely used for studies of the atmosphere [1–3], local and global weather forecasts [4, 5], air traffic control services [1, 6], for monitoring the atmosphere at wind farms [7–10], near potential sources of hazardous emissions [1], etc.

As a rule, monostatic three-beam sodars with one vertical beam and two beams, deviated from the vertical by 20...30° in mutually perpendicular directions are used [1, 2, 5–10]. The range to scattering volume is determined by the time of an echo signal delay, and wind projection onto the direction of sounding is determined by the Doppler frequency shift. Turbulence intensity is estimated by the power of a return signal and the width of the Doppler spectrum. The general tendency of sodars improvement is to increase reliability and operative measurements [1–10]. This is especially important when detecting hazardous meteorological phenomena, for example, in the aircraft takeoff-landing area [6]. Taking into consideration the increasing requirements for measurement accuracy and temporal resolution of sodars, the relevant task is to improve the methods for obtaining meteorological information from parameters of echo signals.

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