Розглядається задача про поширення внутрішніх хвиль для ідеальної нестисливої рідини. Гідродинамічна система складається з трьох шарів скінченної товщини, які не змішуються та обмежені зверху твердою кришкою, а знизу твердим дном. На поверхнях розділу рідких середовищ діє сила поверхневого натягу. Постановка задачі була здійснена в безрозмірному вигляді. В якості малого параметру використовується коефіцієнт нелінійності, який дорівнює відношенню характерної амплітуди до характерної довжини хвилі.

Розв'язки лінійної задачі шукаються вигляді прогресивних хвиль. На основі цих розв'язків отримано дисперсійне співвідношення як умову розв'язуванності системи лінійних диференціальних рівнянь. Виявлено існування двох характерних мод (дійсних коренів дисперсійного рівняння). Проаналізовано графіки коренів дисперсійного рівняння в залежності від різних фізичних та геометричних параметрів системи. Встановлено, що товщини шарів не впливають на дисперсію хвиль, тоді як зміна поверхневого натягу та відношення густин значно впливають на умови проходження хвиль. Розглянуто хвильові пакети в лінійній постановці, що є суперпозицією гармонічних хвиль близької довжини. Виявлено, що амплітуда обвідної хвильового пакету на нижній поверхні контакту залишається синусоїдальною, тоді як на верхній змінюється за більш складним законом.

Задача поширення внутрішніх хвиль вздовж поверхні трьох рідких шарів можуть моделювати сильно стратифікований термоклин в океані. Вивчення впливу поверхневого натягу також може бути використане при розробці нових технологій, пов'язаних з використанням трьох рідких шарів, що не змішуються

Ключові слова: ідеальна нестислива рідина, внутрішні хвилі, тришарова гідродинамічна система, дисперсійне рівняння

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ANALYSIS OF CONDITIONS FOR THE PROPAGATION OF INTERNAL WAVES IN A THREE-LAYER FINITE-DEPTH LIQUID

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1. Introduction

Fluids with discrete or continuous density stratification are encountered in numerous applied problems. Perturbation of such an inhomogeneous fluid causes propagation of internal waves with interesting characteristic properties. As a rule, mathematical formulas of such problems consist of a system of nonlinear differential partial derivative equations. Nonlinearity of such systems brings about significant mathematical difficulties in the problem solution. Therefore, some small parameter that permits rejection of the equation members having insignificant effect and the problem statement linearization itself is singled out in the problem statement. The so-called dispersion relation which binds the frequency and the wave number is the condition for solvability of such a system. In a general case, the dispersion relation has several real roots which are called modes. In linear problems, solution is considered as a superposition of various modes. Thus, the dispersion relation indicates conditions of waves propagation for a concrete hydrodynamic system. The conditions of internal wave propagation are quite sensitive in relation to physical and geometric parameters of the system.

Relevance of this study direction was determined by the necessity of developing a theoretical apparatus for the problems of detecting internal waves in various media. The problems of internal wave propagation along the surfaces of contact of three liquid layers can model strongly depth-stratified thermocline regions in the ocean. In shallow seas, vertical density stratification has a three-layered structure with a prominent seasonal pycnocline at a depth of about several tens of meters and a major pycnocline at a greater depth. For example, the Black Sea and the Baltic Sea have a more or less permanent three-layer structure formed by flow of fresh water on the surface and penetration of the saltiest water to the bottom layers. A noticeable three-layer density stratification is also found in the South China Sea. However, many important issues remain unexplored. In a three-layer fluid, specific classes of nonlinear waves, so called envelope solitons, can spread. They are still poorly studied both analytically and numerically. All this shows urgency of the problem of studying the internal waves in a three-layer fluid. The knowledge of the effect of surface tension can also be applied in the development of new technologies associated with the use of three non-mixing liquid layers.

2. Literature review and problem statement

A new type of stability of wave solutions related to the models of Korteweg-de-Vries type was studied in [1] under condition of a low dispersion of waves. Existence of exponentially increasing solutions of the linearized problem with the use of asymptotic perturbation methods was established. The criteria of spectral instability of single waves in the linear case were found. Besides, nonlinear stability and spectral instability of solutions for a specific set of parameters were analyzed. However, this paper was devoted to the study of mathematical aspects of wave propagation with no relation to concrete models.

A study of propagation of a single internal wave down a sloping bottom in a three-layer hydrodynamic system based on the Korteweg-de-Vries equation with variable coefficients was presented in [2]. Precise and numerical solutions of the problem were considered and compared. Two model configurations for which the system energy change was numerically analyzed were considered. Existence of the phenomenon of fading waves was established. This model is based on the Korteweg-de-Vries equation which enables study of classical solitons but does not take into account the potential appearance of envelope solitons described by the nonlinear Schrödinger equation.

Spread of internal waves over obstacles was studied in [3]. It was established for narrow obstacles that the energy of the wave flow increases in an upstream flow. The situation is just opposite for wide ridges. It was shown that the pressure field makes a significant contribution to the total energy flow. A linear theory was developed that is valid for general stratification and the surface flow that accurately predicts the wave field. This paper has described internal waves in the presence of a flow and obstacles but did not take into account formation of internal waves under the influence of gravitation and capillary forces.

Overturning regular waves were studied in [4] using a numerical model of fluid dynamics based on the Navier-Stokes equations. The constructed model well predicts vortices and overlapping of waves but does not describe the phenomena of propagation and interaction of internal waves.

Dynamics of collision of two internal waves in a two-layer fluid in a model based on the Navier-Stokes equation was investigated in [5]. It was shown that collision of two waves of small and medium amplitude leads to a phase shift and generation of a dispersive wave. It was established that the maximum amplitude of a wave formed after collision can exceed the cumulative amplitude of the collided waves. Wave interaction can also be studied by other methods. One of such methods is the multi-scale method which leads to evolutionary envelope equations.

Work [6] was devoted to the study of single Rosby waves in a stratified fluid. A new model was constructed with the use of the generalized Boussinesq equation. Based on this equation, a generalized Euler-Lagrange equation was obtained with the use of the Agraval methods and reduction of perturbations. Exact solutions were found for the latter. To analyze the dissipation phenomenon, approximate solutions were constructed in the model with the use of a new iterative method. The waves that have a considerable length as regards thickness of the layers were considered in this paper but the presence of waves with lengths commensurate with the geometric parameters of the system was not considered.

Exact solutions were constructed in [7] in the form of a progressive wave for the generalized connected nonlinear Korteweg-de-Vries equations based on an extended algebraic method. The results obtained describe generation and evolution of such waves as well as their interaction and stability. Korteweg-de-Vries equations of higher orders were considered in this paper. Solution of these equations gives soliton waves. Conditions for propagation of such waves were not taken into account

Solutions in the form of singular waves of the fifth-order Corteweg-de-Vries equation were studied in [8]. In this equation, in addition to the quadratic nonlinearity and dispersion of the third order, there is a cubic nonlinearity and a linear dispersion of the fifth order as well as two nonlinear dispersion members. An exact solution was obtained and the dependence of its amplitude and velocity on the parameters of the equation was studied. It was shown that the resulting solution can be a soliton. It was indicated that the presence of nonlinear dispersion members may materially affect the single wave existence, profile and resistance to small perturbations. The results obtained are applicable to the study of internal and surface waves in liquids as well as waves in other media. A single-layer model with a free surface and propagation of internal waves at the interface of two media were considered in this paper. However, the presence of two or more contact surfaces required construction and study of new models.

Interaction of small internal wave packets with a large internal wave was studied in [9]. The main result was that the indicated interaction with a single wave leads to a complete destruction of the packet which consists of waves having small length in comparison with the length of single waves. In the future, a study of interaction of wave packets in hydrodynamic systems of a more complex structure would be of interest.

Capillary-gravitational and interphase waves in a two-layer fluid were studied in [10] with the use of the Stokes wave theory. The study was conducted by the method of perturbations of the third order. It was established that solutions of the third order depend essentially on the surface tension, density and thickness of the layers. In a linear approximation, the constructed model coincides with classical results. Similar studies are also important for three-layer liquids in which a significant dependence of wave amplitudes on various system parameters were also revealed.

Propagation of waves in a two-layer liquid of finite depth with a free surface was studied in [11]. Internal and surface waves were considered with the use of numerical simulation by the finite element method. The ratio of amplitudes of two waves for various values of density and thickness of liquids was analyzed. It should be noted that the analysis of the amplitude ratio is facilitated when the solutions are obtained in an analytical form.

Internal weakly nonlinear waves of finite amplitude in a three-layer liquid of a finite depth were studied in [12]. A pair of Korteweg-de-Vries equations with coefficients depending on the liquid parameters was obtained. As a result of introduction of an additional small parameter, two Gardner equations for two contacting surfaces were obtained. Soliton solutions were constructed and studied for the obtained equations depending on various parameters of the system. Cases are considered when Gardner equations can be written without nonlinear terms. This model does not take into account propagation of wave packets with their inertia described by nonlinear Schr dinger equations.

Work [13] is devoted to the study of single waves in a three-layer fluid with a constant density of each layer. A completely nonlinear numerical scheme based on integral equations was presented. Based on the constructed scheme, various types of waves were studied. The phenomenon of wave overturning was established when the upper layer thickness is much larger than that of the lower layers. This study was devoted to the numerical analysis of wave propagation but analytical solutions describing wave motions in a three-layer liquid are more suitable for revealing new effects.

Linear waves and nonlinear wave interactions in a three-layer fluid were studied in [14] with the use of analytical and numerical methods. It was established that single waves cannot be described mathematically as solitons because of insignificant energy loss. However, since these losses are small and the waves in such a system are dispersionally connected, the single waves manifest themselves as solitons. In this case, an issue of studying another type of wave motion in three-layer systems appear, in particular, propagation of waves whose lengths are commensurate with the geometric parameters of the system.

A model of a two-layer incompressible ideal fluid was studied in [15]. Solution of the original nonlinear problem was constructed in the form of power series for a small parameter. For the coefficients of these series, there are corresponding linear (homogeneous and inhomogeneous) problems solution of which is represented in the form of series of previously constructed eigenfunctions of a nonlinear boundary value problem. Another approach with the use of power series for a small parameter is a multi-scale method which enables obtaining of information about wave motions for various time and spatial scales.

Propagation of internal separated waves in a two-layer liquid medium bounded above and below with solid surfaces without taking into account surface tension was studied in [16]. Solution was found in the form of generalized power series for a parameter depending on the magnitude inverse to the Froude number. It is interesting to study influence of surface tension on conditions of wave propagation in a three-layer system.

The attention deserving theoretical studies in [17] where the model equations derived from the Euler equations describing evolution of internal gravitational waves in a two-layer liquid that are non-viscous and non-mixed have been solved without imposing any limitation on the amplitude. Also, areas of suitability of asymptotic approximations for strongly-nonlinear wave packets of internal waves in the layer - layer system were indicated in this paper A similar study of application of the multi-scale method is relevant for three-layer systems as well.

The problems of propagation and stability of waves in a two-layer hydrodynamic system were considered in [18, 19]. The issue of modulation stability were considered and the system parameters at which wave packets will be modulation-unstable were established. Evolution of wave packets is described in these papers by means of a nonlinear Schrödinger equation using the multi-scale method. Similar studies are also important for a three-layer liquid.

A model of propagation and interaction of waves in a two-layered hydrodynamic system with a free surface was constructed in [20–22]. In particular, conditions for the propagation of waves at various density ratios in the hydrodynamic system were revealed in the first approximation. Dependencies of potential frequencies on the lower layer thickness and the wave number were also analyzed. Interaction of internal and surface waves was studied. Influence of presence of surface tension on the contact surfaces was revealed. Study of the dispersion modes makes it possible to determine structure of the wave motions on the contact surfaces which is also relevant for a three-layer system of a finite depth.

A problem statement for an ideal incompressible three-layer liquid with a solid bottom and a solid cover was considered in this study. At the same time, the characteristic scales of physical quantities were introduced which enabled the mathematical equations to be presented in a dimensionless form and the small parameter (coefficient of nonlinearity) be highlighted. Solution of a linear system leads to a fairness of the relation which is a dispersion relation. The dependence of the modes of the dispersion relation on physical and geometric parameters of the system was analyzed and the wave motion on the contact surfaces was studied.

3. The aim and objectives of the study

This study objective was to analyze propagation and interaction of waves along the contact surfaces in a three-layer hydrodynamic "a layer with solid bottom – a layer – a layer with a cover" system. This will enable a more detailed assessment of the wave processes and interaction in three-layer systems.

To achieve the objective, the following tasks were formulated:

– to perform mathematical statement of the problem and linearize it;

– to find solutions of the linearized problem and derive the dispersion relation;

– to construct dispersion diagrams and analyze the roots of the dispersion relation depending on the physical and geometric parameters of the system being studied;

– to construct wave packets on the contact surfaces and study their shape for various modes of the dispersion relation.

4. The statement and solution of the problem of propagation of waves in a three-layer hydrodynamic system

4. 1. The mathematical problem statement and the method of study

The problem of propagation of three-dimensional wave packets of finite amplitude on the surface of the liquid layer was studied.

$$\Omega_1 = \{(x,z):\ |x| < \infty,\ -h_1 \le z < 0\}$$

with density ρ_1, of the medium liquid layer

$$\Omega_2 = \{(x,z):\ |x| < \infty,\ 0 \le z < h_2\}$$

with density ρ_2, and the upper liquid layer

$$\Omega_3 = \{(x,z):\ |x| < \infty,\ h_2 \le z < h_2 + h_3\}$$

with density ρ_3. Velocities of the liquids, υ_j, in Ω_j are expressed in terms of gradient of the potential φ_j, j=1, 2, 3.

The layers Ω_1 and Ω_2 are separated by the contact surface $z=\eta_1(x, t)$ and the layers Ω_2 and Ω_3 are separated by the contact surface $z=h_2+\eta_2(x, t)$. When solving, the forces of surface tension on the contact surfaces are taken into consideration. Gravitational force is directed perpendicular to the interface in the negative z-direction. Liquids are considered to be ideal and incompressible and the wave motions under study are vortex-free and potential (Fig. 1):

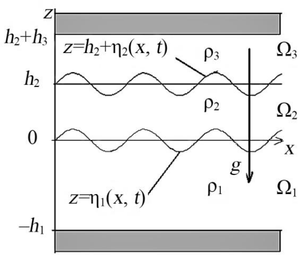


Fig. 1. The problem statement

The mathematical problem statement in a plane case has the following form:

Laplace equation

$$\Delta\varphi_j \equiv \frac{\partial^2\varphi_j}{\partial x^2}+\frac{\partial^2\varphi_j}{\partial z^2}=0,\quad \upsilon_j=\overline{\nabla}\varphi_j \text{ in } \Omega_j,\quad j=1,2,3,$$

kinematic conditions on contact surfaces

$$\frac{\partial\eta_1}{\partial t}-\frac{\partial\varphi_j}{\partial z}=-\frac{\partial\eta_1}{\partial x}\frac{\partial\varphi_j}{\partial x} \text{ at } z=\eta_1(x,t),\quad j=1,2,$$

$$\frac{\partial\eta_2}{\partial t}-\frac{\partial\varphi_j}{\partial z}=-\frac{\partial\eta_2}{\partial x}\frac{\partial\varphi_j}{\partial x} \text{ at } z=h_2+\eta_2(x,t),\quad j=2,3, \qquad (1)$$

dynamic conditions on contact surfaces

$$\rho_1\frac{\partial\varphi_1}{\partial t}-\rho_2\frac{\partial\varphi_2}{\partial t}+g(\rho_1-\rho_2)\eta_1+0.5\rho_1\left(\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\left(\frac{\partial\varphi_1}{\partial z}\right)^2\right)-$$

$$-0.5\rho_2\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-T_1\left[1+\left(\frac{\partial\eta_1}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_1}{\partial x^2}=0$$

at

$$z=\eta_1(x,t),$$

$$\rho_2\frac{\partial\varphi_2}{\partial t}-\rho_3\frac{\partial\varphi_3}{\partial t}+g\eta_2(\rho_2-\rho_3)+$$

$$+h_2(\rho_2-\rho_3)+0.5\rho_2\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-$$

$$-0.5\rho_3\left(\left(\frac{\partial\varphi_3}{\partial x}\right)^2+\left(\frac{\partial\varphi_3}{\partial z}\right)^2\right)-$$

$$-T_2\left[1+\left(\frac{\partial\eta_2}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_2}{\partial x^2}=0$$

at

$$z=h_2+\eta_2(x,t),$$

boundary conditions (conditions of impenetrability)

$$\frac{\partial\varphi_3}{\partial z}=0 \text{ at } z=h_2+h_3,$$

$$\frac{\partial\varphi_1}{\partial z}=0 \text{ at } z=-h_1.$$

Introduce dimensionless quantities in (1) using the following formulas:

$$x=Lx',\quad z=Hz',$$

$$\eta_{1,2}=a\eta'_{1,2},\quad t=\frac{L}{\sqrt{gH}}t',$$

$$\varphi_{1,2,3}=\frac{gaL}{\sqrt{gH}}\varphi'_{1,2,3},$$

$$T_{1,2}=L^2\rho_1 gT'_{1,2},\quad \rho_{1,2,3}=\rho_1\rho'_{1,2,3},$$

where stroked variables are dimensional, and H, L, a are characteristic thickness of the layer, length, and wave amplitude, respectively, ρ_1 is the density of the lower layer and g is the acceleration of gravity. In a dimensionless form, the statement (1) will take the following form (the strokes are omitted below):

$$\frac{\partial^2\varphi_j}{\partial x^2}+\frac{1}{\beta}\frac{\partial^2\varphi_j}{\partial z^2}=0$$

in Ω_j, $j=1,2,3$,

$$\frac{\partial\eta_1}{\partial t}-\frac{1}{\beta}\frac{\partial\varphi_j}{\partial z}=-\alpha\frac{\partial\eta_1}{\partial x}\frac{\partial\varphi_j}{\partial x}$$

at $z=\alpha\eta_1(x,t)$, $j=1,2$,

$$\frac{\partial\eta_2}{\partial t}-\frac{1}{\beta}\frac{\partial\varphi_j}{\partial z}=-\alpha\frac{\partial\eta_2}{\partial x}\frac{\partial\varphi_j}{\partial x}$$

at

$$z=h_2+\alpha\eta_2(x,t),\quad j=2,3, \qquad (2)$$

$$\rho_1\frac{\partial\varphi_1}{\partial t}-\rho_2\frac{\partial\varphi_2}{\partial t}+(\rho_1-\rho_2)\eta_1+$$

$$+0.5\rho_1\alpha\left(\left(\frac{\partial\varphi_1}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_1}{\partial z}\right)^2\right)-$$

$$-0.5\rho_2\alpha\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-$$

$$-T_1\left[1+\left(\alpha\sqrt{\beta}\frac{\partial\eta_1}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_1}{\partial x^2}=0$$

at

$$z=\alpha\eta_1(x,t),$$

$$\rho_2\alpha\frac{\partial\varphi_2}{\partial t}-\rho_3\alpha\frac{\partial\varphi_3}{\partial t}+\alpha\eta_2(\rho_2-\rho_3)+$$
$$+h_2(\rho_2-\rho_3)+0.5\rho_2\alpha^2\left(\left(\frac{\partial\varphi_2}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_2}{\partial z}\right)^2\right)-$$
$$-0.5\rho_3\alpha^2\left(\left(\frac{\partial\varphi_3}{\partial x}\right)^2+\frac{1}{\beta}\left(\frac{\partial\varphi_3}{\partial z}\right)^2\right)-$$
$$-T_2\alpha\left[1+\left(\alpha\sqrt{\beta}\frac{\partial\eta_2}{\partial x}\right)^2\right]^{-3/2}\frac{\partial^2\eta_2}{\partial x^2}=0$$

at

$$z=h_2+\alpha\eta_2(x,t),$$

boundary conditions (conditions of impenetrability)

$$\frac{\partial\varphi_3}{\partial z}=0 \text{ at } z=h_2+h_3,$$

$$\frac{\partial\varphi_1}{\partial z}=0 \text{ at } z=-h_1.$$

In the statement (2), there are two parameters: $\alpha=\frac{a}{L}$ (the coefficient of nonlinearity), and $\beta=\frac{H^2}{L^2}$ (the coefficient of dispersion). Nonlinear problem (2) will be studied using the method of multi-scale developments. Let us assume for the future that the coefficient of nonlinearity $\alpha<<1$, and $\beta=1$. Then the unknown potentials of velocity and deviations of the contact surfaces can be represented as:

$$\eta_i(x,t)=\sum_{n=1}^{3}\alpha^{n-1}\eta_{in}(x_0,x_1,x_2,t_0,t_1,t_2)+O(\alpha^3),\quad i=1,2,$$

$$\varphi_j(x,z,t)=\sum_{n=1}^{3}\alpha^{n-1}\varphi_{jn}(x_0,x_1,x_2,t_0,t_1,t_2)+O(\alpha^3),$$

$$j=1,2,3, \tag{3}$$

where $x_k=\alpha^k x$ and $t_k=\alpha^k t$ (k=0, 1, 2) are scale variables.

Substitute (3) into (2) and equate expressions with the same degrees α to obtain linear approximations of the problem under study.

The problem of the first approximation (at α^0)

$$\varphi_{j1,x_0x_0}+\varphi_{j1,zz}=0 \text{ in } \Omega_j,\quad j=1,2,3,$$

$$\eta_{11,t_0}+\varphi_{j1,z}=0 \text{ at } z=0,\quad j=1,2,$$

$$\eta_{21,t_0}+\varphi_{j1,z}=0 \text{ at } z=h_2,\quad j=2,3,$$

$$\rho_1\varphi_{11,t_0}-\rho_2\varphi_{21,t_0}+(\rho_1-\rho_2)\eta_{11}-T_1\eta_{11,x_0x_0}=0 \text{ at } z=0,$$

$$\rho_2\varphi_{21,t_0}-\rho_3\varphi_{31,t_0}+(\rho_2-\rho_3)\eta_{21}-T_2\eta_{21,x_0x_0}=0 \text{ at } z=h_2,$$

$$\varphi_{31,z}=0 \text{ at } z=h_2+h_3,$$

$$\varphi_{11,z}=0 \text{ at } z=-h_1. \tag{4}$$

The problem of the second approximation (at α^1)

$$\varphi_{j2,x_0x_0}+\varphi_{j2,zz}=-2\varphi_{j1,x_0x_1} \text{ in } \Omega_j,\quad j=1,2,3,$$

$$\eta_{12,t_0}-\varphi_{j2,z}=-\eta_{11,t_1}-\eta_{11,x_0}\varphi_{j1,x_0}+\eta_{11}\varphi_{j1,zz} \text{ at } z=0,\quad j=1,2,$$

$$\eta_{22,t_0}-\varphi_{j2,z}=-\eta_{21,t_1}-\eta_{21,x_0}\varphi_{j1,x_0}+\eta_{21}\varphi_{j1,zz}$$

at $z=h_2$, $j=2,3$,

$$\begin{aligned}&\rho_1\varphi_{12,t_0}-\rho_2\varphi_{22,t_0}+(\rho_1-\rho_2)\eta_{12}-T_1\eta_{12,x_0x_0}=\\&=-\rho_1(\varphi_{11,t_1}+\eta_{11}\varphi_{11,t_0z})+\rho_2(\varphi_{21,t_1}+\\&+\eta_{11}\varphi_{21,t_0z})-0.5\rho_1(\varphi^2_{11,x_0}+\varphi^2_{11,z})+\\&+0.5\rho_2(\varphi^2_{21,x_0}+\varphi^2_{21,z})+2T_1\eta_{11,x_0x_1}\end{aligned} \tag{5}$$

at $z=0$,

$$\begin{aligned}&\rho_2\varphi_{22,t_0}-\rho_3\varphi_{32,t_0}+(\rho_2-\rho_3)\eta_{22}-T_2\eta_{22,x_0x_0}=\\&=-\rho_2(\varphi_{21,t_1}+\eta_{21}\varphi_{21,t_0z})+\rho_3(\varphi_{31,t_1}+\\&+\eta_{21}\varphi_{31,t_0z})-0.5\rho_2(\varphi^2_{21,x_0}+\varphi^2_{21,z})+\\&+0.5\rho_3(\varphi^2_{31,x_0}+\varphi^2_{31,z})+2T_2\eta_{21,x_0x_1}\end{aligned}$$

at $z=h_2$,

$$\varphi_{32,z}=0 \text{ at } z=h_2+h_3,$$

$$\varphi_{12,z}=0 \text{ at } z=-h_1.$$

The problem of the third approximation (at α^2)

$$\varphi_{j3,x_0x_0}+\varphi_{j3,zz}=-\varphi_{j1,x_1x_1}-2\varphi_{j1,x_0x_1}-2\varphi_{j1,x_0x_2}$$

in Ω_j, $j=1,2,3$,

$$\begin{aligned}&\eta_{13,t_0}-\varphi_{j3,z}=-\eta_{11,t_2}-\eta_{12,t_1}-\eta_{11,x_0}\varphi_{j1,x_1}-\\&-\eta_{11,x_1}\varphi_{j1,x_0}-\eta_{12,x_0}\varphi_{j1,x_0}-\eta_{11,x_0}\varphi_{j2,x_0}+\\&+\eta_{11}\varphi_{j2,zz}+\eta_{12}\varphi_{j1,zz}+0.5\eta^2_{11}\varphi_{j1,zzz}-\eta_{11}\eta_{11,x_0}\varphi_{j1,x_0z}\end{aligned}$$

at $z=0$, $j=1,2$,

$$\begin{aligned}&\eta_{23,t_0}-\varphi_{j3,z}=-\eta_{21,t_2}-\eta_{22,t_1}-\eta_{21,x_0}\varphi_{j1,x_1}-\\&-\eta_{21,x_1}\varphi_{j1,x_0}-\eta_{22,x_0}\varphi_{j1,x_0}-\eta_{21,x_0}\varphi_{j2,x_0}+\\&+\eta_{21}\varphi_{j2,zz}+\eta_{22}\varphi_{j1,zz}+0.5\eta^2_{21}\varphi_{j1,zzz}-\eta_{21}\eta_{21,x_0}\varphi_{j1,x_0z}\end{aligned}$$

at $z=h_2$, $j=2,3$,

$$\begin{aligned}&\rho_1\varphi_{13,t_0}-\rho_2\varphi_{23,t_0}+(\rho_1-\rho_2)\eta_{13}-T_1\eta_{13,x_0x_0}=\\&=-\rho_1(\varphi_{11,t_2}+\varphi_{12,t_1}+\eta_{12}\varphi_{11,t_0z}+\\&+\eta_{11}\varphi_{11,t_1z}+0.5\eta^2_{12}\varphi_{11,t_0zz})+\\&+\rho_2(\varphi_{21,t_2}+\varphi_{22,t_1}+\eta_{12}\varphi_{21,t_0z}+\eta_{11}\varphi_{21,t_1z}+\\&+0.5\eta^2_{12}\varphi_{21,t_0zz})-\rho_1(\eta_{11}\varphi_{12,t_0z}+\varphi_{11,x_0}\varphi_{11,x_1}+\\&+\varphi_{11,x_0}\varphi_{12,x_1}+\eta_{11}\varphi_{11,x_0}\varphi_{11,x_0z}+\\&+\varphi_{11,z}\varphi_{11,z}+\eta_{11}\varphi_{11,z}\varphi_{11,zz})+\rho_2(\eta_{11}\varphi_{22,t_0z}+\\&+\varphi_{21,x_0}\varphi_{21,x_1}+\varphi_{21,x_0}\varphi_{22,x_1}+\\&+\eta_{11}\varphi_{21,x_0}\varphi_{21,x_0z}+\varphi_{21,z}\varphi_{21,z}+\eta_{11}\varphi_{21,z}\varphi_{21,zz})+\\&+T_1\eta_{11,x_1x_1}+2T_1\eta_{12,x_0x_1}+2T_1\eta_{12,x_0x_2}-\\&-1.5T_1(\eta_{11,x_0})^2\eta_{11,x_0x_0}\end{aligned} \tag{6}$$

at $z=0$,

$$\rho_2\varphi_{23,t_0}-\rho_3\varphi_{33,t_0}+(\rho_2-\rho_3)\eta_{23}-T_2\eta_{23,x_0x_0}=$$
$$=-\rho_2(\varphi_{21,t_2}+\varphi_{22,t_1}+\eta_{22}\varphi_{21,t_0z}+$$
$$+\eta_{21}\varphi_{21,t_1z}+0.5\eta_{22}^2\varphi_{21,t_0zz})+$$
$$+\rho_3(\varphi_{31,t_2}+\varphi_{32,t_1}+\eta_{22}\varphi_{31,t_0z}+\eta_{21}\varphi_{31,t_1z}+0.5\eta_{22}^2\varphi_{31,t_0zz})-$$
$$-\rho_2(\eta_{21}\varphi_{22,t_0z}+\varphi_{21,x_0}\varphi_{21,x_1}+\varphi_{21,x_0}\varphi_{22,x_1}+$$
$$+\eta_{21}\varphi_{21,x_0}\varphi_{21,x_0z}+\varphi_{21,z}\varphi_{21,z}+\eta_{21}\varphi_{21,z}\varphi_{21,zz})+$$
$$+\rho_3(\eta_{21}\varphi_{32,t_0z}+\varphi_{31,x_0}\varphi_{31,x_1}+\varphi_{31,x_0}\varphi_{32,x_1}+\eta_{21}\varphi_{31,x_0}\varphi_{31,x_0z}+$$
$$+\varphi_{31,z}\varphi_{31,z}+\eta_{21}\varphi_{31,z}\varphi_{31,zz})+T_2\eta_{21,x_1x_1}+$$
$$+2T_2\eta_{22,x_0x_1}+2T_2\eta_{22,x_0x_2}-1.5T_2(\eta_{21,x_0})^2\eta_{21,x_0x_0}$$

at $z=h_2$,

$$\varphi_{33,z}=0 \text{ at } z=h_2+h_3,$$

$$\varphi_{13,z}=0 \text{ at } z=-h_1.$$

The problems (4) to (6) are linear with respect to the functions that are summands in the decompositions (3). Since all terms in the decompositions (3), except for the first, enter with coefficients α in the first degree and above, therefore the solutions of the first linear approximation (4) give the main contribution to solution of the problem (2).

4. 2. Solutions to the problem of the first approximation and the dispersion relation

In a solution of the first approximation (4), a dispersion relation is obtained which simultaneously is a condition of solvability of the problem (4) in the form:

$$\frac{\rho_2^2\omega^4}{\sinh^2(kh_2)}-\begin{pmatrix}(1-\rho_2)k+T_1k^3-\\-\omega^2\left(\coth(kh_1)+\rho_2\coth(kh_2)\right)\end{pmatrix}\times\begin{pmatrix}(\rho_2-\rho_3)k+T_2k^3+\\+\omega^2\left(\coth(kh_2)-\rho_3\coth(kh_3)\right)\end{pmatrix}=0. \quad (7)$$

The dispersion relation (7) can be written as a biquadratic equation relative to ω which has two pairs of roots:

$$\omega_1^2=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \omega_1^2=\frac{-b-\sqrt{b^2-4ac}}{2a}, \quad (8)$$

where a, b, c are the coefficients at ω^4, ω^2 and a free term in the corresponding biquadratic equation

$$a=\rho_2^2-\sinh^2(kh_2)\left(\coth(kh_1)+\rho_2\coth(kh_2)\right)\times\left(\coth(kh_2)-\rho_3\coth(kh_3)\right),$$

$$b=\sinh^2(kh_2)\begin{pmatrix}\left((1-\rho_2)k+T_1k^3\right)\left(\coth(kh_2)-\rho_3\coth(kh_3)\right)-\\-\left((\rho_2-\rho_3)k+T_2k^3\right)\left(\coth(kh_1)+\rho_2\coth(kh_2)\right)\end{pmatrix},$$

$$c=\sinh^2(kh_2)\left((1-\rho_2)k+T_1k^3\right)\left((\rho_2-\rho_3)k+T_2k^3\right).$$

According to (8), there are two pairs of independent solutions to the problem (3):

– for the ω_1 mode:

$$\eta_{11}^{(1)}=A\exp(i(kx-\omega_1t));$$

$$\eta_{21}^{(1)}=-\frac{\sinh(kh_2)\left((1-\rho_2)k+T_1k^3-\omega_1^2\coth(kh_1)-\rho_2\omega_1^2\coth(kh_2)\right)}{\omega_1^2\rho_2}\times A\exp(i(kx-\omega_1t));$$

$$\varphi_{11}^{(1)}=-\frac{i\omega_1}{k\sinh(kh_1)}\cosh(k(h_1+z))A\exp(i(kx-\omega_1t)); \quad (9)$$

$$\varphi_{21}^{(1)}=\begin{pmatrix}\frac{i\omega_1}{k\sinh(kh_2)}\cosh(k(h_2-z))+\\+\frac{i\left((1-\rho_2)k+T_1k^3-\omega_1^2\coth(kh_1)-\rho_2\omega_1^2\coth(kh_2)\right)}{k\rho_2\omega_1}\cosh(kz)\end{pmatrix}\times A\exp(i(kx-\omega_1t));$$

$$\varphi_{31}^{(1)}=-\frac{i\sinh(kh_2)\left((1-\rho_2)k+T_1k^3-\omega_1^2\coth(kh_1)-\rho_2\omega_1^2\coth(kh_2)\right)}{k\rho_2\omega_1\sinh(kh_3)}\times\cosh(k(h_2+h_3-z))A\exp(i(kx-\omega_1t));$$

– for the ω_2 mode:

$$\eta_{11}^{(2)}=-\frac{\omega_2^2\rho_2}{\sinh(kh_2)\left((1-\rho_2)k+T_1k^3-\omega_2^2\coth(kh_1)-\rho_2\omega_2^2\coth(kh_2)\right)}\times B\exp(i(kx-\omega_2t));$$

$$\eta_{21}^{(2)}=B\exp(i(kx-\omega_1t));$$

$$\varphi_{11}^{(2)}=\frac{i\omega_2^3\rho_2}{k\sinh^2(kh_1)\left((1-\rho_2)k+T_1k^3-\omega_2^2\coth(kh_1)-\rho_2\omega_2^2\coth(kh_2)\right)}\times B\cosh(k(h_1+z))\exp(i(kx-\omega_2t)); \quad (10)$$

$$\varphi_{21}^{(2)}=-\frac{i\omega_2}{k\sinh(kh_2)}\times\begin{pmatrix}\frac{\omega_2^2\rho_2}{\sinh(kh_2)\left((1-\rho_2)k+T_1k^3-\omega_2^2\coth(kh_1)-\rho_2\omega_2^2\coth(kh_2)\right)}\times\\\times\cosh(k(h_2-z))+\cosh(kz)\end{pmatrix}\times B\exp(i(kx-\omega_2t))$$

$$\varphi_{31}^{(2)}=\frac{i\omega_2}{k\sinh(kh_3)}\cosh(k(h_2+h_3-z))A\exp(i(kx-\omega_1t)).$$

Solutions (9) and (10) are independent while $\eta_{21}^{(1)}$ in (9) is a response wave on the upper contact surface to a wave $\eta_{11}^{(1)}$, which has amplitude A and propagates with a frequency ω_1 on the lower contact surface. In (10), $\eta_{11}^{(2)}$ is a response wave on the bottom contact surface to a wave $\eta_{21}^{(2)}$, which has amplitude B and propagates with a frequency ω_2 on the upper surface of the contact.

Analysis of the roots of the dispersion relation will be made hereinafter depending on various geometric and physical parameters of the system under study. The issue of wave motion on the contact surfaces is also considered.

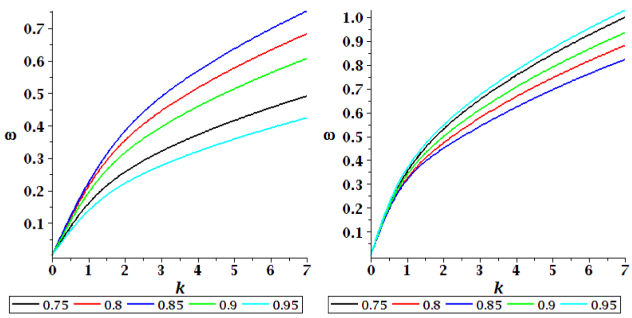
5. Analysis of the dispersion relation

5. 1. Dispersion diagrams for various physical and geometric parameters of the system

Let us study dependence of the dispersion relation (8) modes, ω_1 and ω_2, on the change of the ratio of layer densities. Fix $\rho_3 = 0.7$ and change ρ_2 in the range from 0.7 to 1 in steps of 0.05 at the following values of the system parameters:

$$h_1 = h_2 = h_3 = 1, \quad T_1 = T_2 = 0.$$

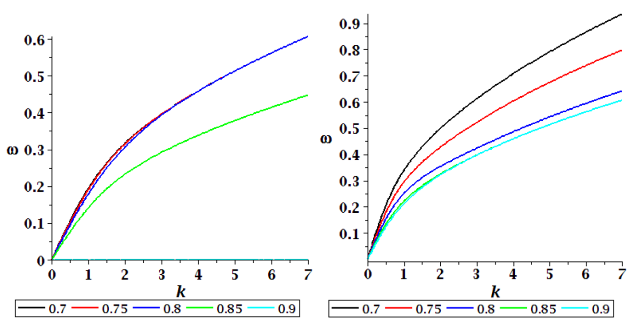
Cases $\rho_2 = 0.7$ and $\rho_2 = 1$ will not be considered because they are marginal and will be considered below.



a *b*

Fig. 2. Dependence of the dispersion relation modes on the wave number for various values $\rho_2 = \{0.75, 0.8, 0.85, 0.9, 0.95\}$: ω_1 (*a*); ω_2 (*b*)

If density of the middle layer, ρ_2, is the mean value of the densities of the lower and upper layers, then the ω_1 graph is higher than the graphs for other ρ_2 values (Fig. 2, *a*) and the ω_2 graph is lower than other graphs (Fig. 2, *b*) (for wave numbers $k > 2$). If the value of ρ_2 becomes closer to ρ_1 or ρ_3, then the ω_1 graph descends and the ω_2 graph ascends.

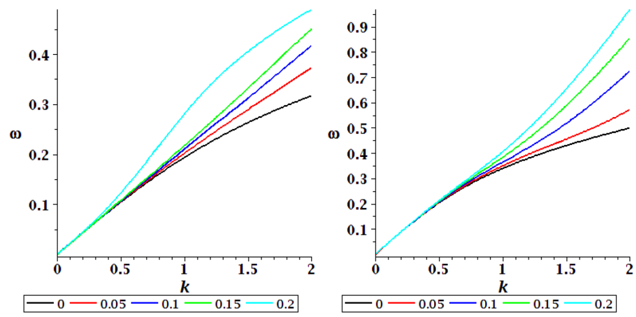


a *b*

Fig. 3. Dependence of the dispersion relation modes on the wave number for various values: ω_1 (*a*); ω_2 (*b*)

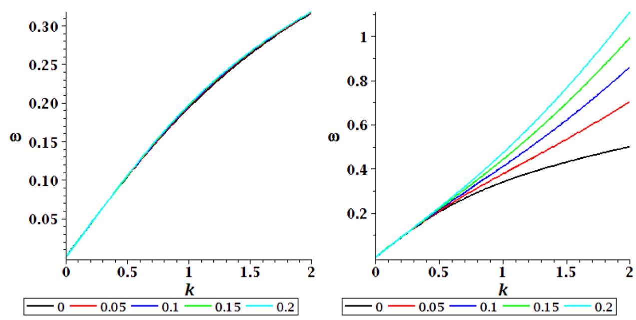
Fix the value of $\rho_2 = 0.9$ and change ρ_3 in a range from 0.7 to 0.9 in 0.05 steps (Fig. 3). At the fixed value of ρ_2, the change of ρ_3 from 0.7 to 0.8 has little effect on the ω_1 value (Fig. 3, *a*). Further approximation of the ρ_3 value to the ρ_2 value leads to the ω_1 zeroing. This will be described when considering the marginal cases. Change of the ρ_3 value from 0.7 to 0.9 leads to a gradual decrease in the ω_2 value (Fig. 3, *b*).

Next, let us analyze the effect of the surface tension on the change of the ω_1 and ω_2 modes. Fix the value of $T_2 = 0$ and change the value of T_1 in a range from 0 to 0.2 in 0.05 steps (Fig. 4). The change of the surface tension, T_1, on the bottom contact surface nearly does not affect the change of the ω_1 mode (Fig. 4, *a*). At the same time, taking into account surface tension on the lower contact surface significantly affects variation of the ω_2 mode for the wave numbers $k > 0.5$, corresponding to the gravitational-capillary and capillary waves (Fig. 4, *b*). A similar pattern was also observed with consideration of the surface tension on the upper surface of the contact, T_2 (Fig. 5).



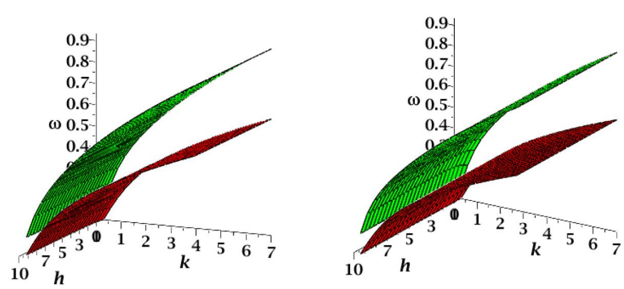
a *b*

Fig. 4. Dependence of the dispersion relation modes on the wave number for various values: ω_1 (*a*); ω_2 (*b*)



a *b*

Fig. 5. Dependence of the dispersion relation modes on the wave number for various values: ω_1 (*a*); ω_2 (*b*)



a *b*

Fig. 6. Dependence of the dispersion relation modes, ω_1 and ω_2, on the wave number and: thickness of the lower layer, h_1 (*a*); thickness of the upper layer, h_3 (*b*)

Analyzing the effect of geometric parameters of the system, i. e. thickness of the lower layer, h_1 and thickness of the upper layer, h_3, the study has shown that the change of these parameters has no significant effect on the change of the ω_1 and ω_2 modes for fixed wave numbers (Fig. 6).

5. 2. Marginal cases of the system under study

Let us consider the marginal cases in which the studied three-layer "a layer with a solid bottom – a layer – a layer with a solid cover" system degenerates into a two-layer "a layer with a solid bottom – a layer with a solid cover" system. The first marginal case arises under conditions $\rho_2=\rho_3$ and $T_2=0$. The second marginal case arises under the conditions $\rho_1=\rho_2$ and $T_1=0$. In both cases, the first pair of roots, ω_1^2, of the dispersion relation becomes equal to zero and the second pair of roots, ω_2^2, degenerates into a pair of roots of the dispersion relation of the "a layer with a solid bottom – a layer with a solid cover" system.

$$\omega^2=\frac{(1-\rho)k+Tk^3}{\coth(kh_1)+\rho\coth(kh_2)} \quad [18].$$

5. 3. Analysis of wave motion on the contact surfaces

According to (9) and (10), the wave motion on the lower and upper contact surfaces can be represented in the first approximation as:

$$\eta_{11}=\eta_{11}^{(1)}+\eta_{11}^{(2)},\quad \eta_{21}=\eta_{21}^{(1)}+\eta_{21}^{(2)}, \qquad (11)$$

that is, as a linear combination of waves for various modes. Let us consider propagation of wave packets on the lower and upper contact surfaces which are a linear combination of waves of the same amplitude and a close length:

$$\eta_1=\eta_{11}(k_1)+\eta_{11}(k_2)+\eta_{11}(k_3),$$

$$\eta_2=\eta_{21}(k_1)+\eta_{21}(k_2)+\eta_{21}(k_3). \qquad (12)$$

Fig. 7 shows appearance of wave packets (12) on the contact surfaces for the following system parameters:

$$h_1=h_2=h_3=1,\quad T_1=T_2=0,\quad \rho_2=0.95,$$

$$\rho_3=0.92,\quad A=0.05,\quad B=0.025.$$

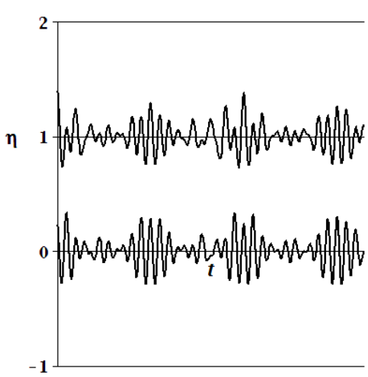


Fig. 7. Wave packets on the contact surfaces at $k_1=0.009,\ k_2=0.01,\ k_3=0.011$

Fig. 8 shows the case of propagation of the wave packet on the lower and upper contact surfaces which is a linear combination of waves with frequency ω_1.

Fig. 9 shows the case of propagation of a wave packet which is a linear combination of waves with a ω_2 frequency on the lower and upper contact surfaces.

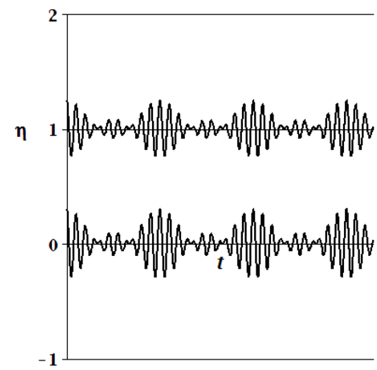


Fig. 8. Propagation of wave packets that correspond to the ω_1 mode in a three-layer system

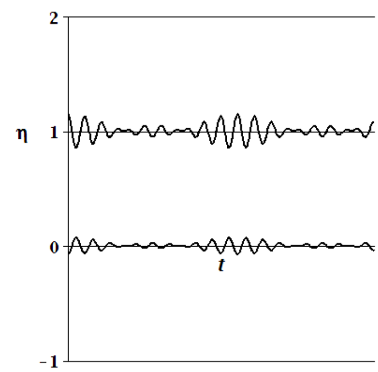


Fig. 9. Propagation of wave packets in a three-layer system corresponding to the ω_2 mode

In the case of waves of a similar length corresponding to the mode ω_1 passing on the lower and upper contact surfaces (Fig. 8), the wave packets formed on both contact surfaces have a clear shape, i. e. amplitude of the envelope of such wave packets is sinusoidal. A similar situation was observed in passing of a group of waves of similar length corresponding to the ω_2 mode (Fig. 9). In the presence of groups of waves of a similar length, for both ω_1 and ω_2 modes on the upper and lower contact surfaces, the wave packets formed on the lower contact surface have a clearer shape than on the upper contact surface (Fig. 7). That is, amplitude of the envelope on the lower contact surface remains sinusoidal whereas it changes according to a more complicated law on the upper contact surface.

6. Discussion of the results obtained in the study of a three-layer hydrodynamic system

The study of a three-layer system with solid cover and bottom was carried out for a model in which the wave amplitude is much smaller than the wave length (the coefficient of nonlinearity $\alpha=\frac{a}{L}$ is much smaller than one) and the wave lengths are close to the thickness of the middle layer (the dispersion coefficient $\beta=\frac{H^2}{L^2}$ is close to one). It is interesting to study other cases, for example, when β is also close to zero. As a result of application of the multi-scale method to the third order, it was possible to obtain the first three approximations of the problem under consideration. Let us derive dispersion relation and solutions of the first approximation constructed on the basis of its roots.

During the study of the dispersion relation roots, it was found that the ω_1 and ω_2 modes differed in their sensitivity to changes of the system physical parameters. In particular, the ω_1 mode was more sensitive to the change of the ratio of the ρ_2 densities than the ω_2 mode while the ω_2 mode was more sensitive to the change of ρ_3, T_1 and T_2 parameters. The study has also shown that both modes were insensitive to the change of geometric parameters of the system, namely thicknesses of the lower, h_1, and upper, h_3, layers.

It was also found that $\omega_2 > \omega_1$ for the same wave numbers. A question arises about the contribution of waves with ω_1 and ω_2 frequencies in their motion on the upper and lower contact surfaces, i. e. at which system parameters the waves with above frequencies make a significant contribution to the wave motion on each contact surface. To answer this question, it is necessary to study the amplitude ratio of the waves on the lower and upper contact surfaces. It is the subject of further studies.

In the course of study of the dispersion relation, two marginal cases have been established in which the three-layer system under study degenerated into a two-layer system. In this case, the ω_1 mode became equal to zero and the dispersion relation degenerated to the corresponding dispersion relation for the "a layer with a solid bottom – a layer with a solid cover" system [18].

The structure of wave motion on the contact surfaces during passage of groups of waves of similar length was analyzed. It was found that the amplitude of the envelope of the wave packets on the lower contact surface was sinusoidal in the first approximation and it changed on the upper surface according to a more complicated law. A detailed study of evolution of the wave-packet envelopes requires an analysis of higher approximations (5) and (6) which should result in the study of nonlinear Schrödinger equations of the second order.

It was planned to continue the study of a three-layer hydrodynamic system for obtaining approximations of the second and third order using the multi-scale method.

7. Conclusions

1. The mathematical statement of the problem of propagation of internal wave packets in a three-layer hydrodynamic "a layer with a solid bottom – a layer – a layer with a free surface" system was fulfilled. A transition to dimensionless quantities was made during which two parameters were found: nonlinearity coefficient and dispersion coefficient. On the basis of this statement, a linear approximation of the problem under study was obtained.

2. Solutions of the linear problem in the form of progressive waves were obtained. A dispersion relation was derived based on these solutions. Existence of two characteristic modes (the dispersion relation roots) was established.

3. The dispersion relation roots of the on the diagrams (ω, k) were analyzed depending on various physical and geometric parameters of the system. It was found that the change of geometric parameters of the system (thickness of the lower and the upper layers) did not affect the dispersion relation modes. The change of the ratio of densities affected both modes but ω_1 to a greater extent. The second mode, ω_2, was more sensitive to the change of parameters ρ_3, T_1 and T_2.

4. The shape of wave packets on the contact surfaces for both modes which are roots of the dispersion relation was investigated. Wave packets in a linear formulation which are a superposition of harmonic waves of a similar length were considered. It was found that the amplitude of the wave packet envelope on the lower contact surface remained sinusoidal whereas it changed according to a more complicated law on the upper contact surface.

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