UDC 621
DOI: 10.15587/1729-4061.2018.127345

INVESTIGATING A PROBLEM FROM THE THEORY OF ELASTICITY FOR A HALF-SPACE WITH CYLINDRICAL CAVITIES FOR WHICH BOUNDARY CONDITIONS OF CONTACT TYPE ARE ASSIGNED

V. Protsenko
Doctor of Physical and Mathematical Sciences, Professor
Department of Mathematics and Systems Analysis
N. E. Zhukovsky National Aerospace University "Kharkiv Aviation Institute"
Chkalova str., 17, Kharkiv, Ukraine, 61070
E-mail: k405@khai.edu

V. Miroshnikov
PhD, Associate Professor
Department of Construction Mechanics
Kharkiv National University of Construction and Architecture
Sumska str., 40, Kharkiv, Ukraine, 61002
E-mail: m0672628781@gmail.com

При проектуванні просторових конструкцій необхідно знати напружено – деформований стан тіла. Серед таких задач зустрічаються розрахункові схеми, в яких є півпростір з циліндричними порожнинами, на межах яких задані умови контактного типу. Сегмент таких задач недостатньо досліджений та потребує уваги.

Запропоновано аналітико-чисельний алгоритм розв'язання просторової задачі теорії пружності для півпростору з циліндричними порожнинами. На межах порожнин задані радіальні переміщення та дотичні напруження, а на межі півпростору заданий один із двох типів граничних умов – переміщення або напруження. Проведеними розрахунками встановлено напружено деформований стан півпростору.

При фіксованих геометричних умовах було проведено чисельний аналіз трьох варіантів задачі, коли на межі півпростору задані переміщення та трьох варіантів задачі, коли на межі півпростору задані напруження. Проведено порівняльний аналіз варіантів з різними граничними умовами між собою.

Встановлено, що при різних видах заданих крайових умов (напруження або переміщення), напруження і на межі прикладення таких умов змінюються на протилежні, тобто з розтягуючих на стискаючі або навпаки. Також встановлено, що крайові умови на межі півпростору у вигляді напружень мають більший вплив на напружений стан ніж крайові умови у вигляді переміщень. Ці твердження мають місце при заданих на межах циліндричних порожнин граничних умов контактного типу, якщо задана функція переміщень та задана функція напружень однакові.

Наведений аналіз можна використовувати при проектуванні конструкцій, в розрахункових схемах яких є межа півпростору із заданими на ній граничними умовами контактного типу та циліндричних порожнин, на поверхні яких задані переміщення або напруження

Ключові слова: циліндричні порожнини в півпросторі, рівняння Ламе, узагальнений метод Фур'є, нескінчені системи лінійних алгебраїчних рівнянь

1. Introduction

In the design of various structures, underground facilities and communications, there is a need in assessing stresses in a half-space with cavities. For this, it is necessary to have a method for calculation of the problems of the theory of elasticity, which makes it possible to find a stressed-strained state of a half-space with cylindrical cavities. We explore the problem, where displacement or stress are assigned at the boundaries of a cavity, and the conditions of a contact type are assigned at the boundaries of a half-space in the form of radial displacement and tangential efforts. Ready calculations of similar problems in the spatial variant were not found, so the problem of calculation of such problems is important. In addition to the proposed algorithm of calculation, an analysis of the stressed state, which enables prediction of the weak points at the stage of design, is presented in this paper.

2. Literature review and problem statement

To evaluate the stressed-strained state of a half-space with cylindrical cavities, several scientific studies apply the finite element method [1, 2], which is an approximated calculation method and does not provide full reliability on the accuracy of calculation when there are the infinite boundaries of an elastic body.

The most common problems for a half-space with cavities are the problems when cavities are perpendicular to the surface of a half-space [3–6]. The calculation of these problems is based on the Weber integral transformation, the method of representations of Papkovich–Neuber, the theory of integral equations of Fredholm, the Neumann series, and the generalized integral equation of Cauchy. However, these methods cannot be used when cylindrical cavities are parallel to the surface of a half-space.

Papers [7–9] consider the problems of diffraction of waves in a half-space with a single cylindrical cavity based on the Helmholtz equation. Since article [7] considers all regions to be proportional to e^{ikx}, in fact a flat problem with the use of the transformation of the plane with a circular opening into a concentric ring is solved. Papers [8, 9] also consider flat problems, the wave equation of which is solved approximately using the comformal transformation and the collocation scheme of the least square. The described algorithms make it possible to calculate the problems of a half-space with only one cavity.

Articles [10, 11] are devoted to determining the stressed state of the final cylinder. The method is based on the superposition of solutions and decomposition into Fourier and Dini series. But the problems for elastic bodies with multiple boundary surfaces cannot be solved within the framework of the classical approach. For such problems it was necessary to create a generalized Fourier method [12], the substantiation of which for spatial problems of the elasticity theory was given in [13]. This method was laid as the basis for the approach to the solution of the considered problem.

The first major problem of the theory of elasticity for transversally-isotropic bodies limited by coordinate surfaces in cylindrical and parabolic coordinates of the stationary parabolic cavity is solved in [14] and with actual parabolic inclusion in [15]. The applied problem about the effect of a concentrated force on a sandstone half-space with parabolic inclusion was considered in [16]. The thermo-elastic boundary problem for a transversally-isotropic half-space with a spherical cavity was considered in [17]. In articles [18, 19], the first and the second basic problems of the elasticity theory for a half-space with a single cylindrical cavity were explored. The second basic problem of the elasticity theory for a half-space with several cylindrical cavities was solved in [20]. All these papers are based on the generalized Fourier method, but the algorithms, presented in them, do not make it possible to directly address the mixed problems with the boundary conditions of the contact type and explore the stressed state of a half-space with such boundary conditions. For this, it is necessary to explore further the possibility of solving the problem with the contact type conditions.

Mixed problems were considered for a space with cylindrical cavities, when displacements are assigned on the boundaries of some cavities, stresses are assigned on the boundaries of other cavities [21]. Mixed problems for space were considered when displacements are assigned on some boundaries, stresses are assigned on the other boundaries, and the conditions of the contact type are assigned on the third boundaries [22]. These works are based on the generalized Fourier method, but can not be applied directly to the problems in a half-space.

It follows from the above that the problems for a half-space with cylindrical cavities with the boundary conditions of the contact type need studying.

That is why it is appropriate to develop the analytical-numerical method for solving a mixed problem of the elasticity theory for a half-space with cylindrical cavities and some boundary conditions of the contact type.

3. The aim and objectives of the study

The aim of this research is the evaluation of the stressed-strained state of a half-space with cylindrical cavities, under the following conditions: radial displacements and tangential stresses are assigned at the boundaries of cavities, and one of the two types of boundary conditions – displacements or stresses – is assigned at the half-space boundary.

To accomplish the aim, the following tasks have been set:

– to develop the analytical-numerical algorithm of calculation;

– to conduct numerical studies for a half-space and two cylinders and to analyze the influence of the type of boundary conditions on the stress in the zone of the isthmus between the cylinders and the isthmus between the half-space boundary and a cylinder.

4. Analytical and numeric algorithm for calculation

4. 1. Problem statement

Elastic homogeneous half-space has N circular cylinder parallel cavities, non-crossing each other and the boundary of a half-space (Fig. 1).

The cavity will be considered in the cylindrical coordinate system (ρ_p, z, where p is the number of the cylinder), a half-space will be considered in Cartesian coordinates (x, y, z), which are equally oriented and combined with the coordinate system of the cylinder with number p=1. The half-space boundary is located at the distance y=h, the equation of the surface of the cylinders S_p: ρ_p=R_p , p=1, 2,..., N. It is necessary to find a solution to the Lame's equation

$$\Delta\vec{u}+(1-2\sigma)^{-1}\nabla\operatorname{div}\vec{u}=0$$

under conditions that at the boundaries of a half-space, one of the two types of boundary conditions – displacement $\vec{U}_d^0(x,z)$ or stress $\vec{f}_d(x,z)$, is assigned at the boundaries of a half-space, and conditions of the contact type are assigned at the boundaries of the cavities

$$\left.\begin{aligned} &U_\rho(\phi_p,z)\big|_{\rho_p=R_p}=U_0^{(p)}(\phi_p,z),\\ &\tau_{\rho\phi|\rho_p=R_p}=\tau_1^{(p)}(\phi_p,z),\\ &\tau_{\rho z|\rho_p=R_p}=\tau_2^{(p)}(\phi_p,z)\end{aligned}\right\}\quad p=1,2,\dots,N, \tag{1}$$

where the right sides of these equalities are the known functions.

All the assigned vectors and functions will be considered descending to zero at long distances from the origin along coordinate z for cylinders and along coordinates x and z for a half-space.

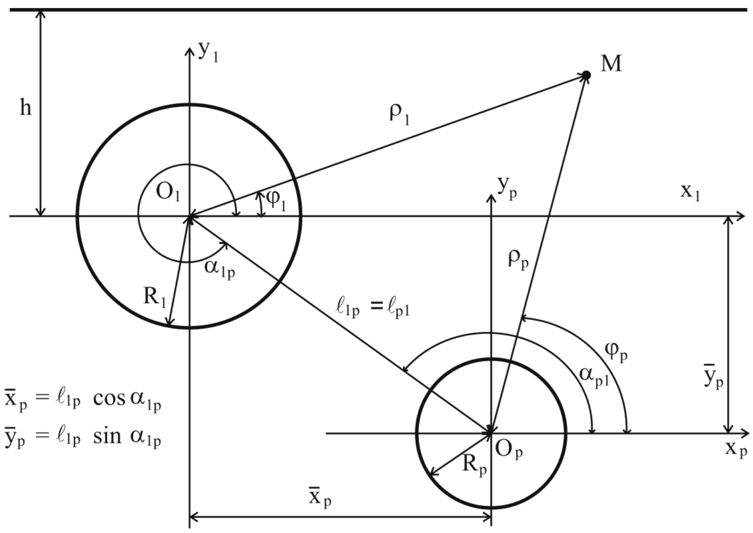


Fig. 1. Half-space with cylindrical cavities

4. 2. The method of solution

Let us select the basic solutions of the Lame's equation for the specified coordinate systems in the form of [12]:

$$\vec{u}_k^{\pm}(M_d;\lambda,\mu)=N_k^{(d)}u^{\pm}(M_d;\lambda,\mu);\quad (k=1,\ 2,\ 3);\tag{2}$$

$$\vec{R}_{k,m}(M_p;\lambda)=N_k^{(p)}I_m(\lambda\rho)e^{i(\lambda z+m\phi)};$$

$$\vec{S}_{k,m}(M_p;\lambda)=N_k^{(p)}\left[s_m(\rho_p;\lambda)e^{i(\lambda z+m\phi)}\right],\quad k=1,2,3;\tag{3}$$

$$N_1^{(d)}=\frac{1}{\lambda}\nabla;\quad N_2^{(d)}=\frac{4}{\lambda}(\sigma-1)\vec{e}_2^{(1)}+\frac{1}{\lambda}\nabla(y\cdot);$$

$$N_3^{(d)}=\frac{i}{\lambda}\mathrm{rot}\left(\vec{e}_3^{(1)}\cdot\right);\quad u^{\pm}(M_d;\lambda,\mu)=e^{i(\lambda z+\mu x)\pm\gamma y};$$

$$N_1^{(p)}=\frac{1}{\lambda}\nabla;\quad N_2^{(p)}=\frac{1}{\lambda}\left[\nabla\left(\rho\frac{\partial}{\partial\rho}\right)+4(\sigma-1)\left(\nabla-\vec{e}_3^{(2)}\frac{\partial}{\partial z}\right)\right];$$

$$N_3^{(p)}=\frac{i}{\lambda}\mathrm{rot}\left(\vec{e}_3^{(2)}\cdot\right);\quad s_m(\rho_p;\lambda)=(\mathrm{sign}\,\lambda)^m K_m(|\lambda|\rho_p);$$

$$\gamma=\sqrt{\lambda^2+\mu^2},\quad -\infty<\lambda,\mu<\infty,$$

where $M_d=(x, y, z)$, $M_p=(\rho_p, \phi_p, z)$ are the points of a space, respectively, in Cartesian coordinates and in the cylindrical coordinate system, connected with p – cylinder; $\vec{e}_j^{(k)}$, $(j=1,2,3)$ is the orts of the Cartesian (k=1) and cylindrical (k=2) coordinate systems; σ is the Poisson coefficient; $I_m(x)$, $K_m(x)$ are the modified Bessel functions; $\vec{R}_{k,m}$, $\vec{S}_{k,m}$, $(k=1,2,3)$ are respectively internal and external solutions to the Lame's equation for a cylinder; $\vec{u}_k^{(-)}$, $\vec{u}_k^{(+)}$ are the solutions to the Lame's equation for a half-space.

The solution to the problem will be represented in the form of

$$\vec{U}=\sum_{p=1}^{N}\sum_{k=1}^{3}\int_{-\infty}^{\infty}\sum_{m=-\infty}^{\infty}B_{k,m}^{(p)}(\lambda)\cdot\vec{S}_{k,m}(M_p;\lambda)d\lambda+$$
$$+\sum_{k=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_k(\lambda,\mu)\cdot\vec{u}_k^{(+)}(M_d;\lambda,\mu)d\mu d\lambda,\tag{4}$$

where $\vec{S}_{k,m}(M_p;\lambda)$ and $\vec{u}_k^{(+)}(M_d;\lambda,\mu)$ are the basic solutions, which are assigned by formulas (2) and (3), and it is necessary to find the unknown functions $H_k(\lambda, \mu)$ and from the boundary conditions.

For transition between the coordinate systems (Fig. 1), we will use the formulas:

– for transition from the coordinates of the cylinder with number p to the coordinates of a half-space, we will generalize formula [18]

$$\vec{S}_{k,m}(M_p;\lambda)=\frac{(-i\cdot\mathrm{sign}\,\lambda)^m}{2}\int_{-\infty}^{\infty}\omega^m\cdot\vec{u}_k^{(-)}\cdot e^{-i\mu\bar{x}_p+\gamma\bar{y}}$$

$$k=1,3;$$

$$\vec{S}_{2,m}(M_p;\lambda)=$$
$$=\frac{(-i\cdot\mathrm{sign}\,\lambda)^m}{2}\int_{-\infty}^{\infty}\omega^m\cdot\left(\left(m\cdot\mu-\frac{\lambda^2}{\gamma}+\lambda^2y_p\right)\vec{u}_1^{(-)}-\lambda^2\vec{u}_2^{(-)}+\right.$$
$$\left.+4\mu(1-\sigma)\vec{u}_3^{(-)}\right)\frac{e^{-i\mu\bar{x}_p+\gamma\bar{y}_p}d\mu}{\gamma^2},\tag{5}$$

where

$$\gamma=\sqrt{\lambda^2+\mu^2},\quad \omega(\lambda,\mu)=\frac{\mu-\gamma}{\lambda},$$

$y>0$, $m=0, \pm1, \pm2,\ldots$;

– for transition from the coordinates of a half-space to the coordinates of the cylinder p, we will generalize formula [18]

$$\vec{u}_k^{(+)}=\sum_{m=-\infty}^{\infty}(i\cdot\omega)^m\vec{R}_{k,m}(M_p)\cdot e^{i\mu\bar{x}_p+\gamma\bar{y}_p},\quad (k=1,3);$$

$$\vec{u}_2^{(+)}=e^{i\mu\bar{x}_p+\gamma\bar{y}_p}\times$$
$$\times\sum_{m=-\infty}^{\infty}\left[(i\cdot\omega)^m\cdot\lambda^{-2}\left(\left(m\cdot\mu+\bar{y}_p\cdot\lambda^2\right)\cdot\vec{R}_{1,m}(M_p)+\right.\right.$$
$$\left.\left.+\gamma\cdot\vec{R}_{2,m}(M_p)+4\mu(1-\sigma)\vec{R}_{3,m}(M_p)\right)\right],\tag{6}$$

where

$$\vec{R}_{k,m}(M_p)=\vec{\tilde{b}}_{k,m}(\rho_p,\lambda)\cdot e^{i(m\phi_p+\lambda z)};$$

$\bar{x}_p, \bar{y}_p$ are the coordinates of cylinder p relative to the cylinder number 1.

$$\vec{\tilde{b}}_{1,n}(\rho_p,\lambda)=\vec{e}_\rho\cdot I_n'(\lambda\rho_p)+i\cdot I_n(\lambda\rho_p)\cdot\left(\vec{e}_\phi\frac{n}{\lambda\rho_p}+\vec{e}_z\right);$$

$$\vec{\tilde{b}}_{2,n}(\rho_p,\lambda)=\vec{e}_\rho\cdot\left[(4\sigma-3)\cdot I_n'(\lambda\rho_p)+\lambda\rho_p I_n''(\lambda\rho_p)\right]+$$
$$+\vec{e}_\phi i\cdot m\left(I_n'(\lambda\rho_p)+\frac{4(\sigma-1)}{\lambda\rho_p}I_n(\lambda\rho_p)\right)+\vec{e}_z i\lambda\rho_p I_n'(\lambda\rho_p);$$

where $\vec{e}_\rho$, $\vec{e}_\phi$, $\vec{e}_z$ are the orts in the cylindrical coordinate system,

– for transition from the coordinates of cylinder p to coordinates of cylinder 1 [12]

$$\vec{S}_{k,m}(M_p;\lambda)=\sum_{n=-\infty}^{\infty}\vec{b}_{k,p1}^{mn}(\rho_1)\cdot e^{i(n\phi_1+\lambda z)},\quad k=1,2,3;$$

$$\vec{b}_{1,p1}^{mn}(\rho_1)=(-1)^n\tilde{K}_{m-n}(\lambda\ell_{p1})\cdot e^{i(m-n)\alpha_{p1}}\cdot\vec{\tilde{b}}_{1,n}(\rho_1,\lambda);$$

$$\vec{b}_{3,p1}^{mn}(\rho_1)=(-1)^n\,\tilde{K}_{m-n}(\lambda\ell_{p1})\cdot e^{i(m-n)\alpha_{p1}}\cdot\vec{\tilde{b}}_{3,n}(\rho_1,\lambda);$$

$$\vec{b}_{2,p1}^{mn}(\rho_1)=(-1)^n\left\{\tilde{K}_{m-n}(\lambda\ell_{p1})\cdot\vec{\tilde{b}}_{2,n}(\rho_1,\lambda)-\frac{\lambda}{2}\ell_{p1}\times\right.$$

$$\left.\times\left[\tilde{K}_{m-n+1}(\lambda\ell_{p1})+\tilde{K}_{m-n-1}(\lambda\ell_{p1})\right]\cdot\vec{\tilde{b}}_{1,n}(\rho_1,\lambda)\right\}\cdot e^{i(m-n)\alpha_{p1}}, \qquad (7)$$

where α_{p1} is the angle between the coordinate axis x_1 and section ℓ_{1p},

$$\tilde{K}_m(x)=(\operatorname{sign}x)^m\cdot K_m(|x|).$$

For transition from the coordinates of cylinder 1 to coordinates of cylinder p in formula (7), the places of indices should be changed

To satisfy the boundary conditions at the boundary of half-space $y=h$, the left part (4) with the help of transition formula (5) will be re-written in the Cartesian coordinates through the basic solutions $\vec{u}_k^{(-)}$. If the boundary conditions on the boundary $y=h$ are assigned in displacements, the resulting vector (at $y=h$) will be equaled to the assigned vector but if boundary conditions are assigned in stresses, we will find the stress for the resulting vector and equal it (at $y=h$) to the assigned vector $\vec{f}_d(x,z)$. Vectors

$$\vec{U}_d^0(x,z)=U_x^{(d)}\vec{e}_1^{(1)}+U_y^{(d)}\vec{e}_2^{(1)}+U_z^{(d)}\vec{e}_3^{(1)}$$

and

$$\vec{f}_d(x,z)=\tau_{yx}^{(d)}\vec{e}_1^{(1)}+\sigma_y^{(d)}\vec{e}_2^{(1)}+\tau_{yz}^{(d)}\vec{e}_3^{(1)}$$

preliminarily represent by the double Fourier integral. From the resulting equations, we will find the functions H_k (λ, μ) through $B_{k,m}^{(p)}(\lambda)$.

Using the formulas of transition from the Cartesian system to the cylindrical system (6), as well as from one cylinder to the other (7), we will rewrite (4) in the coordinates of the cylinder number p through basic solutions $\vec{R}_{k,m}$, $\vec{S}_{k,m}$. If now we find U_ρ (ϕ_ρ, z) and stresses $\tau_{\rho\phi}$, $\tau_{\rho z}$, for the right side of equation (4) on the surface of each cylindrical cavity and take into consideration boundary conditions (1), we will obtain the system of equations for coefficients $B_{k,m}^{(p)}(\lambda)$, which includes functions H_k (λ, μ), for each cavity p. The determinant of this system is not equal to zero, moreover

$$\text{– for } m=0\ \left|\Delta_0\right|=8(1-\sigma)\cdot x^2\cdot K_1^2(x)\cdot K_2(x)\ , \qquad (8)$$

$$\text{– for } m\geq 1\left|\Delta_m\right|>4m\cdot K_{m-1}(x)K_m(x)K_{m+1}(x),\quad x=|\lambda|\rho,\quad \lambda\neq 0.$$

Functions H_k (λ, μ), which were expressed above through $B_{k,m}^{(p)}(\lambda)$, will be substituted in the equation with expressions $B_{k,m}^{(p)}(\lambda)$. As a result, we will obtain the totality from N·3 of non-finite systems of linear algebraic equations for determining unknown $B_{k,m}^{(p)}(\lambda)$.

For the derived systems, using inequality (8), a definite possibility of solution on conditions of not touching the boundaries was proved. Moreover, these systems can be solved with the truncating method and approximate solutions converge to the exact ones. Functions $B_{s,m}^{(p)}(\lambda)$, found from the infinite system of equations, will be substituted in expressions for H_k (λ, μ). This will determine all the unknown problems.

5. Numerical studies for a half-space and two cylinders

We have two parallel cylindrical cavities in a half-space (Fig. 1), p=2. A half-space is isotropic material, Poisson ratio σ=0.35, the elasticity modulus E=2 kN/cm^2. The boundary of a half-space is located at the distance h=40 cm, the cylinders, the radii of which are R_1=$R2$=10 cm, are located on the horizontal axis (α_{12}=0) at the distance of $\ell_{12}=50$ cm.

Several variants of the problems with various boundary conditions were calculated – three variants, when displacement was assigned at the boundary of a half-space and three variants when stresses were assigned at the boundary of a half-space. In all variants, conditions of the contact type are assigned at the boundaries of the cylinders.

An infinite system of equations was reduced to the finite by parameter m – the order of the system. The influence of the value of parameter m was studied in [18]. The integration boundaries for the assigned boundary functions were taken from –1...1. Calculation of integrals was performed using quadratic formulas of Filon and Simpson. Accuracy of the implementation of the boundary conditions at specified values of geometrical parameters was brought to 10^{-3} (m=8).

Variant 1

Displacement $U_x^{(d)}=U_y^{(d)}=U_z^{(d)}=0$ is assigned at the boundary of a half-space. At the boundary of cylinder 1, radial displacement is assigned

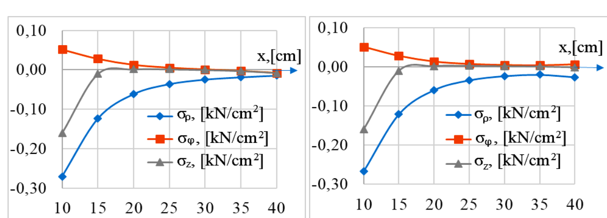
$$U_0^{(1)}(\phi_1,z)=10^{-4}\cdot\left(z^2+10^2\right)^2$$

and tangential stresses $\tau_{\rho\phi}^{(1)}=\tau_{\rho z}^{(1)}=0$, are assigned at the boundary of cylinder 2, boundary conditions are

$$U_0^{(2)}(\phi_2,z)=0;\quad \tau_{\rho\phi}^{(2)}=\tau_{\rho z}^{(2)}=0.$$

Fig. 2 shows the diagram of normal stresses on the isthmus between cylinder 1 and the boundary of a half-space (Fig. 2, a) and on the isthmus between the cylinders 1 and 2 (Fig. 2, b) in plane z=0.

The most stressed state is on the surface of the "loaded" cylinder 2, where stress σ_ρ=–0.271 kN/cm^2, σ_ϕ=0.052 kN cm^2, σ_z=–0.161 kN/cm^2. On the isthmuses, the stressed state at the boundary of a half-space (Fig. 2, a, distance is 40 cm) and on the boundary of the cylinder 2 (Fig. 2, b, distance is 40 cm) differ from each other: thus stress σ_ρ=–0.0261 kN/cm^2 increases near the surface of cylinder 2, unlike the half-space, where σ_ρ decreases (σ_ρ=–0.014 kN cm^2), at the boundary of a half-space has compressing forces (σ_ϕ=–0.008 kN/cm^2), and at the boundary of cylinder 2, vice versa, stretching forces (σ_ϕ=0.006 kN/cm^2).



a *b*

Fig. 2. Normal stresses in the coordinates of cylinder 1 in plane z=0: a – on straight line x=0 between cylinder 1 and the boundary of a half-space; b – along section O_1O_2 between cylinders

Variant 2

At the boundary of a half-space, the displacement is assigned

$$U_x^{(d)} = U_y^{(d)} = U_z^{(d)} = 0.$$

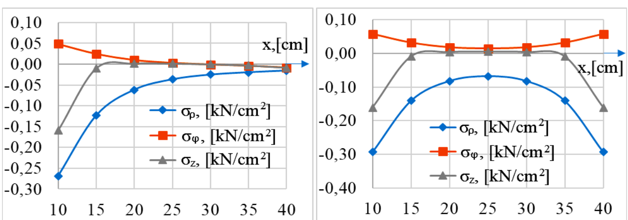
At the boundaries of cylinders 1 and 2, radial displacement

$$U_0^{(1)}(\phi_1, z) = U_0^{(2)}(\varphi_2, z) = 10^{-4} \cdot \left(z^2 + 10^2\right)^2$$

and tangential stresses are assigned

$$\tau_{\rho\phi}^{(1)} = \tau_{\rho z}^{(1)} = \tau_{\rho\phi}^{(2)} = \tau_{\rho z}^{(2)} = 0.$$

The diagram of normal stresses in Fig. 3 shows how stresses changed at loading cylinder 1 and cylinder 2.



a *b*

Fig. 3. Normal stresses in coordinates of cylinder 1 in plane z=0: *a* — along the straight line x=0 between cylinder 1 and the boundary of the half-space; *b* — along section O_1O_2 between cylinders

Thus, in comparison with variant 1 (Fig. 2), on the isthmus between cylinder 1 and the boundary of a half-space (Fig. 3, *a*), the stress almost did not change (σ_ρ= =–0.269 kN cm^2, σ_ϕ=0.048 kN/cm^2, σ_z=–0.160 kN/cm^2 on the cylinder and σ_ρ=–0.016 kN/cm^2, σ_ϕ=–0.008 kN/cm^2, σ_z=–0.008 kN cm^2 on the half-space), however, on the isthmus between cylinders 1 and 2 (Fig. 3, *b*), stresses have symmetric location relative to $\ell_{12}/2$, at the surface of the cylinders, extreme values (σ_ρ=–0.293 kN/cm^2, σ_ϕ=0.058 kN/cm^2, σ_z=–0.161 kN cm^2), between cylinders stresses decrease (σ_ρ=–0.068 kN/cm^2, σ_ϕ=0.014 kN/cm^2, σ_z=0.005 kN/cm^2).

Variant 3

Displacement $U_x^{(d)} = U_z^{(d)} = 0$ is assigned at the boundary of a half-space

$$U_y^{(d)} = -\left(10^{-4} \cdot \left(z^2 + 10^2\right)^2\right) \cdot \left(10^{-4} \cdot \left(\left(x - \ell_{12}/2\right)^2 + 10^2\right)^2\right),$$

and graphically shown in Fig. 4. At the boundary of cylinders 1 and 2, radial displacement

$$U_0^{(1)}(\phi_1, z) = U_0^{(2)}(\phi_2, z) = 0$$

and tangential stresses are assigned

$$\tau_{\rho\phi}^{(1)} = \tau_{\rho z}^{(1)} = \tau_{\rho\phi}^{(2)} = \tau_{\rho z}^{(2)} = 0.$$

In this variant, the boundary of the half-space with the maximum values in the middle between the cylinders is "loaded", which influenced the stressed state on the isthmuses (Fig. 5).

Thus, on the line between cylinder 1 and the half-space boundary (Fig. 5, *a*) σ_ρ=–0.005 kN/cm^2 has an extreme value at the border of cylinder 1, at the same time σ_ϕ and σ_z have the extreme values between the cylinder and the half-space boundary (σ_ϕ=–0.003 kN/cm^2, σz=0.002 kN/cm^2). On the isthmus between the cylinders (Fig. 5*b*), due to the vertical pressure of the boundary points of the half-space, stress σ_ϕ, which is directed perpendicularly to the half-space boundary, has extreme compressing values contrary to the maximum assigned displacements of the half-space (σ_ϕ=–0.004 kN/cm^2). Stress σ_ρ=–0.001 kN/cm^2 also has a small increase in compressing forces on the surface of the cylinders.

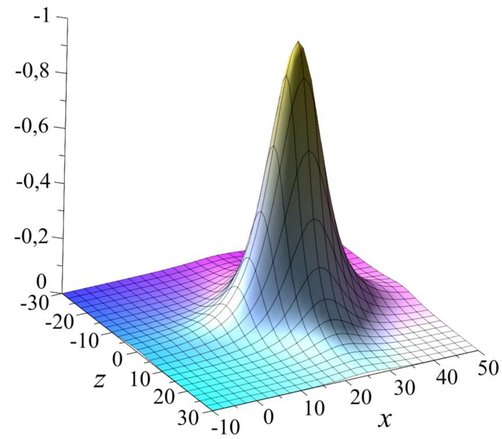
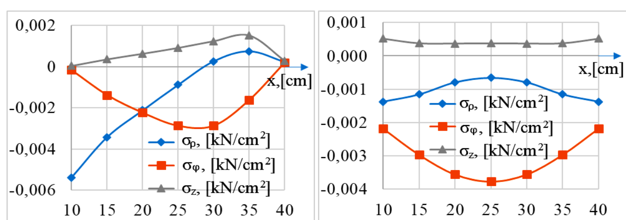


Fig. 4. Function of normal displacement which is set on the surface of the half-space



a *b*

Fig. 5. Normal stresses in the coordinates of cylinders 1 in plane z=0: *a* — on straight line x=0 between cylinder 1 and the half-space boundary; *b* — on section O_1O_2 between cylinders

Variant 4

Stress is assigned at the half-space boundary

$$\sigma_y^{(d)} = \tau_{yx}^{(d)} = \tau_{yz}^{(d)} = 0.$$

Radial displaceme

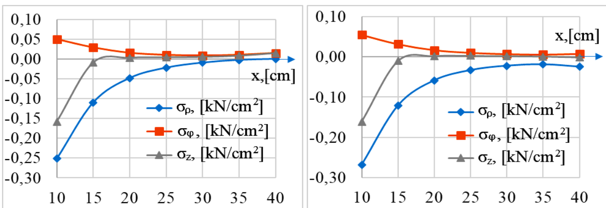
$$U_0^{(1)}(\phi_1, z) = 10^{-4} \cdot \left(z^2 + 10^2\right)^2$$

and tangential stresses $\tau_{\rho\phi}^{(1)} = \tau_{\rho z}^{(1)} = 0$, are assigned at the boundary of cylinder 1, at the boundary of cylinder 2 boundary conditions

$$U_\rho^{(2)}(\phi_2, z) = 0; \quad \tau_{\rho\phi}^{(2)} = \tau_{\rho z}^{(2)} = 0.$$

Compared to variant 1, in this case stress is assigned at the boundary of a half-space (diagram of the stressed state is shown in Fig. 6).

Fig. 6 shows that a change in boundary conditions at the half-space boundary did not affect the stressed state of the isthmus between the cylinders (Fig. 6, *b*), but affected the stressed state of the isthmus between cylinder 1 and half-space boundary (Fig. 6, *a*). Thus, at the boundary half-space σ_ρ=0, σ_ϕ and σ_z have now not compressing but stretching values (σ_ϕ=0.015 kN/cm^2, σz=0.015 kN/cm^2).



a *b*

Fig. 6. Normal stresses in coordinates of cylinder 1 in plane *z*=0: *a* – on straight line *x*=0 between cylinder 1 and half-space boundary; *b* – on section O_1O_2 between cylinders

Variant 5

Stress is assigned at the boundary of the half-space

$$\sigma_y^{(d)} = \tau_{yx}^{(d)} = \tau_{yz}^{(d)} = 0.$$

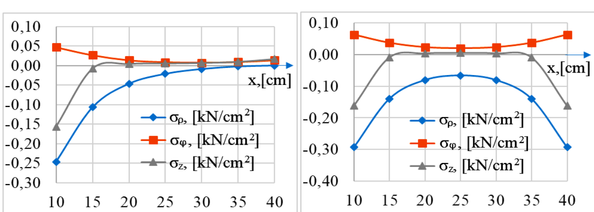
At the boundary of cylinders 1 and 2, tangential stresses

$$\tau_{\rho\phi}^{(1)} = \tau_{\rho z}^{(1)} = \tau_{\rho\phi}^{(2)} = \tau_{\rho z}^{(2)} = 0$$

and radial displacement are assigned

$$U_0^{(1)}(\phi_1, z) = U_0^{(2)}(\phi_2, z) = 10^{-4} \cdot \left(z^2 + 10^2\right)^2.$$

The stressed state on the isthmuses is shown in Fig. 7



a *b*

Fig. 7. Normal stresses in the coordinates of cylinder 1 in the plane *z*=0: *a* – along the straight line *x*=0 between cylinder 1 and half-space boundary; *b* – along section O_1O_2 between cylinders

Compared to variant 2, as in the previous case, changes affected the isthmus between cylinder 1 and half-space boundary (Fig. 7, *a*): at the half-space boundary, stresses σ_ρ=0, σ_ϕ and σ_z now have not compressing but stretching values (σ_ϕ=0.014 kN/cm^2, σ_z=0.017 kN/cm^2).

Variant 6

At the half-space boundary, stress $\tau_{yx}^{(d)} = \tau_{yz}^{(d)} = 0$ is assigned.

$$\sigma_y^{(d)} = -\left(10^{-4} \cdot \left(z^2 + 10^2\right)^2\right) \cdot \left(10^{-4} \cdot \left(\left(x - \ell_{qp}/2\right)^2 + 10^2\right)^2\right).$$

At the boundaries of cylinders 1 and 2, radial displacement

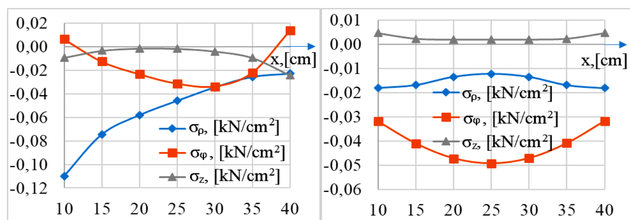
$$U_0^{(1)}(\phi_1, z) = U_0^{(2)}(\phi_2, z) = 0$$

and tangential stresses are assigned

$$\tau_{\rho\phi}^{(1)} = \tau_{\rho z}^{(1)} = \tau_{\rho\phi}^{(2)} = \tau_{\rho z}^{(2)} = 0.$$

In contrast to variant 3, at the half-space boundary, stresses are applied instead of displacements, which influences the stressed state of the isthmuses (Fig. 8).

Load in the form of a single stress, unlike the load in the form of a single displacement, have a greater impact on the stressed state. Thus, the stress on the isthmus between the cylinders (Fig. 8, *b*) has the same form as in Fig. 5, *b*, but is larger in values, for example, the stress between the cylinders in variant 6 σ_ρ=–0.012 kN/cm^2, σ_ϕ=–0.049 kN cm^2, σ_z=0.002 kN/cm^2, in variant 3 σ_ρ=–0.001 kN/cm^2, σ_ϕ= =0.004 kN/cm^2, σ_z=0.0004 kN/cm^2



a *b*

Fig. 8. Normal stresses in the coordinates of cylinder 1 in plane *z*=0: *a* – along the straight line *x*=0 between cylinder 1 and the half-space boundary; *b* – along section O_1O_2 between the cylinders

On the isthmus between cylinder 1 and the half-space boundary, unlike variant 3 (Fig. 5, *a*), the values of stresses are higher, moreover, σ_z in this case is compressing and increases at the half-space boundary (σ_z=–0.024 kN/cm^2). Stress σ_ρ also, unlike variant 3 (Fig. 5, *a*), at the half-space boundary is compressing (σ_ρ=–0.023 kN/cm^2). Stress σ_ϕ, in comparison with variant 3 (the maximum value σ_ϕ= =–0.003 kN/cm^2), has higher values (the maximum value σ_ϕ=–0.034 kN/cm^2).

6. Discussion of the obtained results for the stressed state and the method for solving the problem

In the framework of the accepted linear model of the homogeneous isotropic medium and precise problem statements, the derived results (distribution of stress fields in a multi-link body) are explained by the response of an elastic body to:

1) existence of some flat and curved surfaces that limit the body;

2) selected system of boundary conditions on these surfaces.

For another system of boundary conditions, at other equal factors, the response of an elastic body will be different.

Based on the generalized Fourier method, the analytical-numerical algorithm of calculation of the spatial problem of the elasticity theory was developed. The algorithm implies the following boundary conditions: one of the two types of boundary conditions – displacement or stress – at the half-space boundary, the contact type conditions in the form of radial displacements and tangential stresses at the boundaries of several parallel cylindrical cavities. The developed algorithm makes it possible to calculate the problems with similar boundary conditions with predetermined accuracy (depending on parameter m) and can be used for the design of various structures.

In comparison with papers [1–11], the proposed algorithm makes it possible to obtain the accurate solution to a similar problem and, in comparison with [13–22], to consider the boundary conditions of the contact type, as well as the half-space boundary.

In terms of shortcomings, it should be noted that at a decrease in the distance between the boundaries of the body, the algorithm becomes less effective as it requires an increase in the order of system m and respectively an increase in time to compute the integrals of matrix elements. The algorithm does not make it possible to solve the problem when the boundaries of a body touch or intersect.

Subsequent development of research in this area is required to solve similar problems in a layer with cylindrical cavities, which in the absence of algorithms with accurate analytical methods was not explored at all but is found in calculation schemes when designing structures.

When considering a layer with cylindrical cavities, it will be necessary to take into consideration the lower boundary of the half-space. In this case, the system of equations is significantly complicated in analytical and numerical terms.

7. Conclusions

1. Numerical analysis of the stressed-strained state of a half-space and two cylindrical cavities shows that

– for different types of boundary conditions (stress or displacement), stresses σ_ϕ and σ_z at the boundary of application of such conditions change for the opposite, that is, from stretching to compressing or vice versa;

– at the conditions of the contact type assigned at the boundary of a cylindrical cavity, specifically, zero tangential stresses and normal displacement in the form of a wave of the height of 1 cm, normal stresses on the surface are equal to: σ_ρ=–0.27 kN/cm^2, σ_ϕ=0.05 kN/cm^2, σ_z=–0.16 kN/cm^2;

– boundary conditions at the half-space boundary in form of stresses have a greater impact on the stressed state than the boundary conditions in the form of displacements. These statements are true at the boundary conditions of the contact type, assigned at the boundaries of cylindrical cavities, if an assigned displacement function and assigned function of stresses are the same.

2. Numerical studies of the algebraic system for a half-space and two cylinders provide an opportunity to argue that its solution can be found with any degree of accuracy by the reduction method. This is proved by a high precision of satisfaction of the boundary conditions. For geometric parameters of the solved problem (R_1/h=0.25; $(R_1+R_1)/\ell_{12}=0{,}4$) at m=8, boundary conditions are satisfied with the accuracy of 10-3. At an increase in the order of system m, the calculation accuracy will be increased

3. The presented diagrams give a pattern of the distribution of stresses in the most interesting areas, such as on the isthmus between cylindrical cavities, and on the isthmus between the half-space boundary and the surface of the cylindrical cavity.

An analysis of the stressed state of the cylindrical cavity with the boundary conditions of the contact type revealed that at approaching its surface, stress σ_ρ increases regardless of the fact whether it is overloaded or not. Thus, at the "loaded" cylinder (Variant 1, Fig. 2, *b*), stress σ_ρ increases up to –0.026 kN/cm^2, at the "unloaded" cylinder in variant 6 (Fig. 8, *a*) σ_ρ increases up to –0.11 kN/cm^2

4. The reliability of the presented algorithm is proved by a high level of satisfaction of the boundary conditions and the resulting diagrams can be used in assessing the stressed state in the structures with similar conditions.

References

1. Karadeniz Z. H., Kumlutas D. A numerical study on the coefficients of thermal expansion of fiber reinforced composite materials // Composite Structures. 2007. Vol. 78, Issue 1. P. 1–10. doi: https://doi.org/10.1016/j.compstruct.2005.11.034
2. Two-dimensional and axisymmetric unit cell models in the analysis of composite materials / Noda N.-A., Nisitani H., Takase Y., Shukuwa Y.-A. // Composite Structures. 2005. Vol. 69, Issee 4. P. 429–435. doi: https://doi.org/10.1016/j.compstruct.2004.08.034
3. Podkovalikhina O. O. Osesymetrychni kraiovi zadachi dlia pruzhnoi bahatosharovoi osnovy z tsylindrychnym vkliuchenniam. Dnipropetr. nats. universytet im. O. Honchara, 2009. 164 p.
4. Popov G. Ya., Vaysfel'd N. D. Osesimmetrichnaya zadacha s cilindricheskim vklyucheniem pri uchete ee udel'nogo vesa // Prikladnaya mekhanika. 2014. Vol. 50, Issue 6. P. 27–38.
5. Zaletov S. V. Osesimmetrichnaya zadacha o deystvii normal'noy nagruzki na izotropnoe poluprostranstvo s uprugo zakreplennoy granicey: dis. ... kand. fiz.-mat. nauk. Taganrogskiy institut imeni A. P. Chekhova, 2016. 157 p.
6. Eskandari-Ghadi M., Mahmoodian M. Analytical Solution of Time-Harmonic Torsional Vibration of a Cylindrical Cavity in a Half-Space // World Academy of Science, Engineering and Technology International Journal of Mechanical and Mechatronics Engineering. 2012. Vol. 6, Issue 1. P. 184–189.
7. Semi-analytical Solution for the Dynamic Response of a Cylindrical Structure Embedded in a Homogeneous Half-Space / Zhao M., van Dalen K. N., Barbosa J. M., Metrikine A. V. // Environmental Vibrations and Transportation Geodynamics. 2017. P. 369–388. doi: https://doi.org/10.1007/978-981-10-4508-0_35
8. Coşkun İ., Engin H., Özmutlu A. Dynamic Stress and Displacement in an Elastic Half-Space with a Cylindrical Cavity // Shock and Vibration. 2011. Vol. 18, Issue 6. P. 827–838. doi: https://doi.org/10.1155/2011/904936
9. Dynamic analysis of a cylindrical cavity in inhomogeneous elastic half-space subjected to SH waves / Yang Z., Jiang G., Tang H., Sun B., Yang Y. // Mathematics and Mechanics of Solids. 2017. P. 108128651773952. doi: https://doi.org/10.1177/1081286517739520

10. Meleshko V. V., Tokovyy Y. V. Equilibrium of an elastic finite cylinder under axisymmetric discontinuous normal loadings // Journal of Engineering Mathematics. 2012. Vol. 78, Issue 1. P. 143–166. doi: https://doi.org/10.1007/s10665-011-9524-y
11. Khoroshun L. P. Mathematical models and method of the mechanics of stochastic composites // International Applied Mechanics. 2000. Vol. 36, Issue 10. P. 1284–1316. doi: https://doi.org/10.1023/a:1009482032355
12. Nikolaev A. G., Procenko V. S. Obobshchennyy metod Fur'e v prostranstvennyh zadachah teorii uprugosti. Kharkiv, 2011. 344 p.
13. Nikolaev A. G., Shcherbakova Yu. A. Obosnovanie metoda Fur'e v osesimmetrichnyh zadachah teorii uprugosti dlya transversal'no-izotropnyh tel, ogranichennyh poverhnost'yu paraboloida // Otkrytye informacionnye i komp'yuternye integrirovannye tekhnologii. 2010. Issue 48. P. 180–190.
14. Shcherbakova Yu. A., Shekhvatova E. M. Sravnitel'niy analiz NDS mnogosvyaznyh transversal'no-izotropnyh tel s razlichnymi uprugimi harakteristikami // Visnyk Zaporizkoho natsionalnoho universytetu. 2015. Issue 2. P. 253–261.
15. Nikolaev A. G., Shcherbakova Yu. A. Apparat i prilozheniya obobshchennogo metoda Fur'e dlya transversal'no- izotropnyh tel, ogranichennyh ploskost'yu i paraboloidom vrashcheniya // Mat. metody ta fiz.-mekh. polia. 2009. Vol. 52, Issue 3. P. 160–169.
16. Nikolaev A. G., Shcherbakova Yu. A., Yuhno A. I. Deystvie sosredotochennoy sily na transversal'no-izotropnoe poluprostranstvo s paraboloidal'nym vklyucheniem // Voprosy proektirovaniya i proizvodstva konstrukciy letatel'nyh apparatov. 2006. Issue 2 (45). P. 47–51.
17. Nikolaev A. G., Orlov E. M. Reshenie pervoy osesimmetrichnoy termouprugoy kraevoy zadachi dlya transversal'no-izotropnogo poluprostranstva so sferoidal'noy polost'yu // Problemy obchysliuvalnoi mekhaniky i mitsnosti konstruktsiy. 2012. Issue 20. P. 253–259.
18. Procenko V. S., Ukrainec N. A. Primenenie obobshchennogo metoda Fur'e k resheniyu pervoy osnovnoy zadachi teorii uprugosti v poluprostranstve s cilindricheskoy polost'yu // Visnyk Zaporizkoho natsionalnoho universytetu. 2015. Issue 2. P. 193–202.
19. Vyznachennia NDS v pruzhnomu pivprostori z tsylindrovymy porozhnynamy / Miroshnikov V. Yu., Medvedieva A. V., Voronchikhina S. O., Oleshkevych S. V. // Naukovyi visnyk budivnytstva. 2012. Issue 68. P. 156–162.
20. Miroshnikov V. Yu. Druha osnovna zadacha teoriyi pruzhnosti u pivprostori z dekilkoma paralelnymy kruhovymy tsylindrychnymy porozhnynamy // Otkrytye informacionnye i komp'yuternye integrirovannye tekhnologii. 2018. Issue 79. P. 88–99.
21. Miroshnikov V. Yu. Tretia osnovna zadacha teoriyi pruzhnosti v prostori z N paralelnymy kruhovymy tsylindrychnymy porozhnynamy // Voprosy proektirovaniya i proizvodstva konstrukciy letatel'nyh apparatov. 2017. Issue 2 (90). P. 89–103.
22. Miroshnikov V. Yu. On computation of the stress-strain state of a space weakened by a system of parallel circular cylindrical cavities with different edge conditions // 4th International Conference «Science and practice: a new level of integration in the modern world». Scope academic house. Sheffield, UK, 2017. P. 77–83.