Запропоновано аналітичний метод розв'язання загального диференціального рівняння Матьє в канонічній формі. Метод ґрунтується на відповідному точному розв'язку, який знайдено для довільних числових параметрів вихідного рівняння a і q. В свою чергу, точний розв'язок виражено через фундаментальні функції, які представляються рядами по степенях параметрів a і q зі змінними коефіцієнтами.

Наряду з рівнянням Матьє, розглядається також рівносильна йому система диференціальних рівнянь. Показано, що матриця Вронського, яка утворена із фундаментальних функцій рівняння, являє собою матрицант системи. Тим самим доведено, що фундаментальні функції рівняння Матьє задовольняють наперед заданим умовам у нульовій точці.

З метою розв'язання проблеми чисельної реалізації знайдених точних формул, фундаментальні функції подано степеневими рядами. Для обчислення коефіцієнтів степеневих рядів виведені відповідні рекурентні співвідношення.

У результаті досліджень отримано остаточні аналітичні формули для обчислення характеристичного показника v, визначення якого є центральною частиною будь-якої задачі, математичною моделлю якої є рівняння Матьє. Фактично встановлено пряму аналітичну залежність v від вихідних параметрів рівняння a, q. Це особливо важливо, оскільки параметр v відіграє роль індикатора таких властивостей розв'язків рівняння Матьє, як обмеженість і періодичність.

Запропонований аналітичний метод являється реальною альтернативою застосуванню наближених методів при розв'язанні будь-яких задач, що зводяться до рівняння Матьє. Наявність остаточних аналітичних формул дозволятиме у подальшому уникати процедури пошуку розв'язків рівняння. Натомість, для розв'язання задачі у кожному конкретному випадку, достатньо лише чисельно реалізувати отримані аналітичні формули

Ключові слова: загальне рівняння Матьє, аналітичний метод, фундаментальні функції, характеристичний показник

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DEVELOPMENT OF THE ANALYTICAL METHOD OF THE GENERAL MATHIEU EQUATION SOLUTION

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1. Introduction

The Mathieu equation is a special case of a second – order linear ordinary differential equation with periodic coefficients. In the initial form, it can be written as [1]:

$$\frac{d^2y}{dz^2}+(a-2q\cos 2z)y=0, \qquad (1)$$

where a, q are constant real parameters, $-\infty<z<\infty$.

Depending on the nature of the original problem, the parameters a, q are determined in different ways. In the majority of physical problems that lead to the equation (1), the value of the parameter q is specified, and the values of the parameter a are found as eigenvalues, which guarantee periodic solutions. Periodic solutions are expressed through special Mathieu functions [2]. When the parameters a and q are given in advance independently of each other, the equation (1) is called the general Mathieu equation [2]. Such an equation arises, for example, in some problems of astronomy [3–5], in the study of a linear parametric oscillator [6], etc. It is the general Mathieu equation that is the object of research.

In addition to the initial equation (1), the modified equation [2] is also used in applications:

$$\frac{d^2y}{dz^2}-(a-2q\operatorname{ch} 2z)y=0. \qquad (2)$$

The equations (1), (2) have numerous applications in physics and engineering [7–11]. Moreover, a set of problems reduced to these equations can be divided into two main categories [11]: boundary value problems and initial value problems.

Boundary value problems arise when solving a two-dimensional wave equation written in elliptic coordinates. The following problems can serve as examples [7]:

– on vibrations of an elliptic membrane;
– on the free oscillation of water in a lake of elliptic boundary;
– on transverse vibrations of gas in a hollow elliptic cylinder;
– on vibrations of an elliptic plate;
– on an elliptic cylinder in a viscous fluid;
– on the electrical and thermal diffusion;
– on the diffraction of sound and electromagnetic waves, etc.

In initial value problems, there is only the equation (1). These include the problems [7]:

– on vibrations in a spring mechanism with a periodic driving force;

– on the stability of the rod (tension string) when applying a periodic component to a constant axial tensile force;

– on the frequency modulation of a sound signal;

– on rotation of an inverted pendulum with a periodically oscillating suspension point [8, 10];

– on ship stability in waves [8, 11];

– on motion of elementary particles in a cyclotron with a periodically changing magnetic field [8, 12];

– on motion stability and resonance phenomena in quadrupole traps and filters used in mass spectrometers [10, 12–23].

The last list also includes some problems of quantum mechanics [11, 24].

According to the Floquet's theorem [25], the equation (1) has a partial solution:

$$y(z)=e^{ivz}\varphi(z), \tag{3}$$

where $\varphi(z)$ is the periodic function with the period π, i is the imaginary unit, and v is the characteristic exponent, which depends on the parameters a and q. For this reason, the representation (3) is used, as a rule [7, 26, 27], to find fundamental solutions of the Mathieu equation.

In addition to finding fundamental solutions, the central problem is the determination of the characteristic exponent v, which depends on the parameters a, q and plays the role of an indicator of such basic properties of solutions of the equation (1) as boundedness and periodicity [7, 8, 27]. Two basic approaches are used to find v.

The first approach is actually based on the formulas [3, 27]:

$$\sin^2\left(\frac{\nu\pi}{2}\right)=\Delta(0)\sin^2\left(\frac{\pi\sqrt{a}}{2}\right),\quad a\neq(2r)^2; \tag{4}$$

$$\cos\pi\nu=2\cdot\Delta(1)-1,\quad a=(2r)^2. \tag{5}$$

Here r is the integer, and $\Delta(v)$ is the Hill determinant [7] of infinite order, which for the equation (1) has the form of [3, 7, 27]:

$$\Delta(\nu)=\begin{vmatrix} . & . & . & . & . & . & . & . & . \\ . & \xi_{-4} & 1 & \xi_{-4} & 0 & 0 & 0 & 0 & . \\ . & 0 & \xi_{-2} & 1 & \xi_{-2} & 0 & 0 & 0 & . \\ . & 0 & 0 & \xi_0 & 1 & \xi_0 & 0 & 0 & . \\ . & 0 & 0 & 0 & \xi_2 & 1 & \xi_2 & 0 & . \\ . & 0 & 0 & 0 & 0 & \xi_4 & 1 & \xi_4 & . \\ . & . & . & . & . & . & . & . & . \end{vmatrix}, \tag{6}$$

where

$$\xi_{2r}=\frac{q}{(\nu+2r)^2-a}\quad(r=...-2,-1,0,1,2,...).$$

In the implementation of formulas (4), (5), the determinant (6) is taken in a finite truncated form, depending on the given accuracy of computations [27].

The second approach [2, 5, 7, 26, 28] consists in solving one of the equations:

$$\cos\pi\nu=y_1(\pi)=y_2'(\pi), \tag{7}$$

$$\cos\pi\nu=1+2y_1'\left(\frac{\pi}{2}\right)y_2\left(\frac{\pi}{2}\right), \tag{8}$$

where $y_1(z)$, $y_2(z)$ are the fundamental functions of the equation (1) satisfying the following conditions:

$$\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

Herewith, $y_1(z)$ is an even function, and $y_2(z)$ is an odd function of z. So in this case, the problem of finding the characteristic exponent v is actually reduced to the determination of fundamental solutions possessing these properties.

In general, it should be noted that a large number of both theoretical and practical problems are reduced to the Mathieu equation. That is why the study of this equation is an urgent scientific and practical problem.

2. Literature review and problem statement

The Mathieu equation has for a long time been an object of systematic research. Recently, the number of publications related to this equation has increased dramatically. In particular, this is due to the successful development of the direction in mass spectrometry associated with quadrupole traps and filters. The point is that the motion of ions in these traps under a superposition of electric fields is described precisely by the equation (1). Herewith, a special case when $a=0$ turned out to be very relevant in these applications.

When $a=0$, there is no constant component in the combination of supply voltages and ions in a trap are under the influence of only a radio-frequency electric field varying according to the harmonic law. On the parameter plane (a, q), the values $a=0$, $0<q<q_{\max}\approx 0.92$ set the lower bound of the first stability zone of ion oscillations.

In [8], two physical processes corresponding to the value $a=0$ are considered, namely: focusing and acceleration of ion motion in a quadrupole radio–frequency electric field; focusing of charged particles in a cyclotron by means of an azimuthally variable magnetic field. In [13], for the values $a=0$, $q=0.36$, parametric resonance excitation of ion oscillations by the biharmonic power supply of the quadrupole filter is studied. In [14], the case of $a=0$ allows considering the equations of ion motion in only one of the coordinates. In [15], for $a=0$, $q=0.2\div 0.85$, the quadrupole field acceptance as a function of the initial phase is studied. The paper [16] is devoted to estimation of the potential energy of an ion in the quadrupole field for $a=0$ and small values of q. With the same values of the parameters, in [17] the time-of-flight mass-separation of ions in two-dimensional radio-frequency fields is considered.

In [28], for the Mathieu equation, a whole set of issues is considered. Including: finding eigenvalues, stability of solutions of differential equations, calculation of eigenvalues, calculation of the characteristic exponent, calculation of the values of the equation solutions for large values of the argument. Computing algorithms are proposed for the numerical solution of these problems.

The authors of [29] propose a combined approach for constructing solutions to the Mathieu equation, combining the Floquet theorem and the describing function method. An approximate solution of the equation is constructed by replacing the infinite Fourier series for the periodic factor in the Floquet representation with the truncated finite sum of the terms of the series. Then this truncated solution is applied to the Mathieu equation with damping. In [30], stable solutions of the homogeneous Mathieu equation in the first stability zone are considered as oscillations modulated in amplitude and frequency. The authors have made an attempt, using computing experiments, to establish the relationship between the nature of the oscillations and the ratio of the equation parameter values. The solutions of the Cauchy problem for the Mathieu equation with various combinations of parameter values are obtained by numerical integration. The publication [31] is devoted to various generalizations and extensions of the Mathieu equation. These extensions include: effects of linear viscous damping, geometric nonlinearity, time lag effect, quasiperiodic excitation or elliptic excitation. The aim is to provide a systematic review of the methods for determining the appropriate stability diagram, its structure and features. In [32], the first integrals of various extensions of the Mathieu equation with damping were obtained. In [33], the generalized Mathieu equation, which describes the behavior of a parametrically excited pendulum system in the two-frequency combined excitation is studied. The approximate boundaries of the regions of stability and instability of the equation solutions are constructed. Cases of periodic and quasiperiodic effects on the system are considered. An analysis of resonant phenomena in the system with the commensurability and incommensurability of the frequencies of the effects is carried out. The main method of research in this paper is the perturbation method. An approximate solution of the generalized Mathieu equation is sought as a uniformly convergent series with respect to a small parameter.

Also, it is necessary to mention a so-called «matrix method» [19–22] of solving the equation (1), which is widely used to find the equations of ion motion in quadrupole traps. This method is also numerical and is suitable for finding only periodic solutions (the case of $v=0$), since the periodicity is the basis of the method algorithm.

As for the general Mathieu equation, the authors know only one universal numerical solution method for any given parameter values, which was presented in [27]. The method is based on the representation of fundamental solutions in the form of (3), Fourier transform and finding the values of the characteristic exponent by the formulas (4)–(6). The proximity of the method is dictated by the need to replace the infinite determinant with a truncated finite option. Other researchers, as a rule, operate within the framework of the traditional approach devoted to finding periodic solutions of equation (1) and its generalizations.

The methods used for research in the above publications have one common property. For each new set of the values a, q, a solution procedure should be repeated. It is quite clear that in this case it is not possible to obtain analytically the dependence of the exponent v on the parameters a, q. It would be possible to determine such a dependence with an analytical method of solution.

There are also other advantages of the analytical method. In particular, as shown in [15, 16], even in the case of periodic solutions, when studying the fundamental properties of physical processes and phenomena, the analytical approach can be more versatile and flexible in comparison with numerical methods. Moreover, solutions can be more complex and can be expressed, for example, through quasiperiodic functions [21].

Therefore, it is promising to develop an analytical method for solving the equation (1), suitable for any values of the parameters a and q. Such method is proposed in the paper. In particular, explicit analytical formulas for the exponent v are derived as functions of the parameters a and q. The Floquet's theorem is not used in this case.

3. The aim and objectives of the study

The aim of the paper is to develop an analytical method for solving the general Mathieu equation.

To achieve the aim, the following objectives were formulated:

– to find the fundamental functions of the equation that satisfy the given conditions;

– to obtain the exact solution of the equation;

– to represent the fundamental functions by power series;

– to obtain analytically the formulas for the characteristic exponent as functions of the initial parameters of the equation.

4. Exact solution of the general Mathieu equation

Together with the equation (1), we consider the equivalent system of differential equations:

$$\frac{d\Phi(z)}{dz}=P(z)\Phi(z). \tag{10}$$

Here the vector of unknowns and the matrix of the coefficients of the system have the form of:

$$\Phi(z)=\begin{pmatrix} y(z) \\ y'(z)\end{pmatrix};$$

$$P(z)=\begin{pmatrix} 0 & 1 \\ -a+p\cos 2z & 0\end{pmatrix},$$

where $p=2q$ is indicated.

The fundamental solutions $y_n(z)$ $(n=1,2)$ of the equation (1) will be sought as series in powers $a^m p^{k-m}$ $(k=0,1,2,...)$ $(m=0,1,...,k)$ with variable coefficients:

$$y_n(z)=\sum_{k=0}^{\infty}\sum_{m=0}^{k} a^m p^{k-m}\beta_{n,m,k-m}(z), \tag{11}$$

where $\beta_{n,m,k-m}(z)$ $(n=1,2)$ $(k=0,1,2,...)$ $(m=0,1,...,k)$ are unknown functions that are assumed to be continuous together with their first and second derivatives. So far we assume that the series (11), as well as similar series composed of the first and second derivatives:

$$y'_n(z)=\sum_{k=0}^{\infty}\sum_{m=0}^{k} a^m p^{k-m}\beta'_{n,m,k-m}(z), \tag{12}$$

$$y''_n(z)=\sum_{k=0}^{\infty}\sum_{m=0}^{k} a^m p^{k-m}\beta''_{n,m,k-m}(z), \tag{13}$$

uniformly converge. In this case, an operation of termwise differentiation of the series will be possible, as a result of which the notation $y'_n(z)$, $y''_n(z)$ for the sums (12), (13) will be true.

We find the unknown functions $\beta_{n,m,k-m}(z)$ from the condition:

$$\frac{d^2 y_n}{dz^2}+(a-p\cos 2z)\,y_n=0\quad (n=1,2). \tag{14}$$

Using the representations (11), (13), after the transformations, we come to the need to fulfill the equality:

$$\beta''_{n,0,0}(z)+\sum_{k=1}^{\infty}a^k(\beta''_{n,k,0}(z)+\beta_{n,k-1,0}(z))+$$
$$+\sum_{k=1}^{\infty}p^k(\beta''_{n,0,k}(z)-\cos 2z\beta_{n,0,k-1}(z))+$$
$$+\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}a^m p^{k-m}\begin{pmatrix}\beta''_{n,m,k-m}(z)+\beta_{n,m-1,k-m}(z)-\\ -\cos 2z\beta_{n,m,k-m-1}(z)\end{pmatrix}=0.$$

To satisfy it, we equate all the coefficients with the powers $a^m p^{k-m}$ $(k=0,1,2,...)$ $(m=0,1,...,k)$ to zero:

$$\beta''_{n,0,0}(z)=0\quad (n=1,2); \tag{15}$$

$$\beta''_{n,k,0}(z)+\beta_{n,k-1,0}(z)=0\quad (k=1,2,3,...); \tag{16}$$

$$\beta''_{n,0,k}(z)-\cos 2z\beta_{n,0,k-1}(z)=0\quad (k=1,2,3,...); \tag{17}$$

$$\beta''_{n,m,k-m}(z)+\beta_{n,m-1,k-m}(z)-\cos 2z\beta_{n,m,k-m-1}(z)=0$$
$$(k=2,3,4,...)(m=1,2,...,k-1). \tag{18}$$

To the obtained differential equations (15)–(18), we attach the corresponding conditions:

$$\begin{pmatrix}\beta_{1,0,0}(0) & \beta_{2,0,0}(0)\\ \beta'_{1,0,0}(0) & \beta'_{2,0,0}(0)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}; \tag{19}$$

$$\beta_{n,k,0}(0)=\beta'_{n,k,0}(0)=0\quad (k=1,2,3,...); \tag{20}$$

$$\beta_{n,0,k}(0)=\beta'_{n,0,k}(0)=0\quad (k=1,2,3,...); \tag{21}$$

$$\beta_{n,m,k-m}(0)=\beta'_{n,m,k-m}(0)=0$$
$$(k=2,3,4,...)(m=1,2,...,k-1). \tag{22}$$

Then, integrating each of the equations (15)–(18) twice and realizing the specified conditions, we have:

$$\beta_{n,0,0}(z)=z^{n-1}\,(n=1,2); \tag{23}$$

$$\beta_{n,k,0}(z)=-\int_0^z\int_0^z\beta_{n,k-1,0}(z)\mathrm{d}z\mathrm{d}z\quad (k=1,2,3,...); \tag{24}$$

$$\beta_{n,0,k}(z)=\int_0^z\int_0^z\cos 2z\beta_{n,0,k-1}(z)\mathrm{d}z\mathrm{d}z\quad (k=1,2,3,...); \tag{25}$$

$$\beta_{n,m,k-m}(z)=-\int_0^z\int_0^z(\beta_{n,m-1,k-m}(z)-\cos 2z\beta_{n,m,k-m-1}(z))\mathrm{d}z\mathrm{d}z$$
$$(k=2,3,4,...)(m=1,2,...,k-1). \tag{26}$$

The formulas (23)–(26) are recurrent. By means of these formulas, the functions $\beta_{n,k,0}(z)$, $\beta_{n,0,k}(z)$, $\beta_{n,m,k-m}(z)$, which we call generating are successively determined from the known initial function $\beta_{n,0,0}(z)$. For such functions, the equality (14) will be satisfied identically by construction.

Now it is necessary to prove that the series (11)–(13) do converge uniformly.

The proofs of the series (11) are based on the following estimates, which follow directly from the formulas (23)–(26):

$$|\beta_{n,0,0}(z)|=|z|^{n-1}\quad (n=1,2); \tag{27}$$

$$|\beta_{n,k,0}(z)|\le\int_0^z\int_0^z|\beta_{n,k-1,0}(z)|\mathrm{d}z\mathrm{d}z\quad (k=1,2,3,...); \tag{28}$$

$$|\beta_{n,0,k}(z)|\le\int_0^z\int_0^z|\beta_{n,0,k-1}(z)|\,\mathrm{d}z\mathrm{d}z\quad (k=1,2,3,...); \tag{29}$$

$$|\beta_{n,m,k-m}(z)|\le\int_0^z\int_0^z\left(|\beta_{n,m-1,k-m}(z)|+|\beta_{n,m,k-m-1}(z)|\right)\mathrm{d}z\mathrm{d}z$$
$$(k=2,3,4,...)(m=1,2,...,k-1). \tag{30}$$

Carrying out the operations prescribed by the recurrent formulas (27)–(30) successively for the values $k=0,1,2,...;$ $m=0,1,...,k$, in general, we come to such a result:

$$|\beta_{n,m,k-m}(z)|\le C_k^m\frac{|z|^{n+2k-1}}{(n+2k-1)!}, \tag{31}$$

where C_k^m is the number of combinations of k by m.

Considering (31), for the series (11) we have:

$$|y_n(z)|\le\sum_{k=0}^{\infty}\sum_{m=0}^{k}|a|^m|p|^{k-m}|\beta_{n,m,k-m}(z)|\le$$
$$\le\sum_{k=0}^{\infty}\frac{|z|^{n+2k-1}}{(n+2k-1)!}\sum_{m=0}^{k}C_k^m|a|^m|p|^{k-m}=|z|^{n-1}\sum_{k=0}^{\infty}\frac{(|a|+|p|)^k}{(n+2k-1)!}|z|^{2k}.$$

Applying the d'Alembert principle to calculate the convergence radius of the last power series, we obtain:

$$R=\frac{1}{|a|+|p|}\lim_{k\to\infty}(n+2k)(n+2k+1)=\infty.$$

This proves that the series (11) converge uniformly.

In much the same way, the uniform convergence of the series (12) is also proved. As regards the series (13), their uniform convergence follows immediately from the identity (14), according to which they are proportional to the series (11).

Thus, two solutions $y_n(z)$ $(n=1,2)$ of the equation (1) are determined by the formulas (11), (23)–(26). In this case, it is easy to verify that the Wronskian matrix:

$$\Lambda(z)=\begin{pmatrix}y_1(z) & y_2(z)\\ y'_1(z) & y'_2(z)\end{pmatrix}$$

satisfies the system (10). In addition, taking into account (19)–(22), we find:

$$\Lambda(0)=\begin{pmatrix}y_1(0) & y_2(0)\\ y'_1(0) & y'_2(0)\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{32}$$

Consequently, $\Lambda(z)$ is the transition matrix [34] of the system (10), and the functions $y_n(z)$ $(n=1,2)$ are the fundamental solutions of the equation (1) satisfying the conditions (9).

The general solution of the system (10) is expressed by the formula:

$$\Phi(z)=\Lambda(z)\Phi(0). \tag{33}$$

From this, we obtain the general solution of the equation (1) in the form of:

$$y(z)=y(0)y_1(z)+y'(0)y_2(z),$$

where the integration constants are expressed through the initial parameters $y(0)$, $y'(0)$.

5. Identity for the fundamental functions of the Mathieu equation

Using the Jacobi formula [34]:

$$|\Lambda(z)|=|\Lambda(0)|\exp\left(\int_0^z Sp\,P(z)\,\mathrm{d}z\right),$$

where $Sp\,P(z)$ is the trace of the matrix $P(z)$, which in our case is zero, for the fundamental functions we obtain the identity:

$$y_1(z)y_2'(z)-y_1'(z)y_2(z)=1. \tag{34}$$

This identity can be very useful in solving specific problems.

6. Representation of fundamental functions by power series

From the point of view of applications, it is expedient to transform the formulas for fundamental functions to a form convenient for their numerical implementation.

As for the generating functions $\beta_{n,k,0}(z)$, they are found by the formula (24) in an explicit form:

$$\beta_{n,k,0}(z)=\frac{(-1)^k}{(n+2k-1)!}\,z^{n+2k-1}\quad (k=1,2,3,\ldots). \tag{35}$$

In the formulas for the generating functions (25), (26), we use the expansion:

$$\cos 2z=\sum_{j=0}^{\infty} c_j\,z^{2j},\quad c_j=(-1)^j\frac{2^{2j}}{(2j)!}. \tag{36}$$

As can be seen from the formula (25), the generating functions $\beta_{n,0,k}(z)$ under the condition (36) are represented by power series, the smallest power of which is equal to $n+2k-1$. Consequently,

$$\beta_{n,0,k}(z)=z^{n+2k-1}\sum_{j=0}^{\infty} d_{n,0,k,j}\,z^{2j}\quad (k=1,2,3,\ldots), \tag{37}$$

where $d_{n,0,k,j}$ are the coefficients to be determined. Herewith,

$$\beta_{n,0,k-1}(z)=z^{n+2k-3}\sum_{j=0}^{\infty} d_{n,0,k-1,j}\,z^{2j}\quad (k=1,2,3,\ldots). \tag{38}$$

Assuming here $k=1$ and comparing the result with the expression (23), we find the initial values:

$$d_{n,0,0,0}=1;\ d_{n,0,0,j}=0\ \ (j=1,2,3,\ldots). \tag{39}$$

Performing the operations prescribed by the formula (25), we multiply the series (36), (38) and integrate the result twice. Thus, we will have:

$$\beta_{n,0,k}(z)=z^{n+2k-1}\sum_{j=0}^{\infty}\frac{e_{n,0,k-1,j}}{f_{n,k,j}}\,z^{2j}\quad (k=1,2,3,\ldots), \tag{40}$$

where

$$e_{n,0,k-1,j}=\sum_{i=0}^{j} c_{j-i}\,d_{n,0,k-1,i},$$

$$f_{n,k,j}=(n+2(k+j)-2)(n+2(k+j)-1).$$

By comparing the formulas (37) and (40), we obtain a recurrent formula for the required coefficients:

$$d_{n,0,k,j}=\frac{\sum_{i=0}^{j} c_{j-i}\,d_{n,0,k-1,i}}{f_{n,k,j}}\quad (k=1,2,3,\ldots)(j=0,1,2,\ldots). \tag{41}$$

Thus, the coefficients of the series (37) are completely determined by the formulas (39), (41).

Analyzing the formula (26), taking into account (35)–(37) we conclude that the generating functions $\beta_{n,m,k-m}(z)$ are also power series with the smallest power $n+2k-1$. Therefore, we can write:

$$\beta_{n,m,k-m}(z)=z^{n+2k-1}\sum_{j=0}^{\infty} d_{n,m,k-m,j}\,z^{2j}$$

$$(k=2,3,4,\ldots)(m=1,2,\ldots,k-1), \tag{42}$$

where $d_{n,m,k,j}$ are the required coefficients. Reducing the indices k and m by unity in the formula (42), we obtain:

$$\beta_{n,m-1,k-m}(z)=z^{n+2k-3}\sum_{j=0}^{\infty} d_{n,m-1,k-m,j}\,z^{2j}$$

$$(k=2,3,4,\ldots)(m=1,2,\ldots,k-1), \tag{43}$$

and reducing only the index k by unity, we have:

$$\beta_{n,m,k-m-1}(z)=z^{n+2k-3}\sum_{j=0}^{\infty} d_{n,m,k-m-1,j}\,z^{2j}$$

$$(k=2,3,4,\ldots)(m=1,2,\ldots,k-1). \tag{44}$$

The value $k=2$ in the formula (43) corresponds to the initial values $d_{n,0,1,j}$, which are already determined by the formula (41). Assuming $k=2$ in the formula (44) and comparing the result with the formula (35), we determine the initial values $d_{n,1,0,j}$:

$$d_{n,1,0,0}=-\frac{1}{(n+1)!};\ \ d_{n,1,0,j}=0\quad (j=1,2,3,\ldots). \tag{45}$$

Substituting the values (43), (44) instead of $\beta_{n,m-1,k-m}(z)$, $\beta_{n,m,k-m-1}(z)$ in the formula (26), multiplying the series and integrating twice, we obtain:

$$\beta_{n,m,k-m}(z)=z^{n+2k-1}\sum_{j=0}^{\infty}\frac{e_{n,m,k-m-1,j}-d_{n,m-1,k-m,j}}{f_{n,k,j}}z^{2j}$$

$$(k=2,3,4,\ldots)(m=1,2,\ldots,k-1), \qquad (46)$$

where

$$e_{n,m,k-m-1,j}=\sum_{i=0}^{j}c_{j-i}d_{n,m,k-m-1,i}.$$

Comparing the formulas (42) and (46), we come to a recurrent formula for the required coefficients:

$$d_{n,m,k-m,j}=\frac{\sum_{i=0}^{j}c_{j-i}d_{n,m,k-m-1,i}-d_{n,m-1,k-m,j}}{f_{n,k,j}}$$

$$(k=2,3,4,\ldots)(m=1,2,\ldots,k-1)(j=0,1,2\ldots). \qquad (47)$$

Thus, the set of formulas (47), (41), (45) completely determines the coefficients of the series (42).

Transforming the formula (11) as:

$$y_n(z)=\beta_{n,0,0}(z)+\sum_{k=1}^{\infty}a^k\beta_{n,k,0}(z)+$$
$$+\sum_{k=1}^{\infty}p^k\beta_{n,0,k}(z)+\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}a^m p^{k-m}\beta_{n,m,k-m}(z)$$

and considering (23), (35), (37), (42), for fundamental functions we have:

$$y_n(z)=z^{n-1}\left(\begin{array}{l}1+\sum_{k=1}^{\infty}a^k\frac{(-1)^k}{(n+2k-1)!}z^{2k}+\\+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k d_{n,0,k,j}z^{2(k+j)}+\\+\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}\sum_{j=0}^{\infty}a^m p^{k-m}d_{n,m,k-m,j}z^{2(k+j)}\end{array}\right)\quad(n=1,2)\qquad(48)$$

or

$$y_1(z)=\cos\sqrt{a}z+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k d_{1,0,k,j}z^{2(k+j)}+$$
$$+\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}\sum_{j=0}^{\infty}a^m p^{k-m}d_{1,m,k-m,j}z^{2(k+j)}, \qquad (49)$$

$$y_2(z)=\frac{1}{\sqrt{a}}\sin\sqrt{a}z+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k d_{2,0,k,j}z^{2(k+j)+1}+$$
$$+\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}\sum_{j=0}^{\infty}a^m p^{k-m}d_{2,m,k-m,j}z^{2(k+j)+1}. \qquad (50)$$

It is obvious that $y_1(z)$ is an even solution, and $y_2(z)$ is odd.

To obtain fundamental solutions for the modified Mathieu equation (2), it is sufficient to replace z with iz in the formula (48) or in the formulas (49), (50).

7. Analytical formulas for the characteristic exponent

Having the fundamental solutions $y_n(z)$ $(n=1,2)$ in an explicit form, based on the formulas (7), (8), (49), (50), we can obtain three equivalent formulas for determining the characteristic exponent v. We write down here only two of them:

$$\cos\pi\nu=\cos\sqrt{a}\pi+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k d_{1,0,k,j}\pi^{2(k+j)}+$$
$$+\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}\sum_{j=0}^{\infty}a^m p^{k-m}d_{1,m,k-m,j}\pi^{2(k+j)}; \qquad (51)$$

$$\cos\pi\nu=1+$$
$$+2\left(\begin{array}{l}-\sqrt{a}\sin\frac{\sqrt{a}\pi}{2}+2\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k(k+j)d_{1,0,k,j}\left(\frac{\pi}{2}\right)^{2(k+j)-1}+\\+2\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}\sum_{j=0}^{\infty}a^m p^{k-m}(k+j)d_{1,m,k-m,j}\left(\frac{\pi}{2}\right)^{2(k+j)-1}\end{array}\right)\times$$
$$\times\left(\begin{array}{l}\frac{1}{\sqrt{a}}\sin\frac{\sqrt{a}\pi}{2}+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k d_{2,0,k,j}\left(\frac{\pi}{2}\right)^{2(k+j)+1}+\\+\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}\sum_{j=0}^{\infty}a^m p^{k-m}d_{2,m,k-m,j}\left(\frac{\pi}{2}\right)^{2(k+j)+1}\end{array}\right). \qquad (52)$$

It should be noted that in the scientific literature, along with the characteristic exponent v, the exponent $\mu=iv$ is also used. To calculate μ, the formulas (51), (52) can also be used considering the equality $\cos\pi v=\mathrm{ch}\,\pi\mu$.

The values of the characteristic exponent μ, calculated in the program mode using the formula (51) or (52), were compared with the previously known values [35]. For some sets of the parameters a, q, such comparisons are given in Table 1.

Table 1

Comparison of the values of μ

	$q=0.2$	$q=0.4$	$q=0.6$	$q=0.8$
Author's method				
$a=0.1$	0.3501	0.4431	0.5901	0.8660
$a=0.3$	0.5732	0.6549	0.8330	0.2816
Table values				
$a=0.1$	0.3502	0.4433	0.5904	0.8670
$a=0.3$	0.5739	0.6565	0.8375	0.2845
Relative error, %				
$a=0.1$	0.03	0.04	0.05	0.12
$a=0.3$	0.12	0.25	0.54	1.04

These results validate the formulas (51), (52). Moreover, the practice of calculations shows that for rather large values of the parameters a and q, it is more expedient to use the formula (52), since the corresponding numerical series converge faster there.

8. Formulas for a special case

In a special case when $a=0$, the formulas (48), (51), (52) are simplified:

$$y_n(z)=z^{n-1}\left(1+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k d_{n,0,k,j}z^{2(k+j)}\right)\quad (n=1,2); \qquad (53)$$

$$\cos\pi\nu=1+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k\, d_{1,0,k,j}\,\pi^{2(k+j)}; \qquad (54)$$

$$\cos\pi\nu=1+4\left(\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k(k+j)d_{1,0,k,j}\left(\frac{\pi}{2}\right)^{2(k+j)-1}\right)\times$$
$$\times\left(\frac{\pi}{2}+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}p^k\, d_{2,0,k,j}\left(\frac{\pi}{2}\right)^{2(k+j)+1}\right). \qquad (55)$$

As was shown above, such a case is common in practice. Therefore, for the convenience of applications, the formulas (53)–(55) are given separately.

9. Discussion of the results of research of the Mathieu equation

Whenever the research of a physical phenomenon is reduced to a differential equation, the key issue is the construction of its exact (analytical) solution. After all, it is the exact solution that carries information of a qualitative nature and forms the most complete picture of the studied physical phenomenon. However, researchers often face the known mathematical problem on this path, which is the lack of a universal method for finding exact solutions for differential equations with variable coefficients. Probably, this can explain the preferential use of approximate methods.

This also applies fully to the general Mathieu equation, the exact solution of which, for arbitrary values of the parameters a and q, was not known. This can explain the lack of an analytical research method in the scientific literature. Such a method is developed.

An analytical method opens up new promising research opportunities for a wide class of astronomy, physics and engineering problems reduced to the Mathieu equation. Moreover, the application of this method in the long term will allow solving certain problems only by the numerical implementation of the finite analytical formulas obtained here.

In general, the work is of a theoretical nature. Comparison of several values of the characteristic exponent, calculated by the author's method, with the previously known table values, although validating the formulas obtained, cannot be considered a full–fledged approbation of the method in practice. This lack of research can be overcome by demonstrating the capabilities of the method on several practical problems. This direction is seen by the authors as further development of the research.

10. Conclusions

1. An analytical method for solving the Mathieu equation, suitable for arbitrary values of the parameters a and q independent of each other is developed. The method is based on the exact solution of the equation.

2. The fundamental functions of the Mathieu equation are found. It is important that these functions satisfy the given conditions at the point $z=0$.

3. To solve the problem of numerical realization, the fundamental functions are represented by power series. The coefficients of these series are calculated by means of the derived recurrent relations.

4. To calculate the characteristic exponent v, finite analytic formulas are obtained. These formulas actually establish a direct functional dependence of v on the initial parameters of the equations a and q. This allows investigating the solutions of the Mathieu equation for boundedness and periodicity.

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