Подано спектральний розклад для несамоспряженої моделі Фрідріхса і наведено узагальнення відомої у самоспряженому випадку функції Вейля на несамоспряжений випадок. Встановлено, що для несамоспряженої моделі Фрідріхса довільний елемент простору можна подати як лінійну комбінацію власних елементів оператора, що відповідають точкам спектра. Побудовано спектральний розклад, тобто представлення довільного елемента простору через власні функції, що свідчить про повноту власних функцій. Це зроблено з врахуванням спектральних особливостей (тобто власних значень на неперервному спектрі) несамоспряженого оператора моделі Фрідріхса. Ця модель виступає важливим інструментом для знаходження розв'язку звичайних диференціальних рівнянь після застосування відповідного перетворення Фур'є.

Запропоновано загальний метод побудови спектрального розкладу (тобто не прив'язаний виключно до моделі Фрідріхса), який грунтується на понятті так званого розгалуження резольвенти і який можна використовувати для довільних несамоспряжених операторів, а також і для самоспряжених операторів.

Доведено, що за умов існування максимального оператора, резольвента допускає відокремлення розгалуження. Вказані достатні умови для існування функції Вейля $m(\zeta)$ для оператора несамоспряженої моделі Фрідріхса та отримано формули для її обчислення через резольвенту.

Показано, що функція Вейля $m(\zeta)$ для самоспряженого оператора співпадає з класичною функцією Вейля у випадку оператора Штурма-Ліувілля на півосі. Наведено два приклади, в яких знайдено узагальнену функцію Вейля $m(\zeta)$ для несамоспряженої моделі Фрідріхса

Ключові слова: оператор Штурма-Ліувілля, модель Фрідріхса, перетворення Фур'є, функція Вейля, неперервний спектр, розгалуження резольвенти, максимальний оператор

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CONSTRUCTION OF SPECTRAL DECOMPOSITION FOR NON-SELF-ADJOINT FRIEDRICHS MODEL OPERATOR

E. Cheremnikh
PhD, Professor*
E-mail: Ivasyk-G@ukr.net
H. Ivasyk
PhD, Senior Lecturer*
E-mail: Ivasyk-G@ukr.net
V. Alieksieiev
PhD, Associate Professor
Department of Applied Mathematics**
E-mail: vladyslav.i.alieksieiev@lpnu.ua
M. Kuchma
PhD, Associate Professor*
E-mail: markuchma@ukr.net
O. Brodyak
PhD, Associate Professor*
*Department of Mathematics**
**National University «Lviv Polytechnic»
S. Bandery str., 12, Lviv, Ukraine, 79013

1. Introduction

The spectral theory is one of the most important directions of the theory of linear operators. Modern trends in the development of the theory of non-self-adjoint operators dictate the need for the appropriate development of spectral decomposition issues.

The branching of the resolvent introduced in [1, 2] allows us to generalize the notion of Weyl function. The completeness of the system of eigenfunctions is important for the practical implementation of the method of separating the variables used in the theory of differential equations.

In the classical version, the Weyl function is determined by the connection between the various solutions of equations in different boundary conditions. It is possible to distinguish between these solutions of equations by the nature of their analyticity. It is known that the solution of differential equations after the application of the corresponding Fourier transform in many cases reduces to the analysis of a non-self-adjoint Friedrichs model, that is, the sum of the operator of multiplication by an independent variable and an operator perturbed by a bounded term [3]. If the continuous spectrum of the operator of the Friedrichs model operating in the Hilbert space contains the half-axis, then it is important to jump during passing through this half-axis.

In [4], we give formulas for calculating the jump of the resolvent, but do not give the representation of the resolvent through the jump.

Having examined and analyzed a number of scientific works, it can be argued that the problem part of the given topic from the spectral theory remains open to research. The logic of the theory of non-self-adjoint operators now challenges the need for appropriate research in constructing a spectral decomposition for the operator of a non-self-adjoint Friedrichs model. To do this, you can use the concept of branching resolvent.

2. Literature review and problem statement

In [5], the Friedrichs model plays an auxiliary role in the study of the family of some operator matrices. This can be applied to a variety of physical problems, using positively defined operators of the Friedrichs model without their spectral decomposition and Parseval's equality.

It was proved in [6] that the spectral features of the Sturm-Liouville operator on the direct axis generate some growing components in the asymptotic behavior over time of solutions of the corresponding evolution equations.

In the Friedrichs model for the Sturm-Liouville operator, we use the calculations of all these components and some scalar functions that characterize the breakpoints of the Fourier transforms of the initial elements of the evolution equations. A real example of an operator with a single spectral feature is given and an auxiliary function is described which describes the nature of the breakpoints. The asymptotic behavior of the solution of the corresponding evolution equation is considered and Parseval's equality is obtained using the Weyl function, but it is not presented in sufficient detail.

The classical definition of the Weyl function is considered in [7] with the help of the boundary triple for the Schrödinger operator, and in [8–10] Weyl theory was investigated for a self-directed Schrödinger operator. The establishment of conditions for the existence of the Weyl function and the resolvent structure was investigated without involving the operator spectral decomposition.

In [11], the direct and inverse problem for the Sturm-Liouville operator with discontinuous coefficients is investigated. Its spectral peculiarities are studied, orthogonality of their own functions and one-time of their own values are established. An asymptotic formula for the eigenvalues and eigenfunctions of the Sturm-Liouville operator is considered, a resolvent of the operator is constructed and a spectral decomposition is obtained. It is shown that the eigenfunctions form a complete system and the Weyl function is found. A uniqueness theorem for the solution of the inverse problem is proved. In this paper, these results are obtained for a finite gap $(0,\pi)$, that is, this refers to the space $L^2(0,\pi)$.

The problem of localization of spectral singularities of dissipative operators in terms of the asymptotics of the corresponding exponential function is considered in [12] and the solution of this problem for the spectral singularities of higher orders is presented. We study the Weyl function for the perturbed Laplacian in space $L^2\left(R^3\right)$, using a traditional classical approach.

In [4], we give conditions for the Friedrichs model, which allow us to write a formula for a jump of a resolvent on a continuous spectrum, but they are bulky and inconvenient for use, and also there is no direct connection with the resolvent.

In [13], the Weyl matrix function is considered using the so-called branching of a resolvent without considering the spectral decomposition of the operator. Therefore, from the viewpoint of scientific literature, the direction of research concerning the theory of non-self-adjoint operators is incompletely studied, requires new approaches to the consideration of the questions of constructing the spectral decomposition of a non-self-adjoint Friedrichs model and the generalization of the Weyl function for a non- non-self-adjoint case.

3. The aim and objectives of the study

The aim of the paper is to provide a spectral decomposition of a non-self-direct operator of the Friedrichs model and generalize the Weyl function in terms of the introduced concept of the branching of the resolvent [13, 1]. Upon reaching this goal, the construction of the spectral decomposition of a non-self-directed operator of the Friedrichs model can be used to solve problems of mathematical physics, which opens up additional possibilities for the case of self-adjoint operators.

To achieve the aim, the following objectives were set:

– to use the concept of the branching of the resolvent and the branching of the vector-function to generalize the Weyl function to a non-self-adjoint case;

– to find sufficient conditions for existence and formulas for calculating the Weyl function $m(\zeta)$;

– to show that the Weyl function for a self-directed operator coincides with the Weyl classical function of the Sturm-Liouville operator on the semicolon;

– to get the formula of the spectral decomposition of the operator of the Friedrichs model.

4. Preliminary notions: branching of the resolvent and Weyl function

Let $H=L^2_\rho(0,\infty)$, $\rho(\tau)>0$. Suppose that the interval $[0,\infty)$ coincides with the continuous spectrum of some operator:

$$T:H\to H,\quad \overline{D(T)}=H. \tag{1}$$

Denote $T_\zeta=(T-\zeta)^{-1}$. The bilinear form of the resolvent $(T_\zeta\phi,\psi)$ is an analytic function if $\zeta\notin[0,\infty)$. Assume that there exists such a linear space $\Phi\subset H$, $\bar{\Phi}=H$, that the form $\Phi\subset H$, $\bar{\Phi}=H$, admits an analytic continuation $\left(T_\zeta\phi,\psi\right)_\pm$ over the axis $(0,\infty)$. Suppose that $T:\Phi\to\Phi$, and the multiplicity m of the continuous spectrum of the operator T is equal to 1, that is, $m=1$.

4. 1. Branching of the resolvent and Weyl function

Denote by $h_\zeta\in H$ $\zeta\in\Omega\setminus[0,\infty)$ analytic by ζ element on space H,

$$\Omega=\left\{\zeta: dist\left(\zeta,[0,\infty)\right)<\varepsilon\right\},\quad \varepsilon>0.$$

Let $a(\zeta)$, $b(\zeta)$, $r(\zeta)$ be some functionals in H and $B(\zeta):\Phi\to\Phi$ – a certain operator.

Definition 1. *Branching of the resolvent* $T_\zeta:H\to H$ is called the representation of the operator-function T_ζ through the vector-function $h_\zeta\in H$, which has the same jump T_ζ, when passing through a continuous spectrum, and the coefficients h_ζ depend analytically on ζ and given by the spectral data of the operator T.

Definition 2. *Branching of the vector-function* $h_\zeta\in H$ is called the representation $h_\zeta\in H$ through the scalar function $m(\zeta)$, which has the same jump as h_ζ with a coefficient analytic to ζ and determined by the spectral data of the operator T.

The construction of the spectral decomposition will be implemented in the form of a chain:

$$T_\zeta\to h_\zeta\to m(\zeta). \tag{2}$$

Consider all the eigenvalues, spectral features and their eigenfunctionals. Their linear shell may be dense in space (so-called completeness of the system of its eigenfunctions). The decomposition of an arbitrary element is required in order to submit any element through the system of its eigenfunctions. This will mean that the system of its eigenfunctions is complete. As is known, the completeness of the system of eigenfunctions is important for the practical realization of the method of separation of variables, which is used for solving differential equations with partial derivatives.

Consider the extension $T_{\max}$ of the operator *T*, that is, the operator is such that:

$$D(T_{\max}) = D(T)\dot{+}L, \quad T_{\max}\big|_{D(T)} = T. \tag{3}$$

Then each value ζ is proper for the operator $T_{\max}$, and the corresponding eigenvector will be the element h_ζ in the chain (2). So, we have the following Definition 3 for the operator T:

$$T: H \to H, \quad \overline{D(T)} = H.$$

Definition 3. We say that an element $h_\zeta, \zeta \in \Omega \setminus [0,\infty)$ *separates* branching of the resolvent $T_\zeta, \zeta \in \Omega \setminus [0,\infty)$, if:

$$T_\zeta \phi = \left(\phi, b\left(\overline{\zeta}\right)\right) h_\zeta + B(\zeta)\phi, \quad \phi \in \Phi, \quad \zeta \in \Omega \setminus [0,\infty), \tag{4}$$

where the functions $\left(\phi, \overline{b(\zeta)}\right)$ and $B(\zeta)\phi, \phi \in \Phi$ are analytic in Ω.

Definition 4. We say that the scalar function $m(\zeta)$, $\zeta \in \Omega \setminus [0,\infty)$ *separates* the branching h_ζ, if:

$$\left(h_\zeta, \psi\right) = m(\zeta)\left(a(\zeta), \psi\right) + \left(r(\zeta), \psi\right),$$

$$\psi \in \Phi, \quad \zeta \in \Omega \setminus [0,\infty), \tag{5}$$

where the functions $\left(a(\zeta), \psi\right)$ and $\left(r(\zeta), \psi\right)$ are analytic in Ω.

The function $m(\zeta)$ is called the Weyl function of the operator *T*.

In other words, the branching of the resolvent T_ζ is given by the element h_ζ, and the branching h_ζ – by the scalar function $m(\zeta)$. Determine:

$$\left(T_\sigma \phi, \psi\right)_\pm = \lim_{\tau \to \pm 0}\left(T_{\sigma + i\tau}\phi, \psi\right), \quad \phi, \psi \in \Phi$$

and similarly define elements $\left(h_\sigma\right)_\pm$ and functions $m_\pm(\sigma), \sigma > 0$. Further, we assume on the operator T that there are elements ϕ, $\psi \in \Phi$ such that:

$$\left(T_\sigma \phi, \psi\right)_+ - \left(T_\sigma \phi, \psi\right)_- \neq 0, \quad \sigma > 0. \tag{6}$$

The contents of the functionals $a(\sigma)$, $b(\sigma)$ are presented in the following Lemma.

Lemma 1. The functionals $a(\sigma)$ and $b(\sigma)$ in the relations (5) and (4) respectively are the eigenfunctionals of the operators *T* and *T**, corresponding to the point $\sigma \in (0,\infty)$ of the continuous spectrum.

Proof. As $T: \Phi \to \Phi$, we replace in (4) the element ϕ with $(T - \zeta)\phi$, then

$$T_\zeta (T - \zeta)\phi = \left((T - \zeta)\phi, b\left(\overline{\zeta}\right)\right) h_\zeta + B(\zeta)(T - \zeta)\phi.$$

Here $T_\zeta (T - \zeta)\phi = \phi$. If $\zeta \to \sigma \pm 0$, then:

$$\phi = \left((T - \sigma)\phi, b(\sigma)\right)\left(h_\sigma\right)_\pm + B(\sigma)(T - \sigma)\phi. \tag{7}$$

From the equations (4), (6), it follows that $\left(h_\sigma\right)_+ - \left(h_\sigma\right)_- \neq$ $\neq 0$. Taking into account the equality (7), we have that $\left((T - \sigma)\phi, b(\sigma)\right) \equiv 0$.

So, $b(\sigma)$ is the eigenfunctional of the operator *T**.

Substituting the equation (5) in the formula (4), we get:

$$\left(T_\zeta \phi, \psi\right) = \left(\phi, b\left(\overline{\zeta}\right)\right)\left(m(\zeta)\left(a(\zeta), \psi\right) + \left(r(\zeta), \psi\right)\right).$$

Due to (6), we will have $m_+(\sigma) - m_-(\sigma) \neq 0$. After substituting $\left(T^* - \overline{\zeta}\right)\psi$ instead of ψ, and calculating the jump over $(0,\infty)$, we get:

$$0 = \left(\phi, b(\sigma)\right)\left(m_+(\sigma) - m_-(\sigma)\right)\left(a(\sigma), \left(T^* - \overline{\sigma}\right)\psi\right).$$

Choosing ϕ such that $\left(\phi, b(\sigma)\right) \neq 0$, $\sigma > 0$. Then, having $m_+(\sigma) - m_-(\sigma) \neq 0$, $\sigma > 0$, we will get

$$\left(a(\sigma), \left(T^* - \sigma\right)\psi\right) = 0, \quad \sigma > 0.$$

So, $a(\sigma)$ is the eigenfunctional of the operator *T*.

Lemma 1 is proved.

5. Definition of maximal operator $T_{\max}$

Some operator $T_1 \supset T$ is called the *extension* of the operator *T*, if:

$$D(T) \subset D\left(T_1\right) \text{ and } T_1\phi = T\phi, \quad \phi \in D(T).$$

Definition 5. The extension $T_{\max} \supset T$ is called the *maximal operator* for *T*:

1) if for each element $\phi \in \Phi$ and each value $\sigma > 0$ there is only one solution of the equation

$$\left(T_{\max} - \sigma\right) f_\sigma = \phi, \quad \sigma > 0, \quad \phi \in \Phi \tag{8}$$

and if $f_\sigma \in \Phi$;

2) if the solution of the equation (8) admits the analytic prolongation f_ζ in the region Ω such that $f_\zeta \in D\left(T_{\max}\right)$ and $\left(T_{\max} - \zeta\right) f_\zeta = \phi$, $\phi \in \Phi$.

We will introduce the operator $T_{\max,\sigma}: \Phi \to \Phi$ in this way:

$$T_{\max,\sigma}\phi = \left(T_{\max} - \sigma\right)^{-1}\phi = f_\sigma, \quad \sigma > 0. \tag{9}$$

Denoting $T_{\max,\zeta}\phi = f_\zeta$, we get

$$\left(T_{\max} - \zeta\right) T_{\max,\zeta}\phi = \phi, \quad \phi \in \Phi, \quad \zeta \in \Omega. \tag{10}$$

Obviously, $T_{\max,\zeta}: \Phi \to \Phi$.

Note that the operator $T_{\max} - \sigma$, $\sigma > 0$ in the equation (9) is reversed.

Lemma 2. The operator has the following properties:

1) the equality is true for the operator:

$$T_{\max,\zeta}T_{\max,z}\phi = T_{\max,z}T_{\max,\zeta}\phi,\quad \zeta, z\in\Omega; \tag{11}$$

2) the formal resolvent $T_{\max,\sigma}$ in the interval $(0,\infty)$ is the pseudoresolvent $T_{\max,\zeta}$, $\zeta\in\Omega$ in the linear space Φ.

Proof.

1) Let μ, $\nu>0$, the operators $T_{\max}-\nu$ and $T_{\max}-\mu$ commute, i. e.

$$T_{\max,\nu}T_{\max,\mu} = T_{\max,\mu}T_{\max,\nu}. \tag{12}$$

By using the analytic prolongation and changing the symbols $\nu\to\zeta\in\Omega$, $\mu\to z\in\Omega$, we obtain the equality (11) from (12).

Using the relation (12), we have:

$$\begin{aligned}&(\zeta - z)T_{\max,\zeta}T_{\max,z}\phi =\\ &=\left[(T_{\max}-z)-(T_{\max}-\zeta)\right]T_{\max,\zeta}T_{\max,z}\phi =\\ &=(T_{\max}-z)T_{\max,z}T_{\max,\zeta}\phi -\\ &-(T_{\max}-\zeta)T_{\max,\zeta}T_{\max,\zeta}\phi = T_{\max,\zeta}\phi - T_{\max,z}\phi.\end{aligned}$$

Therefore,

$$T_{\max,\zeta}\phi - T_{\max,z}\phi = (\zeta - z)T_{\max,\zeta}T_{\max,z}\phi,\quad \zeta, z\in\Omega. \tag{13}$$

So, $T_{\max,\zeta}$ is a pseudoresolvent.

Lemma 2 is proved.

6. Functional $c(\phi)$ and branching T_ζ

The concepts, which we want to apply to the Friedrichs model, we consider in the case if the maximal operator domain differs in dimension from the area of the operator by one unit. Suppose that the operator $T_{\max}$ exists also for some $e\in H$:

$$D(T_{\max}) = D(T)\dot{+}\{e\},\quad e\notin D(T). \tag{14}$$

In this case, we define the functional $c(\phi)$.

Definition 6. Let $c(\phi)$ be the functional in $D(T_{\max})$, determined by the condition

$$\phi + c(\phi)e\in D(T),\quad \phi\in D(T_{\max}). \tag{15}$$

For example, if $\phi\in D(T_{\max})$, then:

1) $\phi\in D(T)\Leftrightarrow c(\phi)=0$;

2) $\phi = e\Rightarrow c(e) = -1$ as

$$e + c(e)e = (1 + c(e))e\in D(T),$$

we have $1+c(e)=0$, if $\phi\in D(T_{\max})$, then $\phi=\phi_0+\alpha e$, $\alpha\in C$, $\phi_0\in D(T)$. Applying the functional $c(\phi)$, we have that:

$$c(\phi) = c(\phi_0) + \alpha c(e) = -\alpha,$$

then ((14), so $c(\phi_0)=0$) $\alpha = -c(\phi)$ and

$$\begin{aligned}&\phi = \phi_0 - c(\phi)e\in D(T_{\max}),\\ &T_{\max}\phi = T\phi_0 - c(\phi)E,\end{aligned} \tag{16}$$

where $T_{\max}e = E$ for some specified element $E\in H$. Having the maximal operator $T_{\max}$ and the functional $c(\phi)$, we can submit the branching of the resolvent T_ζ.

Theorem 1. Let $T_{\max}\supset T$ be the maximal operator. Then the resolvent T allows separation of the branching:

$$T_\zeta\phi = (\phi, b_{\bar{\varsigma}})h_\varsigma + T_{\max,\zeta}\phi,\quad \varsigma\in\Omega\setminus[0,\infty), \tag{17}$$

where

$$(\phi, b_{\bar{\varsigma}}) = c(T_{\max,\zeta}\phi), \tag{18}$$

and the element:

$$h_\varsigma = e - T_\varsigma(T_{\max}-\varsigma)e \tag{19}$$

is the eigenvector $T_{\max}$, namely:

$$(T_{\max}-\varsigma)h_\varsigma = 0,\quad c(h_\zeta) = -1,\quad \varsigma\in\Omega\setminus[0,\infty). \tag{20}$$

Proof. Considering the equation (10):

$$(T_{\max}-\varsigma)T_{\max,\varsigma}\phi = \phi$$

and despite the fact that $T_{\max,\varsigma}\phi\in D(T_{\max})$, we get:

$$T_{\max,\varsigma}\phi = \phi_0 + \alpha e,\quad \phi_0\in D(T). \tag{21}$$

Then:

$$\begin{aligned}&(T_{\max}-\varsigma)(\phi_0+\alpha e) = \phi \Rightarrow (T-\varsigma)\phi_0 + \alpha(T_{\max}-\varsigma)e = \phi,\\ &\phi_0 + \alpha T_\varsigma(T_{\max}-\varsigma)e = T_\varsigma\phi.\end{aligned}$$

Substitute ϕ_0 from the equation (21), and

$$\begin{aligned}&(T_{\max,\varsigma}\phi - \alpha e) + \alpha T_\varsigma(T_{\max}-\varsigma)e = T_\zeta\phi,\\ &T_\zeta\phi = -\alpha(e - T_\varsigma(T_{\max}-\varsigma)e) + T_{\max,\zeta}\phi,\\ &T_\varsigma\phi = -\alpha h_\varsigma + T_{\max,\varsigma}\phi.\end{aligned} \tag{22}$$

As $c(e)=-1$, then $c(h_\zeta)=-1$ (19). From the formula (22), with $c(T_\varsigma\phi)=0$ and $c(h_\varsigma)=-1$ we have that:

$$0 = -\alpha\cdot(-1) + c(T_{\max,\varsigma}\phi).$$

So, $\alpha = -c(T_{\max,\varsigma}\phi)$ and the relations (17), (18) are proved.

It remains to prove that the element h_ς is the eigenvector $T_{\max}$, i. e. the relations (19), (20) are true. We have:

$$(T_{\max}-\varsigma)h_\varsigma=(T_{\max}-\varsigma)\left[e-T_\varsigma(T_{\max}-\varsigma)e\right]=$$
$$=(T_{\max}-\varsigma)e=(T_{\max}-\varsigma)\big|_{D(T)}T_\varsigma(T_{\max}-\varsigma)e=$$
$$=(T_{\max}-\varsigma)e-(T_{\max}-\varsigma)e=0.$$

Moreover, $c(h_\varsigma)=-1$ uniquely identifies h_ς as the eigenvector of the operator $T_{\max}$.

Theorem 1 is proved.

Note 1. The Weyl function exists, if the operator $T_{\max}$ exists (see (43), (44)).

7. Friedrichs model

Let

$$H=L^2_\rho(0,\infty),\quad \rho(\tau)=\frac{1}{\pi}\sqrt{\tau}.$$

We denote Φ, $\overline{\Phi}=H$ as the subspace of the functions $\phi(\tau)$, which admit the analytic prolongation $\phi(\zeta)$, $\zeta=\tau+i\mu$ in the domain Ω.

We denote by $S:H\to H$ the operator $S\phi(\tau)=\tau\phi(\tau)$, $\tau>0$ with the maximal definition domain $D(S)$. Let G be the Hilbert space and $V=A^*B$, where $A,B:H\to G$ are limited integral operators. The operator

$$T=S+V,\quad V=A^*B,\quad D(T)=D(S),$$
$$R(A^*),R(B^*)\subset\Phi \tag{23}$$

is called the *Friedrichs model*. We obtain another definition of the maximal operator $S_{\max}$ (Definition 5 and calculations in (14), (15) where $T=S$).

If $S_\zeta=(S-\zeta)^{-1}$, $\zeta\notin[0,\infty]$, then:

$$S_\zeta\psi(\tau)=\frac{\psi(\tau)}{\tau-\zeta}=\psi(\zeta)\frac{1}{\tau-\zeta}+\frac{\psi(\tau)-\psi(\zeta)}{\tau-\zeta}.$$

This decomposition coincides with (17), therefore,

$$S_{\max,\zeta}\psi(\tau)=\frac{\psi(\tau)-\psi(\zeta)}{\tau-\zeta}. \tag{24}$$

In order to get $S_{\max,}\phi$, we solve the equation $S_{\max,\sigma}\psi(\tau)=\phi(\tau)$ or

$$\frac{\psi(\tau)-\psi(\sigma)}{\tau-\sigma}=\phi(\tau). \tag{25}$$

The solution of the equation (25) gives:

$$\psi(\tau)=(S_{\max}-\sigma)\phi(\tau)=(\tau-\sigma)\phi(\tau)+\psi(\sigma).$$

Thus, we have the following definition $S_{\max}$, ϕ, which is different from (16).

$$S_\zeta\psi(\tau)=\frac{\psi(\tau)}{\tau-\zeta}=\psi(\zeta)\frac{1}{\tau-\zeta}+\frac{\psi(\tau)-\psi(\zeta)}{\tau-\zeta}.$$

Definition 7. Domain of definition:

$$D(S_{\max})=\{\phi\in H:\exists c(\phi):\tau\phi(\tau)+c(\phi)\in H\} \tag{26}$$

and the maximal operator:

$$S_{\max}\phi(\tau)=\tau\phi(\tau)+c(\phi),\quad \tau>0. \tag{27}$$

Keep the same decomposition (14), where $e(\tau)=\dfrac{1}{\tau+1}$:

$$D(S_{\max})=D(S)\dot{+}\left\{\frac{1}{\tau+1}\right\}. \tag{28}$$

Note that the definition of the functional $c(\phi)$ according to (26) is equivalent to the same definition according to (15).

Indeed,

$$\tau\phi(\tau)+c(\phi)\in H\Leftrightarrow(\tau+1)\phi(\tau)+c(\phi)\in H\Leftrightarrow$$
$$\Leftrightarrow\phi(\tau)+\frac{c(\phi)}{\tau+1}\in D(S)\Leftrightarrow\phi+c(\phi)e\in D(S).$$

Find the value of $|c(\phi)|$. Multiplying the equation (27) by $\chi_{[0,1]}$, we obtain:

$$\|\chi_{[0,1]}\|\cdot|C(\phi)|\le\|\chi_{[0,1]}S_{\max}\phi(\tau)\|+\|\chi_{[0,1]}\tau\phi(\tau)\|\le$$
$$\le\left(\int_0^1|S_{\max}\phi(\tau)|^2\rho(\tau)d\tau\right)^{\frac{1}{2}}+\left(\int_0^1|\tau\phi(\tau)|^2\rho(\tau)d\tau\right)^{\frac{1}{2}}\le$$
$$\le\left(\int_0^\infty|S_{\max}\phi(\tau)|^2\rho(\tau)d\tau\right)^{\frac{1}{2}}+\left(\int_0^\infty|\phi(\tau)|^2\rho(\tau)d\tau\right)^{\frac{1}{2}}\le$$
$$\le\|S_{\max}\phi\|+\|\phi\|.$$

From here:

$$|c(\phi)|\le C(\|S_{\max}\phi\|+\|\phi\|). \tag{29}$$

Thus, the functional $c(\phi)$ is continuous in the sense of the norm of the operator $S_{\max}$.

As the operator $S_{\max}$ is an extension of the operator S, then the next Lemma 3 will be used in Theorem 2.

Lemma 3. Let $\dot{S}\supseteq S$ and the operator $\dot{S}_z=(\dot{S}-z)^{-1}$ be limited. Let $V=A^*B$ and the operator $\dot{K}(z)=1+B\dot{S}_zA^*$ has a limited inverse operator. If $\dot{T}=\dot{S}+A^*B$, then for $\dot{T}_z=(\dot{T}-z)^{-1}$ we have:

$$\dot{T}_z=\dot{S}_z-\dot{S}_zA^*\dot{K}^{-1}(z)B\dot{S}_z. \tag{30}$$

Proof. The equation $(\dot{T}-z)f=g$ or $(\dot{S}-z)f+A^*Bf=g$ means:

$$f+\dot{S}_zA^*Bf=\dot{S}_zg. \tag{31}$$

Applying B, we get $(1+B\dot{S}_zA^*)Bf=B\dot{S}_zg$, so

$$Bf=\dot{K}^{-1}(z)B\dot{S}_zg.$$

Substitution of Bf in the equation (31) gives f, i. e. (30).

Lemma 3 is proved.

The second equation (5) of the branching (4), (5) can be determined from the equation (4), which is written for the adjoint operator.

Indeed, through formal transformations we have:

$$(S_\varsigma 1,\psi)=\overline{(S_{\bar{\varsigma}}\psi,1)}. \tag{32}$$

For the equation (32) we need to identify the functional $(\bullet,1)$ on the elements of the sum (28).

Definitions 8. Denote $(\bullet,1)$ or «1» as the functional defined in:

$$D(S_{\max}) = D(S) \dot{+} \left\{ \frac{1}{\tau+1} \right\}$$

by the relations:

$$(\phi,1) = \lim_{N\to\infty} (\phi, 1_N),$$

$$1_N(x) = \chi_{[0,N]}(x), \quad \phi \in D(S) \tag{33}$$

and

$$\left(\frac{1}{\tau+1}, 1 \right) = -1. \tag{34}$$

The value $(\phi,1)$ of the functional «1» on the item ϕ exists in the domain of elements ϕ, which is dense in H (for example, if ϕ is a finite function or rapidly declining function $\phi(\tau)$ at $\tau\to\infty$). This means that the functional «1» is unbounded (46).

If $A: H \to H$ is the bounded operator $\left|(A^* f,1)\right| \le C\|f\|$, $f\in H$, then we define the element $A\cdot 1$ by the relation:

$$(f, A1) = (A^* f, 1), \quad f \in H. \tag{35}$$

According to the formula (35), the value of the operator A on the functional «1» is determined provided that the value $(A^* f,1)$ is the bounded functional from f, so, according to the Riesz representation, there exists the element $A_1 \in H$.

Suppose, for example, $A = S_\zeta$, then $A^* f = S_{\bar\zeta} f \in D(S)$ and

$$(S_{\bar\zeta} f, 1) = \lim_N (S_{\bar\zeta} f, 1_N) = \lim_N (f, S_\zeta 1_N),$$

where

$$\|S_\varsigma 1_N\|^2 = \frac{1}{\pi}\int_0^N \frac{1}{|\tau-\varsigma|^2}\sqrt{\tau}\,d\tau \le \frac{1}{\pi}\int_0^\infty \frac{1}{|\tau-\varsigma|^2}\sqrt{\tau}\,d\tau = C^2.$$

Finally, $\left|(A^* f,1)\right| \le C\|f\|$.

For example, the relation (32) means that:

$$h_{0,\varsigma} = S_\zeta 1 \text{ or } h_{0,\varsigma}(\tau) = \frac{1}{\tau-\varsigma}. \tag{36}$$

Regarding the calculation $(\phi,1)$ and $c(\phi)$, see also (47).

8. Operators $T_{\max,\zeta}$ and $(T^*)_{\max,\zeta}$

Using Lemma 3 for getting the maximal operator for the operator T from the relations (23).

Theorem 2. Let

$$N(\zeta) = 1 + BS_{\max,\zeta}A^* \left(N_*(\zeta) = 1 + AS_{\max,\zeta}B^* \right).$$

According to (24)). If the conditions hold:

$$\|BS_{\max,\zeta}A^*\|_G < 1, \quad \|AS_{\max,\zeta}B^*\|_G < 1, \quad \zeta\in\Omega, \tag{37}$$

then the operator $T = S + A^*B$ $(T^* = S + B^*A)$ has the maximal operator:

$$T_{\max} = S_{\max} + A^*B \quad \left((T^*)_{\max} = S_{\max} + B^*A \right). \tag{38}$$

Proof. Let $T = S + A^*B$, $\tau > 0$. Taking into consideration the conditions (37), there exists the inverse operator $N(\sigma)^{-1}$ and by Lemma 3, there exists the inverse operator:

$$T_{\max,\sigma} = (T_{\max} - \sigma)^{-1} = S_{\max,\sigma} - S_{\max,\sigma}A^* N(\sigma)^{-1} BS_{\max,\sigma} \tag{39}$$

and the relations (8), (9) are valid.

The analytic prolongation (39) gives:

$$T_{\max,\zeta}\phi = S_{\max,\zeta}\phi - S_{\max,\zeta}A^* N(\zeta)^{-1} BS_{\max,\zeta}. \tag{40}$$

Similarly,

$$(T^*)_{\max,\zeta}\phi = S_{\max,\zeta}\phi - S_{\max,\zeta}B^* N_*(\zeta)^{-1} AS_{\max,\zeta}\phi.$$

We have that:

$$T_\zeta = S_\zeta - S_\zeta \left(A^* K(\zeta)^{-1} B \right) S_\zeta,$$

$$(T_\zeta)^* = S_{\bar\zeta} - S_{\bar\zeta} \left(A^* K(\zeta)^{-1} B \right)^* S_{\bar\zeta},$$

$$\left((T_\zeta)^* f, 1_N \right) = \left(f - \left(A^* K(\zeta)^{-1} B \right)^* S_{\bar\zeta} f, S_\zeta 1_N \right).$$

As $\|S_\zeta 1_N\| \le C$, $N = 1,2\ldots$, then:

$$\left| (T_\zeta)^* f, 1_N \right| \le C_1 \|f\| \cdot \|S_\zeta 1_N\| \le C_1 C \|f\|.$$

According to the equation (35), the element $T_\zeta 1$ is defined. As $R(A^*) \subset \Phi$, then $T_{\max,\zeta}: \Phi \to \Phi$ and the relation (10) is made of (in addition, (8), (9) are also performed).

Theorem 2 is proved.

If $T = S + A^*B$ and the operator $K(\zeta) = 1 + BS_\zeta A^*$, $\zeta \in [0,\infty)$ is reverse, then (according to (30)):

$$(T-\zeta)^{-1}\phi = T_\zeta\phi = S_\zeta\phi - S_\zeta A^* K(\zeta)^{-1} BS_\zeta\phi,$$

$$K(\zeta) = 1 + BS_\zeta A^*.$$

From the equations $(T_{\max} - \zeta)h_\zeta = 0$, (27) and definition 8, it follows that

$$(S-\zeta)h_\zeta + Vh_\zeta + c(h_\zeta) = 0,$$

$$(T-\zeta)h_\zeta = -c(h_\zeta)\cdot 1, \quad h_\zeta = -c(h_\zeta)T_\zeta 1.$$

Given (20), we obtain $c(h_\zeta) = -1$. From here $h_\zeta = T_\zeta 1$.

So the decomposition (17) of the resolvent for the operators T and T^*:

$$T_\zeta\phi = (\phi, b_{\bar\zeta}) T_\zeta 1 + T_{\max,\zeta}\phi,$$

$$(T^*)_{\bar\zeta}\psi = (\psi, a_{\bar\zeta})(T^*)_{\bar\zeta} 1 + (T^*)_{\max,\bar\zeta}\psi. \tag{41}$$

Theorem 3. Let $T=S+A^*B$ (according to (23)). Then the condition (37) is sufficient for the existence of the Weyl function $m(\zeta)$ of the operator T (def. 4). If the conditions (37) are satisfied, then:

$$m(\zeta)=\left(T_\zeta 1,1\right),\ \ \zeta\in\Omega\setminus[0,\infty). \tag{42}$$

Proof. According to Theorem 2 and condition (37), there exist the operators $T_{\max}$ and $(T^*)_{\max}$. According to definition 5 ((9), (10)) we define the operators $T_{\max}$, $(T^*)_{\max,\zeta}$. By Theorem 1 and definition 8, we obtain the relation (41), where the element $h_\zeta=T_\zeta 1$ separates the branching $T_\zeta\phi$. Finally,

$$\begin{aligned}&\left(h_\zeta,\psi\right)=\left(T_\zeta 1,\psi\right)=\left(1,\left(T^*\right)_{\bar\zeta}\psi\right)=\\&=\overline{\left(\psi,a_{\bar\zeta}\right)}\left(1,\left(T^*\right)_{\bar\zeta}1\right)+\left(1,\left(T^*\right)_{\max,\bar\zeta}\psi\right).\end{aligned} \tag{43}$$

According to definition 4, we have the relation (42), where:

$$m(\zeta)=\left(1,\left(T^*\right)_{\bar\zeta}1\right)=\left(T_\zeta 1,1\right).$$

Theorem 3 is proved.

Note 2. Functions:

$$\left(R(\zeta),\psi\right)=\left(1,\left(T^*\right)_{\max,\bar\zeta}\psi\right),\ \ \left(a(\zeta),\psi\right)=\overline{\left(\psi,a_{\bar\zeta}\right)},\psi\in\Phi$$

are analytic functions of $\zeta\in\Omega,$ and the functionals $a(\sigma)$ and $R(\sigma)$ are the eigenfunctionals of the operators T and $T_{\max}$, corresponding to the point $\sigma>0$ (see [2]).

The relation (43), namely:

$$a(\zeta)\bullet m(\zeta)+R(\zeta)=h_\zeta\in H, \tag{44}$$

determines the function $m(\zeta)$ clearly if the functionals $a(\zeta)$ and $R(\zeta)$ are clearly defined, too.

9. Sturm-Liouville operator on the semiaxis

Let us pay attention to the branching of the resolvent of the Sturm-Liouville operator on the semiaxis. Our goal is to show that the separation of the branching (according to definition 3–4) is a natural way to reduce in two stages that part which is the branching $T_\zeta=(T-\zeta)^{-1}$ itself.

Of course, there is the question about the relation between analytic functions in Ω and analytic in $\Omega\setminus[0,\infty)$, where $[0,\infty)$ is the continuous spectrum of T.

To begin with the non-self-adjoint operator L, generated by the differential expression $ly=-y''$, $y(0)=0$ in the space $L^2(0,\infty)$. Using the unitary operator:

$$F\colon\ L^2(0,\infty)\to L^2_\rho(0,\infty),\ \ \rho(\tau)=\frac{1}{\pi}\sqrt{\tau},\tau>0,$$

given that the relations:

$$\phi(\tau)=Fy(\tau)=\int_0^\infty y(x)\frac{\sin x\sqrt{\tau}}{\sqrt{\tau}}\mathrm{d}x,$$

$$y(x)=F^{-1}\phi(x)=\frac{1}{\pi}\int_0^\infty\phi(\tau)\sin x\sqrt{\tau}\mathrm{d}\tau. \tag{45}$$

Diagonalize the operator L, namely $FLF^{-1}=S$.

Lemma 4.

1) Let:

$$D(L_{\max})=\left\{y\in L^2(0,\infty):y'-\text{abs.cont.},y''\in L^2(0,\infty)\right\},$$

$$L_{\max}y=-y''$$

and the operator:

$$\phi(\tau)=Fy(\tau)=\int_0^\infty y(x)\frac{\sin x\sqrt{\tau}}{\sqrt{\tau}}\mathrm{d}x$$

be given by the formulas (26), (27). Then:

$$FL_{\max}F^{-1}=S_{\max}. \tag{46}$$

2) If $\phi=Fy,$ then:

$$c(\phi)=-y(0),\ \ (\phi,1)=y'(0). \tag{47}$$

Proof.

1) If $y\in D\left(L_{\max}\right),$ then the equality:

$$\begin{aligned}&-\int_0^\infty y''(x)\frac{\sin x\sqrt{\tau}}{\sqrt{\tau}}\mathrm{d}x=\\&=-y(0)+\tau\int_0^\infty y(x)\frac{\sin x\sqrt{\tau}}{\sqrt{\tau}}\mathrm{d}x=\\&=-y(0)+\tau Fy(\tau),\end{aligned}$$

$$y,y''\in L^2(0,\infty)$$

is true if and only if $FL_{\max}y=S_{\max}Fy,$ coincides with the equality (46) and, in addition, $c(\phi)=-y(0).$

2) The derivative by the variable x in the relations (45) is $y'(0)=(\phi,1).$

So, conditions (47) are proved.

Lemma 4 is proved.

Note. As:

$$F\left(e^{-x}\right)(\tau)=\frac{1}{1+\tau},$$

then

$$\left(\frac{1}{\tau+1},1\right)=\left(e^{-x}\right)'\Big|_{x=0}=-1.$$

Therefore, the equality (34) is proved.

Let us calculate $(\phi,1),$ where $\phi=Fy,$ using the formula (47), we have

$$(\phi,1)=y'(0)=\left(y(x)\right)'\Big|_{x=0}=\left(F^{-1}\phi(x)\right)'\Big|_{x=0}. \tag{48}$$

Similarly,

$$c(\phi)=-F^{-1}\phi(x)\Big|_{x=0}$$

if $\phi\notin D(S)$ (otherwise $c(\phi)=0$).

Consider the differential expression:

$$ly = -y'' + q(x)y,\ y(0) = 0.$$

Mark the operator through $M = L + Q$. Fourier transform (45) M gives the operator:

$$T = FMF^{-1} = F(L+Q)F^{-1} = S + V,\ \ V = FQF^{-1}. \tag{49}$$

Consider the differential expression:

$$ly = -y'' + q(x)y,\ \ y(0) = 0.$$

Suppose $q(x)$ is a complex-measurable function and

$$|q(x)| < C\exp(-\varepsilon x),\ \ x > 0,\ \ \varepsilon > 0. \tag{50}$$

Let:

$$q(x) = \overline{q_1(x)}q_2(x),\ \ |q_1(x) = |q_2(x)||.$$

Let:

$$Qy(x) = q(x)y(x),\ \ x > 0,$$

then similarly, $Q = Q_1^* Q_2$. Let: $M = L + Q$. Fourier transformation (45) of the operator M gives the operator:

$$T = FMF^{-1} = F(L+Q)F^{-1} = S + V,$$

$$V = A^* B,\ \ A^* = FQ_1^*,\ \ B = Q_2 F^{-1}. \tag{51}$$

Theorem 4.

1) There exist values $C_0 > 0$, $\varepsilon_0 > 0$ such that if $0 < C < C_0$, $0 < \varepsilon < \varepsilon_0$, then for the operator T (provided (50)) the sufficient condition (37) of the existence of the Weyl function $m(\zeta)$ is true.

2) The Weyl Function $m(\zeta) = (T_\zeta 1, 1)$ (according to (42)) in the case of the self-adjoint operator T coincides with the well-known Weyl function of the Sturm-Liouville operator on the semiaxis.

Proof.

1) Taking into consideration the conditions (37), we denote by

$$N(\zeta)c = c + BS_{\max,\zeta}A^* c = c + Q_2 F^{-1} S_{\max,\zeta} FQ_1^* c.$$

We have:

$$FQ_1^* c(\tau) = \int_0^\infty \overline{q_1(y)}c(y)\frac{\sin y\sqrt{\tau}}{\sqrt{\tau}}\,\mathrm{d}y,$$

so according to the formula (24):

$$S_{\max,\zeta} FQ_1^* c(\tau) = \int_0^\infty q_1(y)c(y)\frac{\dfrac{\sin y\sqrt{\tau}}{\sqrt{\tau}} - \dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\tau - \zeta}\,\mathrm{d}y.$$

As:

$$F^{-1}\phi(x) = \frac{1}{\pi}\int_0^\infty \phi(\tau)\sin x\sqrt{\tau}\,\mathrm{d}\tau,$$

then:

$$Q_2 F^{-1} S_{\max,\zeta} FQ_1^* c(x) =$$

$$= \frac{1}{\pi}q_2(x)\int_0^\infty\left(\int_0^\infty \overline{q_1(y)}c(y)\frac{\dfrac{\sin y\sqrt{\tau}}{\sqrt{\tau}} - \dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\tau - \zeta}dy\right)\sin x\sqrt{\tau}\,\mathrm{d}\tau.$$

$$S_{\max,\zeta} FQ_1^* c(\tau) = \int_0^\infty q_1(y)c(y)\frac{\dfrac{\sin y\sqrt{\tau}}{\sqrt{\tau}} - \dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\tau - \zeta}dy.$$

Changing the order of integration, while:

$$Q_2 F^{-1} S_{\max,\zeta} FQ_1^* c(x) =$$

$$= \frac{1}{\pi}q_2(x)\int_0^\infty \overline{q_1(y)}c(y)\left[\int_0^\infty \frac{\dfrac{\sin y\sqrt{\tau}}{\sqrt{\tau}} - \dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\tau - \zeta}\sin x\sqrt{\tau}d\tau\right]dy =$$

$$= \frac{1}{\pi}q_2(x)\int_0^\infty \overline{q_1(y)}c(y)I(y,x)\,\mathrm{d}y, \tag{52}$$

$$I(y,x) = \int_0^\infty \frac{\dfrac{\sin y\sqrt{\tau}}{\sqrt{\tau}} - \dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\tau - \zeta}\sin x\sqrt{\tau}\,\mathrm{d}\tau =$$

$$= \int_0^\infty h(y,\sqrt{\tau})\sin x\sqrt{\tau}\,\mathrm{d}\tau,$$

$$h(y,\sqrt{\tau}) = \frac{\dfrac{\sin y\sqrt{\tau}}{\sqrt{\tau}} - \dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\tau - \zeta}.$$

Having done the replacement $\sqrt{\tau} = \theta$, we get:

$$I(y,x) = \int_0^\infty h(y,\theta)\sin x\theta \cdot 2\theta\,\mathrm{d}\theta =$$

$$= 2\int_0^\infty \theta h(y,\theta)\,\mathrm{d}\left(\frac{1-\cos x\theta}{x}\right).$$

Using the formula:

$$\theta h(y,\theta) = \frac{\sin y\theta - \theta\dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\theta^2 - \zeta},$$

integrate by parts:

$$I(y,x)=$$

$$=2\left[\begin{array}{l}\theta h(y,\theta)\cdot\dfrac{1-\cos x\theta}{x}\Big|_0^{\infty}-\\ -\displaystyle\int_0^{\infty}\dfrac{1-\cos x\theta}{x}\left(\dfrac{\sin y\theta-\theta\dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\theta^2-\zeta}\right)'_{\theta}\mathrm{d}\theta\end{array}\right]=$$

$$=-2\int_0^{\infty}\frac{1-\cos x\theta}{x}\left(\frac{\sin y\theta-\theta\dfrac{\sin y\sqrt{\zeta}}{\sqrt{\zeta}}}{\theta^2-\zeta}\right)'_{\theta}\mathrm{d}\theta.$$

Thus, the expression $I(y,x)$ is bounded $|I(y,x)|<K$ for all y, x. According to the relation (52), we obtain:

$$BS_{\max,\zeta}A^{*}c(x)=\frac{1}{\pi}q_2(x)\int_0^{\infty}q_1(y)c(y)I(y,x)\mathrm{d}y,$$

therefore,

$$\left\|BS_{\max,\zeta}A^{*}\right\|\le\frac{K}{\pi}\|q_1\|\cdot\|q_2\|.$$

Obviously, there exist C_0 and ε_0 (according to (50)) such that $\left\|BS_{\max,\zeta}A^{*}\right\|<1$ for $C<C_0$, $\varepsilon<\varepsilon_0$. Thus, paragraph:

1) proved:

$$BS_{\max,\zeta}A^{*}c(x)=\frac{1}{\pi}q_2(x)\int_0^{\infty}q_1(y)c(y)I(y,x)\mathrm{d}y,$$

2) It is known that:

$$M_{\zeta}f(x)=(M-\zeta)^{-1}f(x)=$$

$$=\frac{e(x,\sqrt{\zeta})}{e(\sqrt{\zeta})}\cdot\int_0^{x}s(t,\zeta)f(t)dt+s(x,\zeta)\cdot\int_x^{\infty}\frac{e(t,\sqrt{\zeta})}{e(\sqrt{\zeta})}f(t)\mathrm{d}t.$$

In result,

$$\left(M_{\zeta}\right)^{*}g(x)=\overline{s(x,\zeta)}\int_x^{\infty}\frac{\overline{e(t,\sqrt{\zeta})}}{\overline{e(\sqrt{\zeta})}}g(t)\mathrm{d}t+$$

$$+\frac{\overline{e(x,\sqrt{\zeta})}}{\overline{e(\sqrt{\zeta})}}\int^{x}s(t,\zeta)g(t)\mathrm{d}t.$$

Taking into consideration (35), we have:

$$\left(\psi,T_{\zeta}1\right)=\left(\left(T_{\zeta}\right)^{*}\psi,1\right)=\left(F\left(M_{\zeta}\right)^{*}F^{-1}\psi,1\right)=y'(0),$$

where $y(x)=\left(\left(M_{\zeta}\right)^{*}g\right)(x),\ g=F^{-1}\psi.$

A simple calculation gives:

$$y'(0)=\int_0^{\infty}\frac{\overline{e(t,\sqrt{\zeta})}}{\overline{e(\sqrt{\zeta})}}g(t)\mathrm{d}t,$$

and

$$\left(\psi,T_{\zeta}1\right)=y'(0)=\left(g,\frac{e(t,\sqrt{\zeta})}{\sqrt{\zeta}}\right)_H=$$

$$=\left(F^{-1}\psi,\frac{e(t,\sqrt{\zeta})}{e(\sqrt{\zeta})}\right)_H=\left(\psi,F\frac{e(t,\sqrt{\zeta})}{e(\sqrt{\zeta})}\right)_{L_{\rho}^2}.$$

Finally,

$$T_{\zeta}1=F\left(\frac{e(t,\sqrt{\zeta})}{e(\sqrt{\zeta})}\right)$$

and (according to (47)) we will have:

$$\left(T_{\zeta}1,1\right)=\left(F,\left(\frac{e(t,\sqrt{\zeta})}{\sqrt{\zeta}}\right),1\right)=\left(\frac{e(t,\sqrt{\zeta})}{e(\sqrt{\zeta})}\right)_t^{1}(0)=\frac{e'(\sqrt{\zeta})}{e(\sqrt{\zeta})},$$

the known expression for the Weyl function $m(\zeta)$.

Thus, the relation (42) is proved for the Sturm-Liouville operator on the semiaxis.

Item 2) is proved.

Theorem 4 is proved.

10. Spectral decomposition based on the branching of the resolvent

The operator $T=S+V$ (according to (23)).

If $T_{\zeta}=(T-\zeta)^{-1}$, then:

$$\left(T_{\zeta}(T-\zeta)\phi,\psi\right)=(\phi,\psi)$$

or

$$\left(T_{\zeta}T\phi,\psi\right)-\zeta\left(T_{\zeta}\phi,\psi\right)=(\phi,\psi).$$

Therefore,

$$\frac{1}{\zeta}(\phi,\psi)=-\left(T_{\zeta}\phi,\psi\right)+\frac{1}{\zeta}\left(T_{\zeta}T\phi,\psi\right).$$

Since

$$\int_{|\zeta|=N}\frac{d\zeta}{\zeta}=2\pi i,$$

then from the condition:

$$\lim_{N\to\infty}\int_{|\zeta|=N}\frac{1}{\zeta}\left(T_{\zeta}T\phi,\psi\right)d\zeta=0 \tag{53}$$

it follows that:

$$(\phi,\psi)=-\frac{1}{2\pi i}\lim_{N\to\infty}\int_{|\zeta|=N}\left(T_{\zeta}\phi,\psi\right)\mathrm{d}s. \tag{54}$$

Branching of the resolvent is given by the formulas (41)–(43), which implies:

$$\begin{aligned}&\left(T_{\zeta}\phi,\psi\right)=\left(\phi,b\left(\overline{\zeta}\right)\right)\times\\&\times\left[m(\zeta)\left(a(\zeta),\psi\right)+\left(r(\zeta),\psi\right)\right]+\left(B(\zeta)\phi,\psi\right).\end{aligned} \tag{55}$$

We assume that the operator T has its eigenvalue λ_k and spectral singularities σ_j. Then, for:

$$\frac{1}{\zeta}(\phi,\psi)=-(T_\zeta\phi,\psi)+\frac{1}{\zeta}(T_\zeta T\phi,\psi)$$

from (55) it follows that:

$$(T_\zeta\phi,\psi)_+-(T_\sigma\phi,\psi)_-=$$
$$=(\phi,b(\sigma))(m_+(\sigma)-m_-(\sigma))(a(\sigma),\psi). \qquad (56)$$

Let $\nu>0$. Denote by C_ν the contour in the ζ-plane, $\zeta=x+iy$, which leads successively through the points $(0,-\nu)$, $(N,-\nu)$, (N,ν), $(0,\nu)$, $(0,-\nu)$, where $N=\text{const}>0$. Deforming the contour $|\zeta|=N$ into the contour C_ν we get that:

$$\int_{|\zeta|=N}(T\phi,\psi)d\zeta=\int_{C_\nu}(T_\zeta\phi,\psi)\mathrm{d}\zeta.$$

Let $\nu\to 0$. Then, given the formula (56), we get:

$$\int_{|\zeta|=N}(T_\zeta\phi,\psi)d\zeta=$$
$$=\int_0^N(m_+(\sigma)-m_-(\sigma))(\phi_1 b(\sigma))(a(\sigma),\psi)d\sigma+\sum_{\lambda_k}+\sum_{\sigma_j}. \qquad (57)$$

Example 1. Let $\alpha,\beta\in\Phi$, $\alpha,\beta\in D(S)$ and $G=C$. Define the operators $A,B:H\to C$ by the equalities:

$$A\phi=(\phi,\alpha)_H,\quad B\phi=(\phi,\beta)_H. \qquad (58)$$

To find the adjoint operator $A^*:C\to H$, we'll use the equality $(A\phi,c)_C=(\phi,A^*c)_H$. Since $A\phi=(\phi,\alpha)_H$, then $(\phi,\alpha)_H\cdot\bar{c}\equiv\equiv(\phi,A^*c)_H$, so $A^*c=c\alpha$. Similarly, we consider the operator B, so:

$$A^*c=c\alpha,\quad B^*d=d\beta, \qquad (59)$$

where $c,d\in C$ are some numbers.

Perturbation looks like:

$$V\phi=A^*B\phi=A^*\left((\phi,\beta)_H\right)=(\phi,\beta)_H\,\alpha.$$

Due to the definition (58)–(59), we have:

$$K(\zeta)c=c+BS_\zeta A^*c=c+B(cS_\zeta\alpha)=c+(S_\zeta\alpha,\beta)_H\,c.$$

So,

$$K(\zeta)=1+(S_\zeta\alpha,\beta)_H. \qquad (60)$$

$$V\phi=A^*B\phi=A^*\left((\phi,\beta)_H\right)=(\phi,\beta)_H\,\alpha.$$

Let $T\phi=S\phi+(\phi,\beta)_H\,\alpha$. If:

$$(T-\zeta)\phi=(S-\zeta)\phi+(\phi,\beta)_H\,\alpha=\psi,$$

we get

$$\phi+(\phi,\beta)_H\,S_\zeta\alpha=S_\zeta\psi.$$

Multiplying by

$$\beta:(\phi,\beta)_H\left(1+(S_\zeta\alpha,\beta)_H\right)=(S_\zeta\psi,\beta).$$

According to the formula (60), we have that:

$$(\phi,\beta)_H=\frac{1}{K(\zeta)}(S_\zeta\psi,\beta).$$

So,

$$T_\zeta\psi=S_\zeta\psi-\frac{1}{K(\zeta)}(S_\zeta\psi,\beta)S_\zeta\alpha. \qquad (61)$$

According to (24):

$$S_{\max,\zeta}\psi(\tau)=\frac{\psi(\tau)-\psi(\zeta)}{\tau-\zeta}.$$

We calculate $\gamma=c(S_{\max,\zeta}\psi)$ according to the definition $c(\phi)$ as (26):

$$\int_0^\infty\left|\tau\frac{\psi(\tau)-\psi(\tau)}{\tau-\zeta}+\gamma\right|^2\rho(\tau)\mathrm{d}\tau<\infty,\quad\psi\in\bar{\Phi}$$

or

$$\int_0^\infty\left|\frac{\tau}{\tau-\zeta}\psi(\tau)+\frac{\tau}{\tau-\zeta}(-\psi(\zeta)+\gamma)-\frac{\zeta}{\tau-\zeta}\right|^2\rho(\tau)\mathrm{d}t<\infty.$$

Since:

$$\frac{\tau}{\tau-\zeta}\to 1,\quad\frac{1}{\tau-\zeta}\to 0,\quad\tau\to\infty,$$

then $\gamma=\psi(\zeta)$, i. e.

$$c\left(\frac{\psi(\tau)-\psi(\zeta)}{\tau-\zeta}\right)=\psi(\zeta). \qquad (62)$$

According to (58)–(59), we get that:

$$N(\zeta)^{-1}BS_{\max,\zeta}\psi=c\in C.$$

From here,

$$A^*N(\zeta)^{-1}BS_{\max,\zeta}\psi=c\alpha$$

and in accordance with the formula (62), we obtain:

$$c\left(S_{\max,\zeta}A^*N(\zeta)^{-1}BS_{\max,\zeta}\psi\right)=$$
$$=\left(N(\zeta)^{-1}BS_{\max,\zeta}\psi\right)c\left(S_{\max,\zeta}\alpha\right)=\alpha(\zeta)N(\zeta)^{-1}BS_{\max,\zeta}\psi.$$

Recall (40):

$$T_{\max,\zeta}\phi=S_{\max,\zeta}\phi-S_{\max,\zeta}A^*N(\zeta)^{-1}BS_{\max,\zeta}\phi,$$

so from the previous formulas according to (18), it follows that:

$$\left(\phi,b_{\bar\zeta}\right)=c\left(T_{\max,\zeta}\phi\right)=\phi(\zeta)-\alpha(\zeta)N(\zeta)^{-1}BS_{\max,\zeta}\phi. \qquad (63)$$

According to (61), we have that:

$$T_\zeta\phi=S_\zeta\phi-\frac{1}{K(\zeta)}\left(\phi,S_{\bar\zeta}\beta\right)S_\zeta\alpha,$$

therefore,

$$T_\zeta^*\phi=S_{\bar\zeta}\phi-\frac{1}{\overline{K(\zeta)}}\left(\phi,S_\zeta\alpha\right)S_{\bar\zeta}\beta.$$

Considering (35) and belonging of $\alpha,\beta\in D(S)$, we get:

$$\left(T_\zeta^*\phi,1\right)=\int_0^\infty\left[S_\zeta\phi(\tau)-\frac{1}{\overline{K(\zeta)}}\left(\phi,S_\zeta\alpha\right)S_{\bar\zeta}\beta(\tau)\right]\rho(\tau)\mathrm{d}\tau=$$

$$=\int_0^\infty\left[\phi(\tau)\cdot\frac{1}{\tau-\bar\zeta}\right]\mathrm{d}\tau-\frac{1}{\overline{K(\zeta)}}\left(\phi,S_\zeta\alpha\right)\left(S_{\bar\zeta}\beta,1\right)=\left(\phi,T_\zeta 1\right). \qquad (64)$$

From here,

$$T_\zeta 1(\tau)=\frac{1}{\tau-\zeta}-\frac{1}{K(\zeta)}\left(\overline{S_{\bar\zeta}\beta,1}\right)S_\zeta\alpha=$$

$$=\frac{1}{\tau+1}-\left(\frac{1}{\tau+1}-\frac{1}{\tau-3}\right)-\frac{1}{K(\zeta)}\left(\overline{S_{\bar\zeta}\beta,1}\right)S_\zeta\alpha, \qquad (65)$$

and according to (42), we obtain:

$$m(\zeta)=\left(T_\zeta 1,1\right)=$$

$$=-1-\int_0^\infty\left(\frac{1}{\tau+1}-\frac{1}{\tau-\zeta}\right)\rho(\tau)\mathrm{d}\tau-\frac{1}{K(\zeta)}\left(\overline{S_{\bar\zeta}\beta,1}\right)\left(S_\zeta\alpha,1\right), \qquad (66)$$

where it is taken into account that:

$$\rho(\tau)=\frac{1}{\pi}\sqrt{\tau}$$

and (33), (34) are used.

Example 2. Let $\alpha_{1,2},\beta_{1,2}\in\Phi\cap D(S)$, $G=C^2$. We define the operators $A,B:H\to C^2$ by the equalities:

$$A\phi=\begin{pmatrix}(\phi,\alpha_1)_H\\(\phi,\alpha_2)_H\end{pmatrix},\quad B\phi=\begin{pmatrix}(\phi,\beta_1)_H\\(\phi,\beta_2)_H\end{pmatrix}. \qquad (67)$$

We'll find the adjoint operator $A^*:C^2\to H$. Let:

$$c=\begin{pmatrix}c_1\\c_2\end{pmatrix}\in C^2.$$

The equality $(A\phi,c)_{C^2}=\left(\phi,A_c^*\right)_H$ becomes:

$$(\phi,\alpha_1)_H\,\bar c_1+(\phi,\alpha_2)_H\,\bar c_2=\left(\phi,A^*c\right)_H.$$

From here,

$$A^*c=c_1\alpha_2+c_2\alpha_2,$$

where we obtain $B^*:C^2\to H$ by analogy,

$$B^*d=d_1\beta_2+d_2\beta_2,\quad d=\begin{pmatrix}d_1\\d_2\end{pmatrix}\in C^2.$$

Perturbation has the following form:

$$V\phi=A^*B\phi=(B\phi)_1\alpha_1+(B\phi)_2\alpha_2=(\phi,\beta_1)_H\alpha_1+(\phi,\beta_2)\alpha_2.$$

Since

$$S_\zeta A^*c=c_1S_\zeta\alpha_1+c_2S_\zeta\alpha_2,$$

then

$$BS_\zeta A^*c=\begin{pmatrix}\left(c_1S_\zeta\alpha_1+c_2S_\zeta\alpha_2,\ \beta_1\right)_H\\\left(c_1S_\zeta\alpha_1+c_2S_\zeta\alpha_2,\ \beta_2\right)_H\end{pmatrix}.$$

Let

$$K(\zeta)c=c+BS_\zeta A^*c=$$

$$=\begin{pmatrix}c_1\\c_2\end{pmatrix}+\begin{pmatrix}(S_\zeta\alpha_1,\beta_1)_H & (S_\zeta\alpha_2,\beta_1)_H\\(S_\zeta\alpha_1,\beta_2)_H & (S_\zeta\alpha_2,\beta_2)_H\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix},$$

i. e.

$$K(\zeta)=1+\begin{pmatrix}(S_\zeta\alpha_1\beta_1)_H & (S_\zeta\alpha_2\beta_1)_H\\(S_\zeta\alpha_1\beta_2)_H & (S_\zeta\alpha_2\beta_2)_H\end{pmatrix}.$$

The equality $\phi=T_\zeta\psi$ means that $(T-\zeta)\phi=\psi$ or $(S-\zeta)+$ $+A^*B\phi=\psi$:

$$\phi+S_\zeta A^*B\,\phi=S_\zeta\psi. \qquad (68)$$

Applying the operator B, we get:

$$B\phi+BS_\zeta A^*B_\phi=BS_\zeta\psi\ \text{ or }\ K(\zeta)B\phi=BS_\zeta\psi.$$

Let $\det K(\zeta)\neq 0$. Then $K(\zeta)^{-1}$ there exists $B\phi=K(\zeta)^{-1}BS_\zeta\psi$. Substituting $B\phi$ in the relation (67), we obtain:

$$T_\zeta\psi=\phi=S_\zeta\psi-S_\zeta A^*K(\zeta)^{-1}BS_\zeta\psi. \qquad (69)$$

From here:

$$T_\zeta^*\phi=S_{\bar\zeta}\phi-S_{\bar\zeta}B^*K(\zeta)^{-1^*}AS_{\bar\zeta}\phi\in D(S).$$

In the dense set of elements in H $\phi\in H$, we have that:

$$\left(T_\zeta^*\phi,1\right)=\lim_{N\to\infty}\left(T_\zeta^*\phi,1_N\right)_H=$$

$$=\int_0^\infty\left[S_\zeta\phi(\tau)-S_{\bar\zeta}B^*K(\zeta)^{-1}AS_{\bar\zeta}\phi(\tau)\right]\rho(\tau)\mathrm{d}\tau. \qquad (70)$$

Let $k_{ij}(\zeta)$ be the analytical functions such that:

$$K(\zeta)^{-1}=\begin{pmatrix}k_{11} & k_{12}\\k_{21} & k_{22}\end{pmatrix}. \qquad (71)$$

Consider (67) and

$$AS_{\bar\zeta}\phi=\begin{pmatrix}(S_{\bar\zeta}\phi,\alpha_1)_H\\(S_{\bar\zeta}\phi,\alpha_1)_H\end{pmatrix},$$

then

$$K(\zeta)^{-1}AS_{\bar\zeta}\phi=\begin{pmatrix}(S_{\bar\zeta}\phi,\alpha_1)_H k_{11}+(S_{\bar\zeta}\phi,\alpha_2)_H k_{12}\\(S_{\bar\zeta}\phi,\alpha_1)_H k_{21}+(S_{\bar\zeta}\phi,\alpha_2)_H k_{22}\end{pmatrix}=$$

$$=\begin{pmatrix}(\phi,\overline{k_{11}(\zeta)}S_\zeta\alpha_1)_H+(\phi,\overline{k_{12}(\zeta)}S_\zeta\alpha_2)_H\\(\phi,\overline{k_{21}(\zeta)}S_\zeta\alpha_1)_H+(\phi,\overline{k_{22}(\zeta)}S_\zeta\alpha_2)_H\end{pmatrix}=\begin{pmatrix}d_1\\d_2\end{pmatrix}\in C^2.$$

As $B^*\alpha=\alpha_1\beta_1+\alpha_2\beta_2$, then:

$$B^*K(\zeta)^{-1}AS_{\bar\zeta}\phi=\left[(\phi,\overline{k_{11}(\zeta)}S_\zeta\alpha_1)_H+(\phi,\overline{k_{12}(\zeta)}S_\zeta\alpha_2)_H\right]\beta_1+$$

$$+\left[(\phi,\overline{k_{21}(\zeta)}S_\zeta\alpha_1)_H+(\phi,\overline{k_{22}(\zeta)}S_\zeta\alpha_2)_H\right]\beta_2,$$

where

$$S_{\bar\zeta}B^*K(\zeta)^{-1}AS_{\bar\zeta}\phi=\begin{bmatrix}(\phi,\overline{k_{11}(\zeta)}S_\zeta\alpha_1)_H+\\+(\phi,\overline{k_{12}(\zeta)}S_\zeta\alpha_2)_H\end{bmatrix}S_{\bar\zeta}\beta_1+$$

$$+\left[(\phi,\overline{k_{21}(\zeta)}S_\zeta\alpha_1)_H+(\phi,\overline{k_{22}(\zeta)}S_\zeta\alpha_2)_H\right]S_{\bar\xi}\beta_2.$$

We denote by $X_{1,2}\in\Phi\cap D(S)$ the following elements:

$$X_1=\overline{k_{11}(\zeta)}S_\zeta\alpha_1+\overline{k_{12}(\zeta)}S_\zeta\alpha_2,$$

$$X_2=\overline{k_{21}(\zeta)}S_\zeta\alpha_1+\overline{k_{22}(\zeta)}S_\zeta\alpha_2. \tag{72}$$

Then from (70), it follows that:

$$(T_\zeta^*\phi,1)=\int_0^\infty\begin{bmatrix}S_{\bar\zeta}\phi(\tau)-(\phi,X_1)_H S_{\bar\zeta}\beta_1-\\-(\phi,X_2)_H S_{\bar\zeta}\beta_2\end{bmatrix}\rho(\tau)\mathrm{d}\tau.$$

For the set of elements ϕ, dense in H we have that:

$$\int_0^\infty S_{\bar\zeta}\phi(\tau)\rho(\tau)d\tau=\left(\phi,\frac{1}{\tau-\zeta}\right)_H,$$

therefore,

$$(T_\zeta^*\phi,1)=\left(\phi,\frac{1}{\tau-\zeta}\right)_H-(\phi,X_1)_H(S_{\bar\zeta}B_1,1)-$$

$$-(\phi,X_2)_H(S_{\bar\zeta}B_2,1)$$

and, as a result,

$$T_\zeta 1(\tau)=\frac{1}{\tau-\zeta}-\overline{(S_{\bar\zeta}\beta^1,1)}X_1-\overline{(S_{\bar\zeta}\beta_2,1)}X_2=$$

$$=\frac{1}{\tau+1}-\left(\frac{1}{\tau+1}-\frac{1}{\tau-\zeta}\right)-\overline{(S_{\bar\zeta}\beta_1,1)}X_1-\overline{(S_{\bar\zeta}\beta_2,1)}X_2.$$

Apply the functional 1:

$$(T_\zeta 1,1)=-1-$$

$$-\int_0^\infty\left(\frac{1}{\tau+1}-\frac{1}{\tau-\zeta}\right)\rho(\tau)\mathrm{d}\tau-\overline{(S_{\bar\zeta}\beta_1,1)}(X_1,1)-\overline{(S_{\bar\zeta}\beta_2,1)}(X_2,1),$$

where to according to (73):

$$(X_1,1)=\overline{k_{11}(\zeta)}(S_\zeta\alpha_1,1)+\overline{k_{12}(\zeta)}(S_\zeta\alpha_2,1),$$

$$(X_2,1)=\overline{k_{21}(\zeta)}(S_\zeta\alpha_1,1)+\overline{k_{22}(\zeta)}(S_\zeta\alpha_2,1)$$

and we obtain the Weyl function:

$$m(\zeta)=(T_\zeta 1,1)=-1-\int_0^\infty\left(\frac{1}{\tau+1}-\frac{1}{\tau-\zeta}\right)\rho(\tau)\mathrm{d}\tau-$$

$$-\left[\overline{k_{11}(\zeta)}(S_\zeta\alpha_1,1)+\overline{k_{12}(\zeta)}(S_\zeta\alpha_2,1)\right]\overline{(S_{\bar\zeta}\beta_1,1)}-$$

$$-\left[\overline{k_{21}(\zeta)}(S_\zeta\alpha_1,1)+\overline{k_{22}(\zeta)}(S_\zeta\alpha_2,1)\right]\overline{(S_{\bar\zeta}\beta_2,1)}, \tag{73}$$

which is similar to the classical case.

11. Discussion of the results of spectral decomposition construction

In terms of the concept of branching the resolvent and the associated concepts of vector function and maximal operator branching, spectral decomposition of an arbitrary non-self-adjoint operator of the Friedrichs model was constructed. In addition, it is shown that the Weyl function $m(\zeta)$ for the self-adjoint operator coincides with the classical Weyl function in the case of the Sturm-Liouville operator on the semiaxis.

The advantage of this study is that the approach can be applied for any non-self-adjoint operators and to any operator in Hilbert space. In practical terms, we have advantages, particularly when the operator is given by an ordinary differential equation, as it was illustrated in the examples.

The received results can be applied:

– to spectral decomposition of an arbitrary non-self-adjoint operator;

– to the analysis of solutions of differential equations after the Fourier transform;

– to the analysis of solutions of evolution equations and consequently to the problems of the theory of scattering.

However, the approach of constructing the spectral decomposition for integral non-self-adjoint operators will have weaknesses, and needs further developments.

The topics of these studies can be developed in the case of the spectral decomposition of the operator, which has a multiple continuous spectrum. For example, the Sturm-Liouville operator has a double aliquot spectrum in the entire axis and the Weyl function in this case is a square matrix of size 2×2.

In addition, for research of the spectral decomposition of the operator, the space of vector functions, for example, can be considered.

12. Conclusions

The method of constructing the spectral decomposition of the Friedrichs model non-self-adjoint operator was implemented under the scheme: $T_\zeta \to h_\zeta \to m(\zeta)$.

With the concepts of branching of the resolvent, vector function branching and maximal operator, the following results were received.

1) Sufficient conditions for the existence of the Weyl function $m(\zeta)$ of the non-self-adjoint operator of the Friedrichs model were found, that is:

$$\left\|BS_{\max,\zeta}A^*\right\|_G < 1,$$

$$\left\|AS_{\max,\zeta}B^*\right\|_G < 1, \quad \zeta\in\Omega.$$

2) Satisfying these conditions, the Weyl function, considering definition 5, becomes as follows:

$$m(\zeta)=\left(T_\zeta 1,1\right), \quad \zeta\in\Omega\setminus[0,\infty).$$

The proposed approach formally uses the element 1, which is actually an unbounded functional in the space H and only in a sense, is close to the number 1 (considering definition 5). But it is possible under certain conditions on some operator A to calculate $A\cdot 1$ as a part of the space.

3) The formula for the spectral decomposition of the Friedrichs model non-self-adjoint operator while integrating the resolvent:

$$(\phi,\psi)=-\frac{1}{2\pi i}\lim_{N\to\infty}\int_0^N \left(m_+(\sigma)-\right.$$
$$\left.-m_-(\sigma)\right)\left(\phi, b(\sigma)\right)\left(a(\sigma),\psi\right)\mathrm{d}\sigma+\sum_{\lambda_k}+\sum_{\sigma_j}.$$

The examples are given, where the generalized Weyl function $m(\zeta)=\left(T_\zeta 1,1\right)$ is found.

These results shows that the classical Weyl function in case of the Sturm-Liouville operator on the semiaxis coincides with the Weyl function $m(\zeta)$ for the self-adjoint operator.

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