Робота присвячена дослідженню можливостей технології чисельно-аналітичного варіанту методу граничних елементів (МГЕ) при визначенні внутрішніх силових факторів і напруг в сингулярних точках при вигині тонких ізотропних пластин. Розглянуто найпростіший тип сингулярності – точки прикладання зовнішніх зосереджених сил і моментів. Важливість даної проблеми полягає в тому, що в цих точках внутрішні силові фактори прагнуть до нескінченності і елементарними методами визначити його розміри не вдається. У той же час, дані сингулярні точки є значними концентраторами напружень (як дотичних, так і нормальних), і обчислення меж, до яких прагнуть внутрішні сили і моменти, вкрай важливо для аналізу міцності пластинчастих конструкцій. Для опису зовнішнього навантаження запропоновано використовувати дельта-функцію Дірака двох змінних. Представлені моделі зовнішніх навантажень. Дана пропозиція дозволяє точно обчислити межі, до яких прагнуть поперечні сили, згинальні і крутний момент в сингулярних точках тонких пластин. Моделювання вигину пластин виконано за допомогою варіаційного методу Канторовича-Власова, який повністю сумісний з моделями зовнішнього навантаження. Визначення внутрішніх силових факторів в сингулярних точках пластин виконано при вирішенні крайових задач, які формувалися за алгоритмом (МГЕ). Для програмування і розрахунків притягувалася середовищі MATLAB. Результати розрахунків характеризуються високою точністю і достовірністю, зокрема похибка визначення прогинів пластин в сингулярних точках не перевищує 2,0 %, а похибка згинаючих моментів – не більше 3,0 %. Дано рекомендації щодо вирішення різних видів крайових задач згину пластин з сингулярними точками запропонованим підходом. Встановлено, що точна модель зовнішнього навантаження у вигляді зосереджених сил і моментів принципово дозволяє визначити внутрішні сили і моменти в сингулярних точках тонких пластин за алгоритмом варіаційного методу Канторовича-Власова. По наступний час дані про значення внутрішніх сил і моментів в сингулярних точках пластин відсутні. Також показано, що при розрахунках внутрішніх сил і моментів пластин недоречно застосовувати один член ряду метода Канторовича-Власова, похибки досягають значних величин порядка 43–44 %

Ключові слова: метод граничних елементів, згин ізотропних тонких пластин, зосереджені навантаження, сингулярні точки

UDC 539.3
DOI: 10.15587/1729-4061.2018.142597

INVESTIGATION OF FORCE FACTORS AND STRESSES AT SINGULAR POINTS OF PLATE ELEMENTS IN SPECIAL CRANES

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1. Introduction

Solving the tasks on bending isotropic thin plates by using various methods (double and single trigonometric series, Bubnov-Galerkin approximation, etc.) under the action of concentrated loads (forces and moments) leads to unexpected results.

Kinematic parameters of plates (deflections and rotation angles) can be quite accurately calculated at points of application of concentrated loads, while static parameters (forces and moments) at these points tend to infinity and it is not possible to determine them by using elementary methods. At the same time, these singular points are the stress concentrators, which is why calculating the limits to which forces and moments tend are essential to analyze the strength of plate structures. In this regard, it is of great scientific and practical interest to devise reliable approaches when addressing this kind of problems. Note that plates and plate structures are widely used in cranes, various ships and submarines, aircraft and missiles, in nuclear and power engineering, construction and transportation. Therefore, it is a relevant task to undertake a research aimed at the development of a numerical-analytical variant of the boundary elements method in order to solve problems on bending the thin isotropic plates.

2. Literature review and problem statement

Paper [1], which addresses studying the bending of thin plates, describes different approaches to solving the problems on bending at arbitrary external load and under various supporting conditions [1]. In this case, examples of calculations and tables of the stressed-strained state of plate bending are given only under the action is relatively simple external loads. For example, it is proposed to use, in order to solve problems on plate bending, a series, the Fourier integrals, Fredholm integral equations, the matrix forms of a finite element method (FEM), variational methods, a theory of functions of a complex variable, the methods of collocation, grids, a mixed method from the construction mechanics [2]. Paper [3] considered the first-order equations for plates at shear deformation, which are derived taking into consideration the kinematic assumptions of the Reissner-Bollé theory, but with respect to the equations of equilibrium in the deformed configuration of a plate. The derived system of differential equations is applicable to the calculation of stresses in isotropic plates and it holds for thin and moderately thick plates. Study [4] addressed solutions to the problems on plate bending using the generalized equations from a finite difference method (FDM).

The serious shortcoming of the considered calculation methods is the lack of a universal approach when dealing with specific problems. Thus, there is a large volume of computational operations, large dimensionality of the resolving system of equations, etc.

Lately, the problems on plate bending have been solved by using professional software for a finite element method (FEM) such as Ansys, Solid Works, Abaqus. The method has been widely applied because of a relatively simple logic in the algorithm, but it is characterized by a large number of arithmetic operations [5] and the complexity of building an exact matrix of stiffness for the flat shape of bending the structural elements. That does not make it possible to obtain accurate and reliable results regardless of the extent of structure discretization. More perfect is the application of the boundary element method algorithm (BEM) [6]. This method employs a precise system of differential equations for a problem, a strict mathematical procedure for building its solution, and a rather simple, in terms of logic, process for forming a resolving system of linear algebraic equations within a boundary-value problem on stability [7]. In addition, as shown in paper [8], BEM makes it possible to obtain the exact values for parameters of the problem (efforts, displacements, strains, frequencies of natural oscillations, critical forces at stability loss) both at the border and inside the region. Papers [9, 10] showed that BEM possesses the simplest algorithm's logic among other numerical methods, a good convergence of the solution, high stability of arithmetic operations, and a rather small accumulation of rounding errors at numerical operations [11]. In this case, BEM demonstrates a simple logic of the algorithm, a good convergence, minimal errors in the results of solution, high stability, and could be applied when calculating the internal force factors in thin isotropic plates.

An analysis of publications [1–8] revealed that none of the known approaches considers determining the internal forces and moments at the points of application of concentrated loads. Given this, there are no values for the shear forces, bending and torsional moments at these singular points. In practical calculations, concentrated forces and moments are distributed over a certain finite area, which eliminates discontinuities in internal efforts and improves the convergence of series [11]. The reason for the singularity of these points is in the model of the force concentrated at a point. If we cut around the point of force application a square element with sides Δx, Δy and direct $\Delta x \to 0$, $\Delta y \to 0$, then, in order to balance the concentrated force F, the intensity of shear forces and moments Q_x, Q_y, M_x, M_y should grow to infinity. This follows from the equations of equilibrium. For example, for lateral forces

$$\sum Q_i = F - 2Q_x\Delta x - 2Q_y\Delta y = 0 \to Q_x = Q_y = \frac{F}{4\Delta x},$$

when

$\Delta x \to 0, Q_x = Q_y \to \infty$ and so on.

The accuracy of the model for the concentrated force, distributed over a finite area, is insufficient, and it is obvious that it is necessary to use more effective models for concentrated loads.

In this regard, the development of a more precise and efficient approach for determining the internal forces of plates using a numerical-analytical variant of the method of boundary elements is a practically required task.

3. The aim and objectives of the study

The aim of this work is to construct rigorous mathematical models for transverse concentrated forces and moments acting on thin plates, and to practically apply the proposed models in the variational Kantorovich-Vlasov method when solving problems on plate bending using an algorithm of the numeral-analytical variant of BEM.

To accomplish the aim, the following tasks have been set:

– to represent an external concentrated load on a plate as a continuous, differentiable and integrable function containing the Dirac delta-function and its derivatives from two variables;

– to construct a one-dimensional model of plate bending using the variational Kantorovich-Vlasov method;

– to solve analytically a one-dimensional model of plate bending (construction of solution to the Cauchy problem) using mathematically rigorous functions of external concentrated forces and moments;

– to calculate deflections and bending moments at the singular points of plates using the programming and simulation environment MATLAB.

4. Development of a mathematical apparatus for calculating the plates

4. 1. Models of concentrated loads

Let us start with models for concentrated loads. The most accurate and mathematically strict are the models of forces and moments with the generalized functions-delta-Dirac functions and its derivatives (Fig. 1).

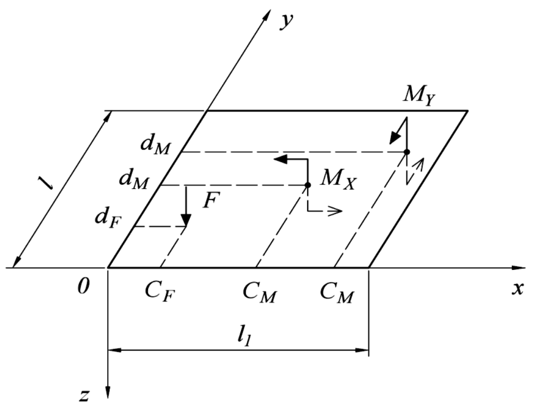

Fig. 1. Concentrated forces and moments on a plate

Load on the plate under the action of concentrated forces and moments can be represented by the following analytical expression

$$q(x,y)=F\delta(x-c_F)\delta(y-d_F)+$$
$$+M_x\delta'(x-c_M)\delta(y-d_M)+M_y\delta(x-c_M)\delta'(y-d_M), \qquad (1)$$

where F, M_x, M_y are the concentrated loads (Fig. 1); $\delta(x-c_F)$ $\delta(y-d_F)$ is the Dirac delta function of two variables; $\delta'(x-c_M)\,\delta'(y-d_M)$ are the first derivatives of delta-functions of one variable.

Expressions (1) follow at executing a limit transition for the concentrated force and moment (Fig. 2).

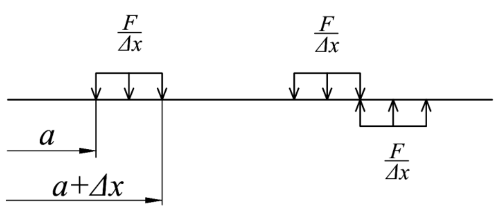

Fig. 2. Models of concentrated force and moment

Thus, for the concentrated force, we obtain

$$q(x)=\lim_{\Delta x\to 0}\frac{F}{\Delta x}\left[\mathrm{H}(x-a)-\mathrm{H}(x-a-\Delta x)\right]=$$
$$=F\delta(x-a), \qquad (2)$$

where $H(x-a)$; $H(x-a-\Delta x)$ are the single Heaviside functions.

For plates, such a limiting transition must be performed for two directions. In this case, the Dirac delta function of two variables will be represented as a product of two delta functions of a single variable [12]. Limiting transitions for the concentrated bending moments are performed in a similar fashion.

Expression (1) is a continuous, differentiable and integrable function throughout the entire region occupied by the plate. According to theory, the Dirac delta function and its derivatives are determined as the limits of the corresponding pulse functions [13]. This provision will make it possible to accurately calculate limits sought, for example, by the transverse forces, bending and torsional moments in plates. However, such a possibility comes only if the plate bending model contains a procedure for double integration. In this regard, the variational method by Kantorovich-Vlasov fully satisfies this requirement. Equation for a bending of isotropic plates is reduced to the form

$$\frac{\partial^4 w(x,y)}{\partial y^4}+2\frac{\partial^4 w(x,y)}{\partial x^2\partial y^2}+\frac{\partial^4 w(x,y)}{\partial x^4}=q(x,y)/D, \qquad (3)$$

where $w(x, y)$ is the deflection of the midplane of a plate; $q(x, y)$ is the transverse load from expression (1); $D=Eh^3/12(1-\mu^2)$ is the cylindrical rigidity; E is the elasticity modulus of first kind; h is the plate thickness; μ is the Poisson's ratio.

In the Kantorovich-Vlasov method, the kinematic and static parameters of the plate are represented by a functional series [14, 15]. For example, deflection and a bending moment take the form

$$w(x,y)=W_1(y)X_1(x)+$$
$$+W_2(y)X_2(x)+...=\sum_{i=1}^{\infty}w_i(x,y); \qquad (4)$$

$$A_{31}=-(1-\mu)^2 r^3 A\phi_4/2; \qquad (5)$$

where $X_i(x)$, $i=1, \infty$ is the assigned system of functions from variable x; $W_i(y)$, $i=1, \infty$ is the desired system of functions from variable y.

It is convenient to consider, as the assigned system of functions $X_i(x)$, the shapes of natural oscillations of a beam with supports, similar to the conditions for supporting the edges, parallel to the Oy axis (Fig. 1). The essence of mathematical transformation of equation (3) in the Kantorovich-Vlasov method is the substitution of a series (4) into equation (3), the multiplication of both sides by the system of functions $X_i(x)$ and the integration within the width of the plate from 0 to l_1. We receive a system of linear differential equations for the desired functions $W_i(y)$. At a pivot of longitudinal edges of the plate

$$X_i(x)=\sin\frac{i\pi x}{l_1}, \qquad (6)$$

and the joint system of linear differential equations is split into separate ordinary differential equations

$$W_1^{IV}(y)-2r_1^2W_1''(y)+S_1^4W_1(y)=q_1(y)/D;$$
$$W_2^{IV}(y)-2r_2^2W_2''(y)+S_2^4W_2(y)=q_2(y)/D; \qquad (7)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where

$$r_i^2=-B_i/A_i;\quad S_i^4=C_i/A_i;$$

$$q_i(y)=\int_0^{l_1}q(x,y)X_i(x)dx/A_i; \qquad (8)$$

$$A_i=\int_0^{l_1}X_i^2(x)\mathrm{d}x;$$

$$B_i=\int_0^{l_1}X_i''(x)X_i(x)\mathrm{d}x;$$

$$C_i=\int_0^{l_1}X_i^{IV}X_i(x)\mathrm{d}x.$$

The system of functions $W_1(y)$, $W_2(y)$, ... is found as a solution to equations (7) taking into consideration supporting conditions at the edges of the plate, parallel to the Ox axis (Fig. 1). Under condition for support of the plate, different from a pivot, one can apply solutions to equations (7). As shown in paper [1], the maximum error in the calculation results does not exceed 3.0 %. The complete solution to equations (7) can be represented in a matrix form as follows (here and below the indices of the terms in a series (4) are omitted).

$$\begin{Vmatrix} DW(y) \\ D\theta(y) \\ M(y) \\ Q(y) \end{Vmatrix} = \begin{Vmatrix} A_{11} & A_{12} & -A_{13} & -A_{14} \\ A_{21} & A_{22} & -A_{23} & -A_{13} \\ -A_{31} & -A_{32} & A_{22} & A_{12} \\ -A_{41} & -A_{31} & A_{21} & A_{11} \end{Vmatrix} \cdot \begin{Vmatrix} DW(0) \\ D\theta(0) \\ M(0) \\ Q(0) \end{Vmatrix} + \int_0^y \begin{Vmatrix} A_{14}(y-\xi) \\ A_{13}(y-\xi) \\ -A_{12}(y-\xi) \\ -A_{11}(y-\xi) \end{Vmatrix} q(\xi)d(\xi), \quad (9)$$

where $W(y)$, $\theta(y)$, $M(y)$, $Q(y)$ are the deflection, rotation angle, bending moment, and transverse force of a conditional beam that replaces the plate in the direction of the oy axis. Two values for the initial parameters are known from the conditions for supporting the edges of the plate, and the other two initial parameters can be determined from a boundary-value problem using the algorithm of BEM. In a general form, function $W(y)$ is determined from expression

$$DW(y) = A_{11}(y)\cdot DW(0) + A_{12}(y)\cdot D\theta(0) - -A_{14}(y)\cdot Q(0) + \int_0^y A_{14}(y-\xi) q\xi \mathrm{d}\xi. \quad (10)$$

The function of a plate deflection $w(x, y)$ is determined in full, and other parameters of the bending are calculated via the appropriate differentiation. Thus, at first integration (8) one computes the limit of the function for variable x; at second integration (10) one calculates the limit of functions for variable y. In general, it makes it possible to calculate the limit of functions of two variables (transverse forces, bending and torsion moments) at singular points of the concentrated loads application. Let us consider examples that confirm our conclusion.

4. 2. A square plate with a hinged support along a contour, loaded with concentrated force F in the center

In this case, function $X_i(x)$ is defined by expression (6), and the coefficients of differential equations (7)

$$r_i = s_i; \quad \omega_i = i\pi, \quad (11)$$

where i is the number of a series (4) term; ω_i are the frequency of natural oscillations.

The fundamental orthonormalized functions of solution (9) and components of the load in Fig. 1 after all integration and transformation operations will take the form

$$\phi_1 = ychry; \quad \phi_2 = chry;$$

$$\phi_3 = shry; \quad \phi_4 = yshry;$$

$$A_{11} = \phi_2 - (1-\mu) r\phi_4 / 2;$$

$$A_{12} = (1-\mu)\phi_1 / 2 + (1+\mu)\phi_3 / (2r);$$

$$A_{13} = \phi_4 / (2rA);$$

$$A_{14} = (r\phi_1 - \phi_3) / (2r^3 A);$$

$$A_{21} = r(1+\mu)\phi_3 / 2 - (1-\mu) r^2 \phi_1 / 2;$$

$$A_{22} = \phi_2 + (1-\mu) r\phi_4 / 2;$$

$$A_{23} = \phi_1 / (2A) + \phi_3 / (2rA);$$

$$A_{31} = -(1-\mu)^2 r^3 A\phi_4 / 2;$$

$$A_{32} = A\left[(1-\mu)^2 r^2 \phi_1 + (1-\mu)(3+\mu) r\phi_3\right] / 2;$$

$$A_{41} = A\left[(1-\mu)^2 r^4 \phi_1 - (1-\mu)(3+\mu) r^3 \phi_3\right] / 2, \quad (12)$$

$$B_{11} = F\gamma_F(\omega)\left[r\phi_1(y-d_F)_+ - \phi_3(y-d_F)_+\right] / (2r^3 A) + +M_y \gamma_{my}(\omega) + \phi_4(y-d_m)_+ / (2rA) + +M_x \gamma_{mx}(\omega)\left[r\phi_1(y-d_m)_+ - \phi_3(y-d_m)_+\right] / (2r^3 A);$$

$$B_{31} = F\gamma_F(\omega)\left[(1-\mu) r\phi_1(y-d_F)_+ + (1+\mu)\phi_3(y-d_F)_+\right] / (2r) + +M_y \gamma_{my}(\omega)\left[2r\phi_2(y-d_m)_+ + (1-\mu) r^2 \phi_4(y-d_m)\right] / (2r) + +M_x \gamma_{mx}(\omega)\left[(1-\mu) r\phi_1(y-d_m)_+ + (1+\mu)\phi_3(y-d_m)_+\right] / (2r);$$

$$B_{41} = F\gamma_F(\omega)\left[\phi_2(y-d_F)_+ - (1-\mu) r\phi_4(y-d_F)_+ / 2\right] + +M_y \gamma_{my}(\omega)\left[(1+\mu) r\phi_3(y-d_m)_+ - (1-\mu) r^2 \phi_1(y-d_m)_+\right] / 2 + +M_x \gamma_{mx}(\omega)\left[\phi_2(y-d_m)_+ - (1-\mu) r\phi_4(y-d_m)_+ / 2\right];$$

$$\gamma_F(\omega) = \sin(i\pi c_F / l_1); \quad \gamma_{my} = \sin(i\pi c_m / l_1).$$

In these expressions, the "+" symbol at the bottom of the parentheses denotes a spline function of the following type

$$\phi_1(y-d_F)_+ = \begin{cases} 0, (y-d_F) \le 0; \\ (y-d_F) ch\left[r(y-d_F)\right], (y-d_F) > 0. \end{cases} \quad (13)$$

When programming, spline-functions are represented in the following form:

$$\phi_1(y-d_F)_+ = \phi(y-d_F)\cdot \mathrm{H}(y-d_F), \quad (14)$$

where $H(y-d_F)$ is the single Heaviside function with a shift to point d_F.

The unknown initial parameters of function $W(y)$ (10) can be determined while solving the boundary value problem for a conditional beam using BEM [1, 16]. The system of linear algebraic equations at initial data μ=0,3, F=1, d_F=l/2, c_F=l_1/2, l_1=l=1 will take the following form

$$\begin{Vmatrix} & A_{12} & & -A_{14} \\ -1 & A_{22} & & -A_{13} \\ & -A_{32} & & A_{12} \\ & -A_{31} & -1 & A_{11} \end{Vmatrix} \cdot \begin{Vmatrix} D\theta(l) \\ D\theta(0) \\ Q(l) \\ Q(0) \end{Vmatrix} = \begin{Vmatrix} -B_{11}(l) \\ -B_{21}(l) \\ B_{31}(l) \\ B_{41}(l) \end{Vmatrix}. \quad (15)$$

By determining, based on the solution to a systems of equations (15), the unknown initial parameters $D\theta(0)$, $Q(0)$, we construct a series for deflection $w(x, y)$ (4). Other parameters are computed using the formulae form a plate bending theory. Table 1 gives the results of calculating five terms of series (4), (5), where there are no even terms since they are equal to zero.

Table 1

Values for deflections and moments in the hinge-supported system

Number of a series term	Frequency of natural oscillations	Deflection in the center of the plate $w(l_1/2, l/2)$	Bending moment in the center of the plate $M_y(l_1/2, l/2)$
1	$\omega_1=\pi$	$107{,}665\cdot10^{-4}Fl_1^2D$	$21{,}756\cdot10^{-2}F$
3	$\omega_3=3\pi$	$5{,}962\cdot10^{-4}Fl_1^2D$	$6{,}901\cdot10^{-2}F$
5	$\omega_5=5\pi$	$1{,}290\cdot10^{-4}Fl_1^2D$	$4{,}138\cdot10^{-2}F$
7	$\omega_7=7\pi$	$0{,}470\cdot10^{-4}Fl_1^2D$	$2{,}956\cdot10^{-2}F$
9	$\omega_9=9\pi$	$0{,}221\cdot10^{-4}Fl_1^2D$	$2{,}299\cdot10^{-2}F$
$\sum$		$115{,}609\cdot10^{-4}Fl_1^2D$	$38{,}049\cdot10^{-2}F$

The error (precise data are taken from reference books) for deflections

$$\Delta_1=\frac{116{,}0-115{,}609}{116{,}0}\cdot100\,\%=0{,}34\,\%. \quad (16)$$

Values for the bending moments at a singular point are missing in the scientific literature. Note that the error of using a single term in a series from the Kantorovich-Vlasov method [1] for a singular point is

$$\Delta_2=\frac{38{,}049-21{,}759}{38{,}049}\cdot100\,\%=42{,}82\,\%. \quad (17)$$

4. 3. A square plate, rigidly fixed along a perimeter, loaded with concentrated force *F* in the center

Under these conditions, $|S|>|r|$; the frequencies of natural oscillations of a beam with rigidly fixed supports are given in Table 2.

Table 2

Dimensionless frequencies of natural oscillations

Conditions for supporting the beams		
$l_1=1$	$l_1=1$	$l_1=1$
$\omega_1=4{,}730040745$	$\omega_1=3{,}926602312$	$\omega_1=\pi$
$\omega_3=10{,}99560784$	$\omega_3=10{,}21017613$	$\omega_3=3\pi$
$\omega_5=23{,}56194490$	$\omega_5=16{,}49336143$	$\omega_5=5\pi$
$\omega_7=23{,}56194490$	$\omega_7=22{,}77654673$	$\omega_7=7\pi$
$\omega_9=29{,}84513021$	$\omega_9=29{,}05973204$	$\omega_9=9\pi$

The assigned system of functions for a beam with rigidly fixed supports is described by the following expression [1]

$$X(x)=\sin(\omega x/l_1)\cdot sh(\omega x/l_1)-$$
$$-a_z\left[\cos(\omega x/l_1)-ch(\omega x/l_1)\right], \quad (18)$$

where coefficient a_z=(sinω–shω)/(cosω–chω).

Fundamental functions and expressions from the external load in a given case take the form

$$\alpha=\sqrt{(S^2+r^2)/2};\quad \beta=\sqrt{(S^2-r^2)/2};$$

$$\phi_1=\mathrm{ch}\alpha y\sin\beta y;\quad \phi_2=\mathrm{ch}\alpha y\cos\beta y;$$

$$\phi_3=\mathrm{sh}\alpha y\cos\beta y;\quad \phi_4=\mathrm{sh}\alpha y\sin\beta y;$$

$$A_{11}=\phi_2-(1-\mu)r^2\phi_4/(2\alpha\beta); \quad (19)$$

$$A_{12}=(S^2-\mu r^2)\phi_1/(2\beta S^2)+(S^2+\mu r^2)\phi_3/(2\alpha S^2);$$

$$A_{13}=\phi_4/(2\alpha\beta A);\quad A_{14}=(\alpha\phi_1-\beta\phi_3)/(2\alpha\beta S^2A);$$

$$A_{21}=(S^2+\mu r^2)\phi_3/(2\alpha)-(S^2-\mu r^2)\phi_1/(2\beta);$$

$$A_{22}=\phi_2+(1-\mu)r^2\phi_4(2\alpha\beta);$$

$$A_{23}=(\alpha\phi_1+\beta\phi_3)/(2\alpha\beta A);$$

$$A_{31}=A\left[\mu r^4(2-\mu)-S^4\right]\phi_4/(2\alpha\beta);$$

$$A_{32}=A\left[-S^4+2(1-\mu)S^2r^2+\mu^2r^4\right]\phi_1/(2\beta S^2)+$$
$$+A\left[S^4+2(1-\mu)S^2r^2-\mu^2r^4\right]\phi_3/(2\alpha S^2);$$

$$A_{41}=A\left[-S^4+2(1-\mu)S^2r^2+\mu^2r^4\right]\phi_1/(2\beta)-$$
$$-A\left[S^4+2(1-\mu)S^2r^2-\mu^2r^4\right]\phi_3/(2\alpha);$$

$$B_{11}=F\gamma_F(\omega)\left[\alpha\phi_1(y-d_F)_+-\beta\phi_3(y-d_F)_+\right]/(2\alpha\beta S^2A)+$$
$$+M_y\gamma_{my}(\omega)\phi_4(y-d_m)_+/(2\alpha\beta A)+$$
$$+M_x\gamma_{mx}(\omega)\left[\alpha\phi_1(y-d_m)_+-\beta\phi_3(y-d_m)_+\right]/(2\alpha\beta S^2A);$$

$$B_{21}=F\gamma_F(\omega)\phi_4(y-d_F)_+/(2\alpha\beta A)+$$
$$+M_y\gamma_{my}(\omega)\left[\alpha\phi_1(y-d_m)_++\beta\phi_3(y-d_m)_+\right]/(2\alpha\beta A)+$$
$$+M_x\gamma_{mx}(\omega)\phi_4(y-d_m)_+/(2\alpha\beta A);$$

$$B_{31}=F\gamma_F(\omega)\begin{bmatrix}(S^2+\mu r^2)\alpha\phi_1(y-d_F)_++\\+(S^2+\mu r^2)\beta\phi_3(y-d_F)_+\end{bmatrix}/(2\alpha\beta S^2)+$$
$$+M_y\gamma_{my}(\omega)\begin{bmatrix}2\alpha\beta\phi_2(y-d_m)_++\\+(1-\mu)r^2\phi_4(y-d_m)_+\end{bmatrix}/(2\alpha\beta)+$$
$$+M_x\gamma_{mx}(\omega)\begin{bmatrix}(S^2-\mu r^2)\alpha\phi_1(y-d_m)_++\\+(S^2+\mu r^2)\beta\phi_3(y-d_m)_+\end{bmatrix}/(2\alpha\beta S^2);$$

$$B_{41}=F\gamma_F(\omega)\left[\phi_2(y-d_F)_+-(1-\mu)r^2\phi_4(y-d_F)_+/(2\alpha\beta)\right]+$$
$$M_y\gamma_{my}(\omega)\begin{bmatrix}(S^2+\mu r^2)\phi_3(y-d_m)_+/(2\alpha)-\\-(S^2-\mu r^2)\phi_1(y-d_m)_+/(2\beta)\end{bmatrix}+$$
$$+M_x\gamma_{mx}(\omega)\left[\phi_2(y-d_m)_+-(1-\mu)r^2\phi_4(y-d_m)_+/(2\alpha\beta)\right];$$

$$\gamma_F(\omega)=\sin(\omega c_F/l_1)-\mathrm{sh}(\omega c_F/l_1)-$$
$$-\alpha_*\left[\cos(\omega c_F/l_1)-\mathrm{ch}(\omega c_F/l_1)\right];$$

$$\gamma_{my}(\omega)=\sin(\omega c_m/l_1)-\mathrm{sh}(\omega c_m/l_1)-$$
$$-\alpha_*\left[\cos(\omega c_m/l_1)-\mathrm{ch}(\omega c_m/l_1)\right];$$

$$\gamma_{mx}(\omega)=\omega\begin{Bmatrix}-\cos(\omega c_m/l_1)+\mathrm{ch}(\omega c_m/l_1)-\\-\alpha_*\left[\sin(\omega c_m/l_1)+\mathrm{sh}(\omega c_m/l_1)\right]\end{Bmatrix}l_1;$$

$$\alpha_*=(\sin\omega-sh\omega)/(\cos\omega-ch\omega).$$

The boundary-value problem for determining the unknown initial parameters of the rigidly fixed beam, according to the BEM algorithm, will take the form (μ=0,3, F=1, $d_F=l/2$, $c_F=l_1/2$, $l_1=l=1$).

$$\begin{Vmatrix} & & -A_{13} & -A_{14}\\ & & -A_{23} & -A_{13}\\ -1 & & A_{22} & A_{12}\\ & -1 & A_{21} & A_{11}\end{Vmatrix}\cdot\begin{Vmatrix}M(l)\\Q(l)\\M(0)\\Q(0)\end{Vmatrix}=\begin{Vmatrix}-B_{11}(l)\\-B_{21}(l)\\B_{31}(l)\\B_{41}(l)\end{Vmatrix}. \qquad (20)$$

Similar to the first example, we calculate values for the deflection and bending moments in a rigidly fixed plate for five terms from a series (4). Table 3 gives the results of computations derived using the MATLAB programming environment.

Table 3

Values for deflections and bending moments in a rigidly fixed plate

Number of a series term	Frequency of natural oscillations	Deflection in the center of the plate $w(l_1/2,l/2)$	Bending moment in the supporting cross section $M_y(l_1/2,0)$	Bending moment in the center of the plate $M_y(l_1/2,l/2)$
1	ω_1	$51{,}52\cdot10^{-4}Fl_1^2/D$	$-11{,}897\cdot10^{-2}F$	$19{,}212\cdot10^{-2}F$
3	ω_3	$3{,}895\cdot10^{-4}Fl_1^2/D$	$-0{,}314\cdot10^{-2}F$	$6{,}148\cdot10^{-2}F$
5	ω_5	$0{,}999\cdot10^{-4}Fl_1^2/D$	$-0{,}0096\cdot10^{-2}F$	$3{,}877\cdot10^{-2}F$
7	ω_7	$0{,}391\cdot10^{-4}Fl_1^2/D$	$-0{,}000266\cdot10^{-2}F$	$2{,}819\cdot10^{-2}F$
9	ω_9	$0{,}191\cdot10^{-4}Fl_1^2/D$	$-6{,}208\cdot10^{-2}F$	$2{,}215\cdot10^{-2}F$
$\sum$		$56{,}966\cdot10^{-4}Fl_1^2/D$	$-12{,}221\cdot10^{-2}F$	$34{,}272\cdot10^{-2}F$

The error for deflections

$$\Delta_3=\frac{56{,}996-56{,}0}{56{,}0}\cdot100\,\%=1{,}78\,\%. \qquad (21)$$

For the bending moment in the supporting cross section

$$\Delta_4=\frac{-12{,}221+12{,}57}{12{,}57}\cdot100\,\%=2{,}78\,\%. \qquad (22)$$

The error of applying the first term from the Kantorovich-Vlasov method is, for a non-singular point,

$$\Delta_5=\frac{-11{,}897+12{,}57}{12{,}57}\cdot100\,\%=5{,}35\,\%, \qquad (23)$$

for a singular point

$$\Delta_6=\frac{34{,}272-19{,}212}{34{,}272}\cdot100\,\%=43{,}94\,\%. \qquad (24)$$

Consequently, the application of a single term in a series of Kantorovich-Vlasov is acceptable (an error of 5.35 %) if the plate's points are not singular. At singular points, the error of using a single term from a series is high (44 %); therefore, here it is necessary to keep at least five terms of a series.

5. Discussion of results of studying the stressed state of plates at singular points

The research reported in this paper demonstrates that a precise, mathematically rigorous model of concentrated loads makes it possible to determine, at high accuracy, the stressed state at the singular points of plates, which cannot be performed using the existing methods. This is the great advantage of the proposed approach. It has become possible to obtain the qualitative and quantitative estimates of the stressed state of different plate structures, which directly affects characteristics such as strength, durability, reliability, maintainability, and reliability. From this point of view, our work is useful in the design, manufacture, and operation of machine-building and other structures. The proposed approach could be developed with respect to shells and shell structures.

Construction of precise models of the concentrated external loads using the Dirac delta function and its derivatives in the variational Kantorovich-Vlasov method makes it possible to calculate the limits sought by the internal force factors of thin plates at singular points. The calculations performed have confirmed this conclusion.

Existing elementary models for calculating the stressed-deformed state of plates do not provide for the possibility to compute the stressed state at points where the concentrated loads are applied. Therefore, the proposed solutions are substantially more accurate to reveal those concentrations of stresses that occur at singular points.

It is worth noting that in this paper in the analysis of the stressed state of thin plates there are almost no constraints for the design of machines and mechanisms, their geometry and materials. This is explained by the great versatility of the proposed solutions to the problems on bending thin plates.

The paper presents the analytical solutions to the differential equation of a thin plate bending for special cases of the external load. For this reason, they are among the most effective representations of the considered problems. The disadvantages of these solutions come down to large cumbersome resolving equations compared to existing solutions to the problems on plate bending. Therefore, programming these models requires special care and careful adjustment of ready programs. We note that these difficulties can be overcome.

A given technology could be applied for the calculation of various shells with respect to the effect of concentrated loads. In this case, significant mathematical difficulties emerge when building the appropriate solutions compared to plates. However, the theory of shells has gained much experience in building analytical models, which allows us to argue on the possibility to overcome these difficulties.

6. Conclusions

1. We have constructed strict mathematical models of concentrated forces and moments as an external transverse load on plates using the Dirac delta function and its derivatives from two variables. This makes it possible to take into consideration, qualitatively and quantitatively, the concentrated loads when calculating the stressed state of plates at singular points.

2. A one-dimensional plate bending model has been built based on the variational Kantorovich-Vlasov method. The derived model allows the simplification of the procedure for obtaining an analytical solution to the Cauchy problems for plates.

3. We present an analytical solution to the one-dimensional model of plate bending (a solution to the Cauchy problem) in a matrix form. That opens up the prospect for applying the algorithm of a numerical-analytical variant of the boundary elements method in order to solve the boundary value problems on plate bending.

4. Calculations of deflections and bending moments at the singular points of plates have been performed when solving the boundary value problems applying the BEM algorithm in the MATLAB programming environment. It has been shown that a combination of the one-dimensional model of plate bending and a mathematically rigorous model of external loads in the variational Kantorovich-Vlasov method makes it possible to qualitatively and quantitatively estimate the magnitudes of stresses at the singular points of plates.

5. It is shown that the accuracy of the calculations is high enough, in particular the accuracy of determining the deflections at the singular points of plates does not exceed 2.0 %, and for bending moments – 3.0 %; one cannot be limited to using only the first term in a series of the variational Kantorovich-Vlasov method; the error reaches 43–44 %. It is required to keep at least five terms of the series.

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