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ANALYTICAL METHOD TO STUDY A MATHEMATICAL MODEL OF WAVE PROCESSES UNDER TWO-POINT TIME CONDITIONS

Z. Nytrebych
Doctor of Physical and Mathematical Sciences, Professor, Head of Department*
E-mail: znytrebych@gmail.com
V. Ilkiv
Doctor of Physical and Mathematical Sciences, Professor*
E-mail: iilkiv@i.ua
P. Pukach
Doctor of Technical Sciences, Professor, Head of Department**
E-mail: ppukach@gmail.com
O. Malanchuk
PhD
Department of Biophysics
Danylo Galytsky Lviv National Medical University
Pekarska str., 69, Lviv, Ukraine, 79010
E-mail: Oksana.Malan@gmail.com
I. Kohut
PhD, Associate Professor**
E-mail: ikohutua@yahoo.com
A. Senyk
PhD, Associate Professor
Department of Engineering Mechanics (Weapons and Equipment of Military Engineering Forces)
Hetman Petro Sahaidachnyi National Army Academy
Heroiv Maidanu str., 32, Lviv, Ukraine, 79012
E-mail: andrij.sen@gmail.com
*Department of Mathematics***
Department of Computational Mathematics and Programming*
***Lviv Polytechnic National University
Bandery str., 12, Lviv, Ukraine, 79013

Дослідження та аналіз динамічних процесів у коливальних системах тісно зв'язані зі встановленням точних або наближених аналітичних розв'язків задач математичної фізики, які моделюють такі системи. Математичні моделі поширення хвиль у коливальних системах за певних початкових умов у фіксований момент часу є добре відомими в літературі. Однак хвильові процеси у довгомірних конструкціях за умови дії лише зовнішньої сили і заданих станів процесу у два моменти часу є мало вивченими. Такі процеси змодельовано двоточковою за часом задачею для неоднорідного хвильового рівняння в необмеженій області $t>0$, $x\in\mathbb{R}^s$. У моделі враховано задання лінійної комбінації невідомої амплітуди коливань та швидкості її зміни у два моменти часу. Двоточкова задача загалом є некоректною крайовою задачею, оскільки відповідна однорідна задача має нетривіальні розв'язки. Встановлено клас квазіполіномів як клас існування єдиного розв'язку задачі. Цей клас не містить нетривіальних елементів ядра задачі, що забезпечує єдиність розв'язку задачі. У вказаному класі запропоновано точний метод побудови розв'язку. Суть методу полягає в тому, що розв'язок задачі зображається у вигляді дії диференціального виразу, символом якого є права частина рівняння, на деяку функцію параметрів. Функція спеціальним чином конструюються за рівнянням та двоточковими умовами і має особливості, пов'язані з нулями знаменника – характеристичного визначника задачі.

Метод проілюстровано для опису коливальних процесів нескінченної струни та мембрани.

Головним практичним застосуванням розробленого методу є можливість адекватного математичного моделювання коливальних систем, яке враховує можливість керування параметрами системи. Таке керування параметрами дозволяє здійснювати оптимальний синтез та проектування параметрів відповідних технічних систем з метою аналізу та врахування особливостей динамічних режимів коливань

Ключові слова: коливальні системи, математичні моделі хвильових процесів, диференціально-символьний метод, двоточкова задача, хвильове рівняння

1. Introduction

Adequate mathematical modeling of wave processes in oscillatory systems plays an important role in modern approaches to solving a series of scientific and engineering tasks, which arise in the problems on analysis, synthesis, and optimization of parameters for machine-building structures. Mathematical and numerical modelling are important in understanding the essence of phenomena and processes that are studied in modern mechanics and physics. Considering the adequacy of mathematical models and methods of analysis contributes to the interpretation of existing, as well as prediction and discovery of new, phenomena. Modern industrial equipment is operated in a wide range of parameters, particularly at high speeds, high pressures and energies, etc. The specified reasons necessitate the study and analysis of complex enough problems from the point of view of the present status of the development of mechanical theory of oscillations and general approaches of mathematical physics [1, 2]. Studying oscillatory and wave phenomena in elastic structures under the action of different kinds of perturbations (force, inertial, and kinematic) is one of the classic tasks of applied mechanics [3–5]. Increased attention to the theoretical research into this field is predetermined not only by the logic in the development of basics of the dynamics of deformed systems, but also by interests of a wide variety of practical applications of oscillatory systems in engineering, construction, and other sectors of national economy.

A relevant problem of the current state of research related to applied mechanics is the development of new, as well as the extension of existing, asymptotic approaches for studying the mathematical models of oscillatory systems. The specified models are described by the problems from mathematical physics for the case of one-dimensional and two-dimensional equations with partial derivatives [6, 7]. Note, on the one hand, the non-perturbed (linear) analogs to such equations do not make it possible to apply a well-known method of separation of variables. On the other hand, the specified approaches to many classes of problems have been, up to now, the only possible analytical method to study complex systems. The asymptotic methods from nonlinear mechanics have made it possible to explore a wide class of mechanical oscillatory systems for the case of a nonlinear dependence of oscillation amplitude on elastic forces and the forces of resistance.

However, in most cases, only if one has precise analytical solutions to linear (non-perturbed) problems [8, 9] it becomes possible to further apply the asymptotic methods from nonlinear mechanics. The task on searching for such solutions has remained relevant.

Among the problems in mathematical physics for partial differential equations of the following form

$$L(\partial_t,\partial_x)U(t,x)\equiv$$
$$\equiv\left[\partial_t^2+c_1(\partial_x)\partial_t+c_2(\partial_x)\right]U(t,x)=f(t,x), \tag{1}$$

in which the coefficients of equation are differential polynomials with complex coefficients,

$$\partial_t=\partial/\partial t,\ \ \partial_x=\left(\partial_{x_1},\ldots,\partial_{x_s}\right),\ \ \partial_{x_j}=\partial/\partial x_j,$$
$$x=(x_1,\ldots,x_s)\in\mathbb{R}^s,\ \ s\in\mathbb{N},$$

$f(t,x)$ is the assigned function, the Cauchy problem has been most fully explored for the time being [8, 9], that is, the task on finding a solution to equation (1) in a domain $(0,\infty)\times\mathbb{R}^s$, which satisfies initial conditions

$$U(0,x)=\varphi_1(x),\quad \partial_t U(0,x)=\varphi_2(x), \tag{2}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are the determined functions.

If magnitude $U(t,x)$ and its derivative $\partial_t U(t,x)$ were determined not at the same time $(t=0)$, but at two moments (for example, when $t=0$ and $t=\tau$, where $\tau>0$), possibly very close moments, we would then obtain the problem on finding a solution to equation (1), which, in domain $t>0,\ x\in\mathbb{R}^s$, satisfies such two-point conditions for time

$$U(0,x)=\psi_1(x),\quad \partial_t U(\tau,x)=\psi_2(x) \tag{3}$$

or

$$\partial_t U(0,x)=\psi_1(x),\quad U(\tau,x)=\psi_2(x). \tag{4}$$

In addition to conditions (3) and (4), other conditions are considered as well, which combine a first condition (3) and a second condition (4) or a first condition (4) and a second condition (3). All these conditions are obviously a particular case of the more general local two-point conditions

$$A(\partial_t)U(t,x)\big|_{t=0}=\psi_1(x),\ \ B(\partial_t)U(t,x)\big|_{t=\tau}=\psi_2(x), \tag{5}$$

where

$$A(\partial_t)=a_0+a_1\partial_t,\ B(\partial_t)=b_0+b_1\partial_t,$$

and complex coefficients $a_0,\ a_1,\ b_0,\ b_1$ satisfy obvious conditions: $|a_0|+|a_1|>0,\ |b_0|+|b_1|>0.$

Two-point conditions (5) at $a_0=b_0=1,\ \ a_1=b_1=0$ have a simple physical interpretation – the observation of magnitude $U(t,x)$ at moments $t=0$ and $t=\tau.$

Two-point problem (1), (5), in contrast to the Cauchy problem (1), (2), possesses new properties. If a homogeneous Cauchy problem

$$L(\partial_t,\partial_x)U(t,x)=0, \tag{6}$$
$$U(0,x)=\partial_t U(0,x)=0$$

has only a trivial solution $U(t,x)\equiv 0,$ then the corresponding homogeneous problem for equation (6) with homogeneous conditions

$$A(\partial_t)U(t,x)\big|_{t=0}=0,\ \ B(\partial_t)U(t,x)\big|_{t=\tau}=0 \tag{7}$$

has in general the non-trivial solutions.

Consider, for example, a homogeneous two-point problem for the following wave equation

$$\left[\partial_t^2-\gamma^2\Delta_s\right]U(t,x)=0, \tag{8}$$

$$U(0,x)=0,\ \ U(\tau,x)=0, \tag{9}$$

where $\Delta_s=\partial_{x_1}^2+\ldots+\partial_{x_s}^2$ is the s-dimensional Laplace operator, and γ is the speed of a wave propagation. Problem (8), (9) is derived from problem (6), (7) at

$$c_1(\partial_x)=0,\ \ c_2(\partial_x)=-\gamma^2\Delta_s,\ \ a_0=b_0=1,\ \ a_1=b_1=0$$

and it has, for $s=1$ and $s=2,$ the non-trivial solutions of the following form

$$U(t,x_1)=\sin\frac{\pi t}{\tau}\sin\frac{\pi x_1}{\gamma\tau},$$

$$U(t,x_1,x_2)=x_2\left\{\sin\frac{\pi(x_1+\gamma t)}{\gamma\tau}-\sin\frac{\pi(x_1-\gamma t)}{\gamma\tau}\right\}.$$

Note, however, that some homogeneous two-point problems, similarly to the Cauchy problem, have only a trivial kernel. For example, a two-point problem for the bicaloric equation

$$\left[\partial_t - \Delta_s\right]^2 U(t,x) = 0$$

under condition (9), which is also a particular case of problem (6), (9) at $c_1(\partial_x) = -2\Delta_s$, $c_2(\partial_x) = \Delta_s^2$ has only the trivial solution.

Studying a set of non-trivial solutions to the homogeneous problem (6), (7) was addressed in paper [10] for s=1, in work [11] for s=1 and conditions (9), as well as in article [12] for s≥1.

Thus, the task on enlarging a circle of oscillatory systems, whose mathematical models allow the possibility to apply precise analytical approaches to study, is relevant at the present stage of the development of mechanical theory of oscillations. The indicated relevance grows in proportion to the emergence of new complex mechanical systems with different structures and the need to synthesize and optimize parameters for appropriate industrial equipment. Construction of a differential-symbol method to study the wave process, carried out in this work, is a pressing issue in the applied mechanics.

2. Literature review and problem statement

Two-point problems for partial differential equations are in general the ill-posed boundary value problems. Such two-point, and the more general n-point, problems with known values $U(t, x)$ at n times for a partial differential equation of order n for time were first examined in papers [13, 14]. In these studies, as well as in subsequent works [15–17] addressing isotropic, and in [18, 19] – anisotropic, partial differential equation, regarding the well-posedness of n-point (multipoint) problems, their authors imposed additional conditions for the 2π-periodicity of solution based on spatial coordinates and used the small denominators lower estimates, inherent to these problems.

The establishment of classes for correct solvability of n-point problems in unbounded domains (without additional constraints for spatial variables) for individual partial differential equation was addressed in papers [20, 21], and for systems of partial differential equation – in [22, 23].

Note that problems with multi-point conditions for partial differential equations are the generalization of n-point problems for ordinary differential equations – the Vallée-Poussin problems [24, 25]. These problems were dealt with, specifically, in studies [26, 27] for a linear case, and in [28, 29] for a nonlinear case.

Understudied as yet is a two-point problem for equation

$$\left[\partial_t^2 - \gamma^2 \Delta_s\right] U(t,x) = f(t,x), \qquad (10)$$

under conditions (9), which models wave processes of different nature that occur under the influence of external force $f(t, x)$, when the process states are assigned at two moments. In addition to conditions (9), of interest are conditions (7) when at two points of time t=0 and t=τ one assigns linear combinations $U(t, x)$ and $\partial_t U(t, x)$, which will be identical and equal to zero.

Note that equation (10) has a wide range of applications. This equation describes the propagation of the forced oscillations of a string and a membrane [30, 31], the propagation of electromagnetic waves [32], waves at sea [33], seismic waves [34, 35]. In medicine, equation (10) describes the propagation of a pulsating wave [36, 37], as well as the process of change in blood pressure [38]. The potential of velocities in an acoustic model of plasma motion [39] is also described by equation (10).

For the problem (10), (7), which is non-correct, we shall establish a class of quasi-polynomials as a class of the existence and uniqueness of solution to the problem. In this class, we shall resolve the problem of small denominators, characteristic of these problems. To solve the problem, it is appropriate to apply a differential-symbol method. Note that a given method has been effectively used earlier to solve similar problems with linear conditions based on the selected time variable (under initial conditions in [40, 41], integral conditions in [42], the Dirichlet conditions in [43], and local two-point conditions in [44, 45]).

Therefore, application of the new differential-symbol method in order to solve problem (10), (7) would make it possible to derive exact solutions to the problem and to establish the character of wave processes. It should be noted that it is the exact solutions that enable the analysis of parameters for the specified systems and control over them. To implement this task, we had to modify the mathematical model of the process taking into consideration the lengthy character of structures and to devise a procedure for adapting the differential-symbol method to respective modified models. Note that an analysis of the scientific literature, given above, reveals the following: such an approach to studying the mathematical models of wave processes is a novelty.

The disadvantage of the method constructed is the impossibility of its application for cases when the spatial dimensions of a respective body are commensurate with the magnitude of an oscillation amplitude. As shown by the conducted numerical experiments, it is possible to effectively and adequately explore the dynamic processes in cases when the spatial dimensions of a body are the quantities that are several orders of magnitude larger than the magnitude of an oscillation amplitude.

3. The aim and objectives of the study

The aim of this study was to model a behavior of wave processes occurring under the influence of an external force, at any time in any point of space, if one knows data on the process at two moments of time. To construct a solution to the modelled two-point time problem, we shall apply a differential-symbol method. That would make it possible to derive an explicit solution to the respective two-point problem and to control parameters in a mathematical model in order to detect and avoid dynamic modes of oscillations, which, are impossible or dangerous for a given technological process (resonance modes, an oscillation beat mode, etc.).

To achieve the set aim, the following tasks have been solved:

– to propose a method to construct a precise analytical solution to the respective two-point problem and the analysis of parameters for the dynamic processes in mathematical models within a wide class of oscillatory systems;

– to illustrate the method for constructing a solution and analysis of oscillatory processes in linear systems for the cases of an infinite string and a membrane.

In combination, the study conducted significantly enlarge the circle of mathematical models of wave processes, which allow a precise analytical description. The differential-symbol method, convenient from the point of view of practical engineering applications, would make it possible to analyze and synthesize parameters for a wide range of technological systems that are described by the specified models.

4. The unique solvability of a two-point problem in the class of quasi-polynomials

For non-empty sets $M \subseteq \mathbb{C}^s$ and $\mathbb{C}^s \setminus M$, consider the class of quasi-polynomials $K_{\mathbb{C},M}$, that is the class of functions of the following form

$$g(t,x)=\sum_{k=1}^{m}\sum_{j=1}^{N}Q_{kj}(t,x)e^{\beta_j t+\alpha_k\cdot x},$$

$$m,N\in\mathbb{N},\quad x\in\mathbb{R}^s,\quad t\in\mathbb{R}, \tag{11}$$

where $\beta_1,\ldots,\beta_N\in\mathbb{C}$ and are pairwise different, vectors

$$\alpha_1=(\alpha_{11},\ldots\alpha_{1s}),\ldots,\ \alpha_m=(\alpha_{m1},\ldots\alpha_{ms})$$

are also pairwise different and belong to the set M,

$$Q_{11}(t,x),\ldots,Q_{mN}(t,x)$$

are the polynomials with complex coefficients,

$$\alpha_k\cdot x= =\alpha_{k1}x_1+\ldots+\alpha_{ks}x_s.$$

Set M is to be selected so that class $K_{\mathbb{C},M}$ is the class of unique solvability of problem (10), (7).

Similarly to papers [45, 46], we consider an entire function of vector-parameter $\nu=(\nu_1,\ldots,\nu_s)\in\mathbb{C}^s$ of the following form

$$\Delta(\nu)=\frac{\sinh[\gamma\|\nu\|\tau]}{\gamma\|\nu\|}\left(a_0b_0-\gamma^2\|\nu\|^2a_1b_1\right)+ +\cosh[\gamma\|\nu\|\tau](a_0b_1-a_1b_0). \tag{12}$$

Note that

$$\|\nu\|=\sqrt{\nu_1^2+\nu_2^2+\ldots+\nu_s^2}\quad\text{and}\quad\Delta(\nu)=a_0b_0\tau+a_0b_1-a_1b_0$$

for $\|\nu\|=0$, as well as the fact that function (12) is a quasi-polynomial with respect to τ for any $\nu\in\mathbb{C}^s$.

Let L be the set of zeros of function (12), then put

$$M=\mathbb{C}^s\setminus L,$$

and force $f(t,x)$ in equation (10), which predetermines a wave process, is to be considered a quasi-polynomial, which belongs to $K_{\mathbb{C},M}$, and takes the form

$$f(t,x)=Q(t,x)e^{\beta t+\alpha\cdot x}, \tag{13}$$

where $Q(t,x)$ is the polynomial with complex coefficients, $\beta\in\mathbb{C},\ \alpha\in M$.

We show that for function $f(t,x)$ of form (13) from class $K_{\mathbb{C},M}$ problem (10), (7) has only one solution which belongs to the same class. This solution can be derived from formula

$$U(t,x)=Q(\partial_\lambda,\partial_\nu)F(t,x,\lambda,\nu)\big|_{\lambda=\beta,\nu=\alpha}, \tag{14}$$

where $Q(\partial_\lambda,\partial_\nu)$ is the differential polynomial that is derived from $Q(t,x)$ by replacing t with ∂_λ and x with ∂_ν, and

$$F(t,x,\lambda,\nu)=\frac{e^{\lambda t}-B(\lambda)e^{\lambda\tau}H_1(t,\nu)-A(\lambda)H_2(t,\nu)}{\lambda^2-\gamma^2\|\nu\|^2}e^{\nu\cdot x}, \tag{15}$$

$$H_1(t,\nu)=\left(a_0\frac{\sinh[\gamma\|\nu\|t]}{\gamma\|\nu\|}-a_1\cosh[\gamma\|\nu\|t]\right)\Delta^{-1}(\nu),$$

$$H_2(t,\nu)=\left(b_0\frac{\sinh[\gamma\|\nu\|(1-t)]}{\gamma\|\nu\|}+b_1\cosh[\gamma\|\nu\|(1-t)]\right)\Delta^{-1}(\nu).$$

Function (14) as a result of the action of differential expression $Q(\partial_\lambda,\partial_\nu)$ on (15) and putting $\lambda=\beta$ and $\nu=\alpha$, is the quasi-polynomial from class $K_{\mathbb{C},M}$. We show first that this quasi-polynomial satisfies equation (10). Indeed,

$$\left[\partial_t^2-\gamma^2\Delta_s\right]U(t,x)=\left[\partial_t^2-\gamma^2\Delta_s\right]\left\{Q(\partial_\lambda,\partial_\nu)F(t,x,\lambda,\nu)\right\}\big|_{\lambda=\beta,\nu=\alpha}=Q(\partial_\lambda,\partial_\nu)\left[\partial_t^2-\gamma^2\Delta_s\right]F(t,x,\lambda,\nu)\big|_{\lambda=\beta,\nu=\alpha}=$$

$$=Q(\partial_\lambda,\partial_\nu)\frac{\left[\partial_t^2-\gamma^2\Delta_s\right]e^{\lambda t+\nu\cdot x}-B(\lambda)e^{\lambda\tau}\left[\partial_t^2-\gamma^2\Delta_s\right]\left(H_1(t,\nu)e^{\nu\cdot x}\right)-A(\lambda)\left[\partial_t^2-\gamma^2\Delta_s\right]\left(H_2(t,\nu)e^{\nu\cdot x}\right)}{\lambda^2-\gamma^2\|\nu\|^2}\Bigg|_{\lambda=\beta,\nu=\alpha}=$$

$$=Q(\partial_\lambda,\partial_\nu)\frac{\left(\lambda^2-\gamma^2\|\nu\|^2\right)e^{\lambda t+\nu\cdot x}-0-0}{\lambda^2-\gamma^2\|\nu\|^2}\Bigg|_{\lambda=\beta,\,\nu=\alpha}=Q(\partial_\lambda,\partial_\nu)e^{\lambda t+\nu\cdot x}\big|_{\lambda=\beta,\,\nu=\alpha}=$$

$$=Q(t,x)e^{\lambda t+\nu\cdot x}\big|_{\lambda=\beta,\,\nu=\alpha}=Q(t,x)e^{\beta t+\alpha\cdot x}=f(t,x).$$

In the chain of expressions, we applied the commutativity of differentiation operators, as well as equalities

$$\left[\partial_t^2-\gamma^2\Delta_s\right]e^{\lambda t+\nu\cdot x}=\left(\lambda^2-\gamma^2\|\nu\|^2\right)e^{\lambda t+\nu\cdot x},$$

$$Q(\partial_\lambda,\partial_\nu)e^{\lambda t+\nu\cdot x}=Q(t,x)e^{\lambda t+\nu\cdot x},$$

$$\left[\partial_t^2-\gamma^2\Delta_s\right]\left(H_1(t,\nu)e^{\nu\cdot x}\right)=\left[\partial_t^2-\gamma^2\Delta_s\right]\left(H_2(t,\nu)e^{\nu\cdot x}\right)=0,$$

which is easily checked via direct differentiation.

Function (14) also satisfies conditions (7). Based on equalities

$$A(\partial_t)H_1(t,\nu)\big|_{t=0}=0,\quad A(\partial_t)H_2(t,\nu)\big|_{t=0}=1$$

we obtain

$$A(\partial_t)U(t,x)\big|_{t=0}=A(\partial_t)Q(\partial_\lambda,\partial_\nu)F(t,x,\lambda,\nu)\big|_{\lambda=\beta,\nu=\alpha,t=0}=$$

$$=Q(\partial_\lambda,\partial_\nu)A(\partial_t)F(t,x,\lambda,\nu)\big|_{\lambda=\beta,\nu=\alpha,t=0}=$$

$$=Q(\partial_\lambda,\partial_\nu)\frac{A(\lambda)e^{\lambda t}-B(\lambda)e^{\lambda\tau}A(\partial_t)H_1(t,\nu)-A(\lambda)A(\partial_t)H_2(t,\nu)}{\lambda^2-\gamma^2\|\nu\|^2}e^{\nu\cdot x}\Bigg|_{\substack{\lambda=\beta,\\ \nu=\alpha,\\ t=0}}=$$

$$=Q(\partial_\lambda,\partial_\nu)\frac{A(\lambda)-0-A(\lambda)}{\lambda^2-\gamma^2\|\nu\|^2}\Bigg|_{\lambda=\beta,\nu=\alpha}=0.$$

By analogy, using equalities

$$B(\partial_t)H_1(t,\nu)\big|_{t=\tau}=1,\ \ B(\partial_t)H_2(t,\nu)\big|_{t=\tau}=0,$$

we thus prove that a second condition from two-point conditions (7) is satisfied.

Proving the uniqueness of solution to problem (10), (7) in class $K_{\mathbb{C},M}$ using a method from the opposite is reduced to proving the trivialness in this class of solution to problem (8), (7). The latter property follows from results of paper [46].

Comment 1. If in equation (10) quasi-polynomial $f(t,x)$ has a more general form (11), then, according to the principle of linear superposition, a solution to problem (10), (7) takes the form of the following sum:

$$U(t,x)=\sum_{k=1}^{m}\sum_{j=1}^{N}Q_{kj}(\partial_\lambda,\partial_\nu)F(t,x,\lambda,\nu)\big|_{\lambda=\beta_j,\nu=\alpha_k}. \tag{16}$$

Comment 2. Solutions to problems (10), (7) in the form of (14) and (16) are the quasi-polynomials. Construction of sets of polynomial and quasi-polynomial solutions to partial differential equations and respective boundary problems has been addressed in numerous studies [47–49].

5. Forced oscillations of an infinite string with its two assigned profiles

Consider the oscillations of an infinite thin string under the action of external force $f(t,x)$ when the profiles of the string at moments $t=0$ and $t=1$ are the same (zero). This process is modeled by a two-point problem

$$\begin{aligned}&\left[\partial_t^2-\gamma^2\partial_x^2\right]U(t,x)=f(t,x),\quad t>0,\quad x\in\mathbb{R},\\&U(0,x)=U(1,x)=0,\quad x\in\mathbb{R}.\end{aligned} \tag{17}$$

For problem (17) as problem (10), (7), we obtain

$$a_0=b_0=1,\ \ a_1=b_1=0,$$

$$\Delta(\nu)=\frac{\sinh[\gamma\nu]}{\gamma\nu},\ \ \Delta(0)=1,$$

$$L=\left\{\nu=\pm i\frac{\pi k}{\gamma},\ \ i^2=-1,\ k\in\mathbb{N}\right\}, \tag{18}$$

$$H_1(t,\nu)=\frac{\sinh[\gamma\nu t]}{\sinh[\gamma\nu]},$$

$$H_1(t,0)=t,\ \ H_2(t,\nu)=H_1(1-t,\nu).$$

If $f(t,x)$ takes the form of (13) where $s=1$ and $\alpha\in M=\mathbb{C}\setminus L$, where L is set (18), then the solution to problem (17) is found from formula (14), where

$$F(t,x,\lambda,\nu)=\frac{e^{\lambda t}-e^{\lambda}H_1(t,\nu)-H_2(t,\nu)}{\lambda^2-\gamma^2\nu^2}e^{\nu x}=$$

$$=\frac{e^{\lambda t}-\dfrac{e^{\lambda}\sinh[\gamma\nu t]+\sinh[\gamma\nu(1-t)]}{\sinh[\gamma\nu]}}{\lambda^2-\gamma^2\nu^2}e^{\nu x}.$$

If an oscillatory process occurs at the expense of a linear external influence, that is

$$f(t,x)=Q(t,x)=at+bx+c,$$

where $a,b,c\in\mathbb{R}$ and $a^2+b^2+c^2>0$, then $\beta=0$,

$$\alpha=0\in M=\mathbb{C}\setminus L,$$

and a solution to problem (17) is derived from formula

$$U(t,x)=(a\partial_\lambda+b\partial_\nu+c)F(t,x,\lambda,\nu)\big|_{\lambda=\nu=0}=$$

$$=aF_\lambda'(t,x,0,0)+bF_\nu'(t,x,0,0)+cF(t,x,0,0)=$$

$$=\frac{1}{6}(t-1)t(at+3bx+3c+a).$$

In particular, the solution to problem (17) for $a=5, b=3$, $c=2$ takes the form

$$U(t,x)=\frac{1}{6}t(t-1)(5t+9x+11).$$

The graph of this function (dependence of the magnitude of an oscillation amplitude on time for each point at the string) is shown in Fig. 1.

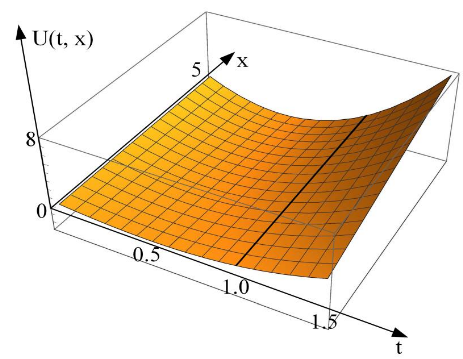


Fig. 1. Graph of function $U(t,x)$ for the case when $a=5$, $b=3$, $c=2$

Therefore, under the action of a linear external force the amplitude of the string has a cubic dependence on time for fixed x and a linear dependence for variable x at any point in time $t>0$.

6. Forced oscillations of an infinite membrane with two assigned profiles

Consider a two-point problem (17) if $s=2$, that is the following problem

$$\begin{aligned}&\left[\partial_t^2-\gamma^2\Delta_2\right]U(t,x)=f(t,x),\ t>0,\ x=(x_1,x_2)\in\mathbb{R}^2,\\&U(0,x)=U(1,x)=0,\ x\in\mathbb{R}^2,\end{aligned} \tag{19}$$

which is a mathematical model of the process of oscillations of an infinite membrane under the action of external force $f(t,x)$ with the same known (zero) positions of the membrane at time $t=0$ and $t=1$. For problem (19), we obtain

$$\Delta(\nu_1,\nu_2)=\frac{\sinh\left[\gamma\sqrt{\nu_1^2+\nu_2^2}\right]}{\gamma\sqrt{\nu_1^2+\nu_2^2}}$$

$$(\Delta(\nu_1,\nu_2)=1\ \text{for}\ \nu_1^2+\nu_2^2=0),$$

$$L=\left\{\nu=(\nu_1,\nu_2)\in\mathbb{C}^2:\quad \gamma^2\left(\nu_1^2+\nu_2^2\right)+\pi^2k^2=0,\ k\in\mathbb{N}\right\},$$

$$F(t,x,\lambda,\nu)=\frac{e^{\lambda t}-e^{\lambda}H_1(t,\nu)-H_2(t,\nu)}{\lambda^2-\gamma^2\left(\nu_1^2+\nu_2^2\right)}e^{\nu_1x_1+\nu_2x_2},$$

$$H_1(t,\nu_1,\nu_2)=\frac{\sinh\left[\gamma\sqrt{\nu_1^2+\nu_2^2}\,t\right]}{\sinh\left[\gamma\sqrt{\nu_1^2+\nu_2^2}\right]},$$

$$H_2(t,\nu)=H_1(1-t,\nu).$$

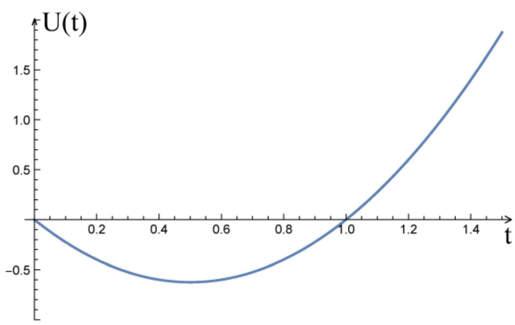
Consider a process of oscillations of an infinite membrane under conditions of a constant external force, that is $f(t,x)=Q(t,x)=b,\ \ b>0.$ Since constant b is a quasi-polynomial, in which $\beta=0,\ \ \alpha=(0,0)\in M=\mathbb{C}^2\setminus L,$ a solution to problem (19) will be derived from formula

$$U(t,x)=bF(t,x,\lambda,\nu)\Big|_{\lambda=0,\nu_1=\nu_2=0}=$$

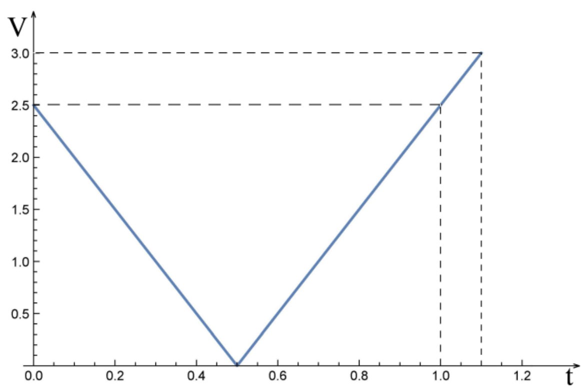
$$=bF(t,x,0,0,0)=b\frac{e^{\lambda t}-(1-t)-e^{\lambda}t}{\lambda^2}\Bigg|_{\lambda=0}=$$

$$=\left[\frac{0}{0}\right]=b\frac{te^{\lambda t}-te^{\lambda}}{2\lambda}\Bigg|_{\lambda=0}=\left[\frac{0}{0}\right]=b\frac{t^2-t}{2}=\frac{1}{2}bt(t-1).$$

We thus obtained that an oscillation amplitude, the solution $U(t,x)$ to problem (19), does not depend on spatial coordinates and takes the form of a time quadratic dependence, and the rate of change in the amplitude, derivative $\partial_tU(t,x),$ is a linear dependence (refer to Fig. 2, *a, b* at $b=5$).



a



b

Fig. 2. Graphs of functions:
a – for $U=U(t,x)$; *b* – for $V=\left|\partial_tU(t,x)\right|$

At different points in time $t\geq0$ the magnitude of amplitude $U(t,x)$ is constant for any $x\in\mathbb{R}^2$. Initial plane of the membrane, while maintaining its shape, moves at rate $|t-0{,}5|b$ first downwards, reaching a minimum $-b/8$, and then upwards, reaching value b at point $t=2$ (Fig. 3).

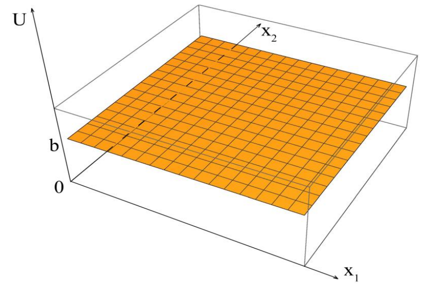


Fig. 3. Graph of function *U*(*t*, *x*) at moment *t*=2

Next, we consider an oscillatory process in a membrane if the external force has an exponential type, namely; $f(t,x)=te^{3x_1+4x_2}$, that is, $Q(t,x)=t,\ \ \beta=0,\ \ \alpha=(3,4)\in M.$

We derive from formula (14)

$$U(t,x)=\partial_\lambda\left\{F(t,x,\lambda,\nu)e^{\nu\cdot x}\right\}\Big|_{\lambda=0,\nu_1=3,\nu_2=4}=$$

$$=e^{3x_1+4x_2}\partial_\lambda\left\{\frac{e^{\lambda t}\sinh[5\gamma]-e^{\lambda}\sinh[5\gamma t]-\sinh[5\gamma(1-t)]}{(\lambda^2-25\gamma^2)\sinh[5\gamma]}\right\}\Bigg|_{\lambda=0}=$$

$$=\frac{e^{3x_1+4x_2}}{25\gamma^2}\left(\frac{\sinh[5\gamma t]}{\sinh[5\gamma]}-t\right).$$

For $\gamma=1$ and

$$U(t,x)=\left(\frac{\sinh 5t}{\sinh 5}-t\right)\frac{e^{3x_1+4x_2}}{25},$$

we obtain such states of square part of the membrane under condition $x\in[0,1]^2$ at times $t=0,\ \ t=0.5,\ \ t=1,\ \ t=2$ (Fig. 4–6).

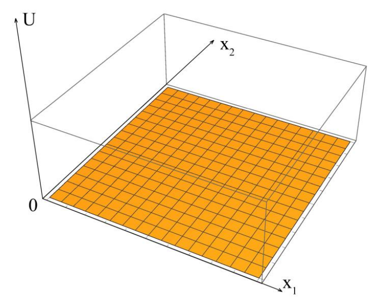


Fig. 4. Graph of function *U*=*U*(*t*, *x*) at moments *t*=0 and *t*=1

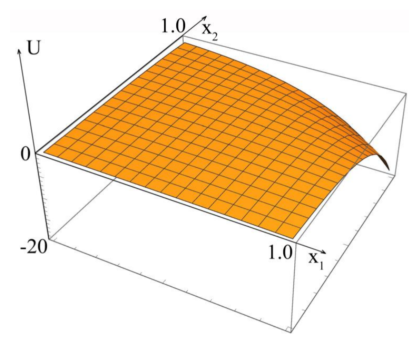


Fig. 5. Graph of function *U*=*U*(*t*, *x*) at time *t*=0.5

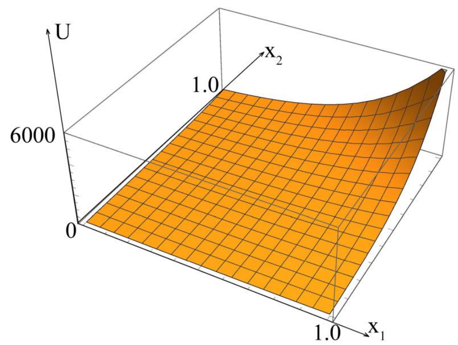


Fig. 6. Graph of function $U=U(t, x)$ at time $t=2$

The greatest speed and the largest deviation from the equilibrium is demonstrated by an angular point with coordinates $x_1=x_2=1$. An increase in time leads to that the initial plane of the membrane deforms, moving downwards to the minimum position, then moves up, coinciding with the initial plane, and then deforms again, moving upwards. In this case, for all $x\in[0,1]^2$, either $U=0$, or $U>0$, or $U<0$.

Consider oscillations of a membrane under the action of external force

$$f(t,x)=\sin(3x_1+4x_2+t),$$

which is a quasi-polynomial of form (11), that is

$$f(t,x)=\frac{1}{2i}e^{(3x_1+4x_2+t)i}-\frac{1}{2i}e^{-(3x_1+4x_2+t)i},$$

where

$$Q_1(t,x)=\frac{1}{2i},\ \ Q_2(t,x)=-\frac{1}{2i},\ \ \beta_1=i,\ \ \beta_2=-i,$$

$$\alpha_1=(3i,4i),\ \ \alpha_2=(-3i,-4i),\ \ \alpha_1,\alpha_2\in M.$$

According to comment 1, a solution to problem (19) is derived from formula (16):

$$U(t,x)=\frac{1}{2i}\left\{F(t,x,\lambda,\nu)e^{\nu\cdot x}\right\}\Big|_{\lambda=i,\nu_1=3i,\nu_2=4i}-\frac{1}{2i}\left\{F(t,x,\lambda,\nu)e^{\nu\cdot x}\right\}\Big|_{\lambda=-i,\nu_1=-3i,\nu_2=-4i}=$$

$$=e^{i(3x_1+4x_2)}\left.\frac{e^{\lambda t}\sin[5\gamma]-e^{\lambda}\sin[5\gamma t]-\sin[5\gamma(1-t)]}{2i(\lambda^2-25\gamma^2)\sin[5\gamma]}\right|_{\lambda=i}-$$

$$-e^{-i(3x_1+4x_2)}\left.\frac{e^{\lambda t}\sin[5\gamma]-e^{\lambda}\sin[5\gamma t]-\sin[5\gamma(1-t)]}{2i(\lambda^2-25\gamma^2)\sin[5\gamma]}\right|_{\lambda=-i}=$$

$$=\frac{\sin[5\gamma]\sin[3x_1+4x_2+t]+\sin[5\gamma(t-1)]\sin[3x_1+4x_2]-\sin[5\gamma t]\sin[3x_1+4x_2+1]}{(25\gamma^2-1)\sin[5\gamma]}.$$

For $\gamma=1$, the solution to problem (19) is the function of form

$$U(t,x)=\frac{\sin[5]\sin[3x_1+4x_2+t]+\sin[5(t-1)]\sin[3x_1+4x_2]-\sin[5t]\sin[3x_1+4x_2+1]}{24\sin[5]}.$$

Let Ouv be a coordinate system created by rotating the coordinate axes Ox_1 and Ox_2 at angle φ, for which $tg\varphi=3/4$. Then $5u=3x_1+4x_2$, $5v=3x_1-4x_2$ and solution $U(t,x)$ to problem (19) does not depend on variable v, and depends only on t and u, in particular,

$$U(t,x)=\frac{1}{24}V(t,u),$$

where

$$V(t,u)=\sin[t+5u]-\frac{\sin[5t]\sin[5u+1]}{\sin[5]}+\frac{\sin[5t-5]\sin[5u]}{\sin[5]}.$$

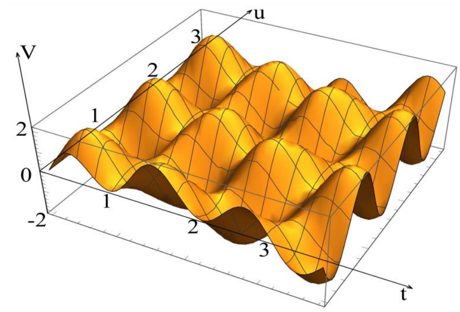


Fig. 7. Graph of function $V(t, u)$

Function $V(t,u)$, shown in Fig. 7, is a two-frequency oscillation with a single frequency over time (phase $5u$) and frequency 5 (phase 0, amplitude $-\sin^{-1}[5]\sin[5u+1]$ and phase (–5) and amplitude $\sin^{-1}[5]\sin[5u]$). In terms of variable u, function $V(t,u)$ is a single-frequency oscillation (frequency 5) with phases $-t$, 1, 0 and amplitudes

$$1,\ (\sin[5])^{-1}\sin[5t],\ (\sin[5])^{-1}\sin[5t-5].$$

Comment 3. Note that the procedure described here enables the accurate assessment of physical-mechanical parameters of a body in the mathematical models of oscillations in order to avoid, in particular, the resonance dynamic modes, etc.

7. Discussion of results of modeling wave processes under two-point time conditions

We have proposed a modified mathematical model for the propagation of wave processes in one- and two-dimensional lengthy structures. Exact analytical solutions to the problems of mathematical physics have been obtained in this work owing to the application of a differential-symbol method, which makes it possible to constructively evaluate the effect of the parameters for an oscillatory system on a wave process.

The benefit of studying the model is that the construction of a solution to the problem employed a differential-symbol method, which is a new and effective when solving problems in unbounded domains under conditions for a single selected variable. In this work, the selected variable is a time variable. Thus, in contrast to approximating approaches that are based on the application of numerical methods, we have managed to analytically describe all the features in the dynamic processes in the specified mathematical models of oscillations. Specifically, we have substantiated a possibility to accurately find the parameters at which the system is under a resonance mode, an oscillation beat mode, etc.

The research procedure, devised in this work, has its limitations and disadvantages. Specifically, it is not applicable to the study of mathematical models of bodies, which cannot be described by the term "lengthy". Particularly, we mean here those oscillations whose maximum values for the amplitudes differ from the linear dimensions of an object by less than an order of magnitude.

Our research into the two-point problems is illustrated for the case of oscillations of an infinite string (s=1) and an infinite membrane (s=2). Graphical and numerical analysis of the respective oscillatory process has been performed, and, which is extremely important, the exact solutions to the examined problems have been constructed. The obtained numerical results confirm a satisfactory, in terms of practical applications, accuracy of the resulting solution and make it possible to select parameters for technological oscillatory systems. Such a selection of parameters, described by the specified mathematical models, ensures the effective operation modes of respective equipment.

The proposed methodology could be used in the future to study more complex mathematical models of oscillatory systems: a nonlinear (perturbed) model of string and membrane oscillations, linear and nonlinear models of oscillations considering dissipative forces, etc.

8. Conclusions

1. The problem on finding a solution to the Poisson equation, which would satisfy homogeneous conditions at moments $t=0$ and $t=\tau$ (problem (10), (7)), is a mathematical model of the oscillatory processes at the assigned states of the process at two points in time. By providing the specified problem for consideration, we have refined and modified the model of linear oscillations of length bodies.

2. Problem (10), (7) is the ill-posed boundary value problem, since the corresponding homogeneous problem has the non-trivial solutions. Therefore, for the unique solvability of the problem, we have introduced a class of quasi-polynomial functions of special form in which elements of the problem's kernel are missing. This was achieved by selecting the set containing no zeros of the characteristic determinant. It is shown that the specified class contains the unique solution to the examined problem.

3. In this work, we have proposed, to study and analyze dynamic processes in the mathematical models of certain oscillatory systems, using such an analytical differential-symbol method, which makes it possible to build constructive solutions to the respective two-point problems within the special classes of quasi-polynomial functions.

4. The analytical, numerical, and graphical results, reported in this paper, confirm the effectiveness of the proposed method when studying wave processes in the lengthy structures, as well as the adequacy of the modified mathematical models to the actual prototypes of oscillatory systems.

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