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SYNTHESIS AND CLASSIFICATION OF PERIODIC MOTION TRAJECTORIES OF THE SWINGING SPRING LOAD

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Продовжено дослідження можливостей геометричного моделювання нехаотичних періодичних траєкторій руху вантажу хитної пружини та її різновидів. В літературі хитною пружиною (swinging spring) називають різновид математичного маятника, який складається з точкового вантажу, приєднаного до невагомої пружини. Другий кінець пружини фіксується нерухомо. Розглядаються маятникові коливання пружини у вертикальній площині за умови збереження прямолінійності її осі. Шукана траєкторія вантажу хитної пружини моделюється з використанням рівнянь Лагранжа другого роду.

Актуальність теми визначається необхідністю дослідження умов відмежування від хаотичних коливань елементів механічних конструкцій, до складу яких входять пружини, а саме визначення раціональних значень параметрів для забезпечення періодичних траєкторій їх коливань. Хитні пружини можна використати як механічні ілюстрації при дослідженні складних технологічних процесів динамічних систем, коли нелінійно зв'язані коливальні компоненти системи обмінюються енергією між собою.

Одержані результати дозволяють долучити до переліку числових параметрів хитної пружини ще й періодичні криві як «параметри» в графічній формі. Тобто визначити числові значення параметрів, які б забезпечили існування наперед заданої форми періодичної траєкторії руху вантажу хитної пружини. Розглянуто приклад обчислення маси вантажу за відомими жорсткістю пружини, її довжиною без навантаження, початковими умовами ініціалізації коливань, а також (увага) формою періодичної траєкторії цього вантажу. Одержано періодичні траєкторії руху вантажу для модифікацій хитної пружини – таких як підвішеної до рухомого візка і вісь якої збігається з математичним маятником. А також двох хитних пружин зі спільним рухомим вантажем і з різними точками кріплення.

Одержані результати проілюстровано комп'ютерними анімаціями коливань відповідних хитних пружин та їх різновидів.

Результати можна використати як парадигму для вивчення нелінійних зв'язаних систем, а також при розрахунках варіантів механічних пристроїв, де пружини впливають на коливання їх елементів. А також у випадках, коли в технологіях використання механічних пристроїв необхідно відмежуватися від хаотичних переміщень вантажів і забезпечити періодичні траєкторії їх руху

Ключові слова: маятникові коливання, траєкторія руху, хитна пружина, рівняння Лагранжа другого роду

1. Introduction

In the present-day sense, a complex technological process can be interpreted as a dynamic system consisting of nonlinearly coupled oscillating components. Besides, within the framework of a dynamic system, its components can exchange energy with each other. An approach to solving the class of problems associated with the phenomenon of energy exchange between components is considered in [1–3]. The issues of dependence of this action on the system control parameters are studied. The method consists in determining total system energy and correct estimation of energy values in time as well as their connection with each component.

To illustrate this approach, a *two-dimensional spring pendulum* is used as a mechanical model of studying several non-linearly connected systems. Such kind of spring pendulum in an idealized form consists of a «point» load of mass *m* attached to the end of a massless spring with stiffness *k* and length *h* in non-loaded state. The other end of the spring is motionlessly fixed. The oscillating system thus formed should move only in a vertical plane while *keeping the spring axis straight*. The point load simultaneously performs two types of oscillations: *similar to the spring* (when moved along the straight axis of the spring) and *similar to the pendulum* (when oscillated together with its axis). This kind of oscillatory system is called in literature a *swinging spring* [4]. In practice, swinging springs can perform role of mechanical illustrations where transverse (pendulum) oscillations and longitudinal (spring) oscillations are analyzed. In a case when ratio of frequencies of above oscillations will differ approximately twice, then the swinging spring will be in a state of *resonance*.

A large number of possible implementations of the idea of oscillation of a swinging spring are given in [1–3]. For example, oscillations of swinging springs are directly related to many mechanical dynamic systems. Effects of violation of stability and controllability of high-speed ships were revealed in the process of calculation of their dynamics in conditions of unfavorable waves. At a ratio of 1:2 between frequencies of quartering and athwart sea, loss of dynamic stability takes place [5]. Also, oscillation of swinging springs helps to study dynamics of supersonic planes when effects of violation of their stability and controllability manifest themselves. It turned out that the most intense oscillation of a swing occurs when oscillations at the angle of attack occur with a frequency twice the frequency of lateral oscillations [4]. These examples explain when it is expedient to take into account energy exchange between its components (longitudinal and transverse oscillations) within the framework of a dynamic system (ship or aircraft).

Modified model of the swinging spring (the model of flexible thread) plays an important role in building mechanics. Flexible thread is a peculiar spring that works only for stretching. In a typical two-dimensional model, flexible thread can simultaneously perform transverse oscillations in its plane (analogous to angular oscillations of the loaded swinging spring) and pendulum oscillations which connect support attachments (analogous to vertical oscillations) [6], for example, wires of high-voltage lines whose state is influenced by wind gusts. Loss of dynamic stability occurs at a ratio of 1:2 of frequencies of the indicated oscillations and then transverse oscillations of the thread appear with amplitude reaching rather large values. The possibility of occurrence of such phenomena must be taken into account in calculation of various structures of building mechanics (suspension bridges, cable and girder systems, cable-ways, power lines, various antennas, cable systems for holding various objects, flexible hoses, etc.) [4].

It is clear that the state of *resonance* of the swinging spring occurs at a certain combination of values of the swinging spring parameters, namely, when the period of vertical oscillations will be approximately two times smaller than that of horizontal oscillations [7]:

$$2T_y = T_x,$$

where

$$T_x = 2\pi\sqrt{\frac{m}{k}}, \quad T_y = 2\pi\sqrt{\frac{h}{g}},$$

where *m* is the load mass, *k* is the spring stiffness, *h* is the spring length in non-loaded state, *g* is acceleration of gravity.

Beside conditions of resonance, there is another possibility to characterize the swinging spring, namely, distinguish *periodic paths* from possible motions during oscillation of a swinging spring load [8, 9]. To do this, it is necessary to elicit regularities of formation of periodic paths depending on parameters of the swinging spring and also classify obtained periodic paths according to the type of schemes of mechanical devices making up them. It is also desirable to bring to conformity a certain number that would characterize its geometric form for each periodic path. In addition, it would be important to study the variety of structures which include swinging springs.

Therefore, it would be advisable to conduct studies aimed at geometric modeling of periodic paths of motion of the swinging spring load as well as varieties of swinging spring designs.

2. Literature review and problem statement

As usual, the methods of dynamic system solution based on ideas of a swinging spring use coordinates that determine spring and pendulum motion [1–3]. In this case, possibility of representing a Hamiltonian in the form of sum of three members corresponding to the energies associated with motions of the spring, the pendulum and the component of their connection are foreseen. As a result, it becomes possible to find out how energy is distributed between above three energy quantities. It is also possible to find out how energy distribution varies according to the total energy and the control parameter which is the ratio of frequencies of a simple pendulum and the spring mass. Energy exchange for individual paths is analyzed with the help of the obtained analytical expressions. Also, global characteristics of distribution of the swinging spring energy are obtained by calculating spatial and time-average components of energy for a large number of paths (periodic, quasiperiodic and chaotic) throughout the phase space.

In [8], changes are studied in behavior of a swinging spring when one response becomes unstable and is replaced by another under the parameter control. Poincare sampling is used to reduce the problem of determining stability of the boundary cycle to a simpler problem of determining stability of a fixed point by Poincaré's mapping. Influence of amplitude and frequency of the basic motion on periodic reaction of the system is investigated and bifurcation characteristics of periodic solution are analyzed in [9]. It has been established that Hopf bifurcation can occur in a periodic response of a system that corresponds to some base frequencies and amplitudes. In [10], dynamics of a spring pendulum is investigated using asymptotic methods. Methods of the theory of nonlinear normal oscillation forms have made it possible to study pendulum dynamics not only for small oscillation amplitudes but for significant ones as well. In [11], experimental observations were made and simple explanations of the spring load motion given including a well-known case where frequency of oscillations in vertical direction is twice as high as for the pendulum motion. Theoretical study of small plane nonlinear oscillations of a swinging spring with a nonlinear dependence of the spring tension on its extension is considered in [12, 13]. The method of the Hamiltonian normal form is used. Solutions of Hamiltonian equations of normal form have shown that periodic reorganization of oscillations between vertical and horizontal modes occurs only in the case of resonances 1:1 and 2:1. In all other cases, both in the presence of resonance and in its absence, oscillations occur with two constant frequencies. In [14], energy crossflow between longitudinal and pendulum oscillations is considered as pulsation. Pulsation and stepped precession are characteristic features of the swinging spring dynamics. Hamiltonian reduction was used to find complete analytical solution. Oscillations of a swinging spring with a moving point of suspension are investigated in [15]. Swinging pendulum is described in [16] as a mechanical system with two degrees of freedom. To do this, a scalar differential equation was formed and numerically solved using Jacobi-Levi-Civita equations. Periodic process of pumping the swinging spring energy from one mode to another is investigated in [17]. The found analytical description reflects with high accuracy the process at any initial deviations of the pendulum. Comparison of this algorithm with algorithms of the classical method of normal form is made. It is shown in [18] that integral approximation of a spatial swinging spring adjusted to the resonance 1:1:2 has a monochromium and the stepwise angle of precession of the plane of oscillation of the resonant spring pendulum is the rotation number of integral approximations. Swinging spring oscillation is analyzed in [19] from the standpoint of energy exchange within the framework of the parametric mechanism. In particular, swinging spring with two degrees of freedom is an auto-parametric system which is the basis for studying nonlinear intertwined systems. An invariant normal form used in [20] has made it possible not to divide oscillations of the swinging spring into autonomous – not autonomous, or resonant – non-resonant cases in the framework of a single approach. Connection of a possible path of a swinging spring load with Lissajou's figures is studied in [21].

However, these profound theoretical works often do not provide clear algorithms of constructing non-chaotic periodic paths of the swinging spring load. Besides theoretical studies, methods for constructing real non-chaotic *periodic* paths of the swinging spring loads are needed for engineering practice. Some of them are described in [22] where examples of periodic paths are given as well as in [23] where conditions for construction of periodic paths were studied. Examples of periodic swinging spring paths are given in [24]. An example of constructing paths of motion of a spring load is given in [25]. The swinging spring dynamics is described in [26] in two different ways: by means of Lagrange equations and with application of Newton's second law. A large number of periodic paths of the swinging spring load are shown in [27]. In this case, motions of the spring pendulum are studied depending on its two control parameters (the ratio of the spring and the pendulum frequencies). It was shown that within the limits of very small and very large values of parameters, path of the spring pendulum load predominantly can be regulated and changes in parameters of most initial conditions lead to chaotic paths. A maple program for constructing path of a swinging spring load is presented in [28]. It is shown in [29] how to construct these paths. Based on a composite program, parametric resonance of the swinging spring is illustrated in [30]. It manifests itself in energy transfer from vertical load oscillations to horizontal and vice versa. It was shown that velocity and amplitude of energy transfer depend essentially on initial conditions.

A method of projection focusing for construction of periodic paths of loads of a variety of mathematical pendulums is considered in [31, 32]. Examples of implementation of this method are given in [33]. A method for finding values of a set of parameters to provide a non-chaotic periodic path of a point load of a swinging spring is given in [34]. Computer animations of corresponding swinging spring oscillations which illustrate the results obtained are shown in [35].

As a result of review of published sources [1–30], issues that have not yet been investigated by other authors were identified, in particular, with regard to development of a universal method for constructing periodic paths of motion of a swinging spring load and classification of these paths depending on basic parameters of the swinging spring.

3. The aim and objectives of the study

The study objective is classification and synthesis of periodic paths of motion of the swinging spring load and some of its varieties.

To achieve this objective, the following tasks should be solved:

– make a table of geometric forms of periodic paths of motion of the swinging spring load which correspond to the specified value of stiffness, k, and mass, m;

– using the ratio of horizontal and vertical periods of load oscillations, determine numbers that will characterize the resulting geometric forms of periodic motion paths;

– classify periodic paths for:

a) a load of a swinging spring whose axis oscillates together with the mathematical pendulum;

b) a load in the common point of attachment of two swinging springs;

c) a load of a swinging spring suspended to a movable carriage;

– construct phase paths of functions of generalized coordinates of the swinging spring and its varieties and provide estimates of the range of change of their values and the load motion velocities.

4. Determination of periodic paths of load of a swinging spring and its varieties

4. 1. Making the table of geometric forms of periodic paths of motion of load of the swinging spring depending on its parameters

In the process of constructing periodic paths, we shall rely on studies [31–35]. Therefore, intermediate results will be omitted as far as possible and only final results will be shown.

Let conditions of idealization of load motion be fulfilled for all tasks:

– parameters and initial conditions are given in conventional numerical units;

– pendulum oscillations of the spring relative to the fixed suspension occur in the vertical plane Oxy;

– axis of the massless spring remains straight during oscillation;

– the load mass is centered in the point located on the spring axis from the nonfixed end;

– there are no supports in nodes and air resistance during oscillation;

– the process of energy dissipation is slow in comparison with the characteristic time scales (the oscillatory system is conservative).

Let us determine paths of motion of the swinging spring load in a vertical plane depending on the load mass, initial length of the spring in the non-loaded state, stiffness of the spring and initial conditions of oscillation occurrence.

The swinging spring diagram according to [34] is shown in Fig. 1.

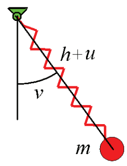


Fig. 1. The swinging spring diagram

Take value of the angle formed by the spring axis and the vertical axis *Oy* as the first generalized coordinate function $v(t)$. Link the second generalized coordinate function $u(t)$ with the spring extension in time; denote length of the spring in non-loaded state through h. Then virtual coordinates of the moving point load can be calculated according to these formulas:

$$x=(h+u)\sin v;\quad y=-(h+u)\cos v. \tag{1}$$

Specify Lagrangian as the difference between kinetic and potential energies:

$$L=0.5m\left(\left(\frac{du}{dt}\right)^2+(h+u)\left(\frac{dv}{dt}\right)^2\right)-$$
$$-0.5ku^2-9.81m(h+u)(1-\cos v)-9.81mu. \tag{2}$$

To form a system of Lagrange differential equations of the second degree, use the following relation (the point means time derivative):

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot u}\right)-\frac{\partial L}{\partial u}=0;\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot v}\right)-\frac{\partial L}{\partial v}=0. \tag{3}$$

As a result, the system of Lagrange equations of the second degree is obtained in the form:

$$(u+h)\frac{d^2v}{dt^2}+2\frac{dv}{dt}\frac{du}{dt}+9.81\sin v=0;$$

$$\frac{d^2u}{dt^2}-(u+h)\left(\frac{dv}{dt}\right)^2+\frac{ku}{m}-9.81\cos v=0. \tag{4}$$

The problem statement. Determine value of the spring stiffness k which would provide periodic path of movement of the load of mass m attached to the spring of length h in a non-loaded state. In its initial position, the swinging spring is arranged vertically, that is, $v(0)=0$. Oscillation is initiated by means of an impulse applied to the spring load in direction of the *Ox* axis: $dv(0)=0.5$. The value 0.5 can be characterized as initial velocity of change in time of the angle value, $v(t)$.

Applying algorithms and programs described in [31–34], solve the system of equations (4) with the values of parameters and initial conditions $v(0)=0$; $dv(0)=0.5$; $u(0)=1$; $du(0)=0$. As a result of solution of the system of equations (4), integral curves and phase paths are obtained. The phase paths are characterized by the number of pixels in their images. Fig. 2 shows graphs of change of the number of pixels *Np* depending on the value of stiffness k for «unit» values $m=1$ and $h=1$. Extreme locally minimum values are obtained. This makes it possible (after refinement) to determine six main *critical values* of the coefficients of stiffness k: 7.99; 9.55; 12.67; 18.12; 22.96; 28.84.

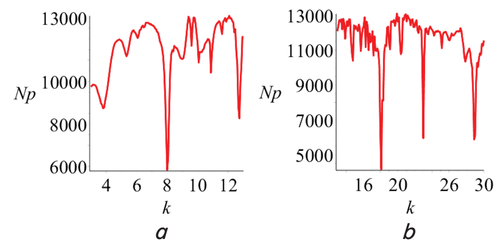


Fig. 2. Graphs of change of the number of pixels depending on k for $m=1$: within $3<k<13$ (*a*); within $13<k<30$ (*b*)

Fig. 3 shows instantaneous positions of the swinging spring and periodic paths of motion of loads of mass $m=1$ depending on values of the coefficient of stiffness k. Note that images of the obtained geometric shapes of the load motion paths correspond to the local minima of the number of pixels in Fig. 2. Computer animations of the corresponding oscillations are given at the Internet site [36].

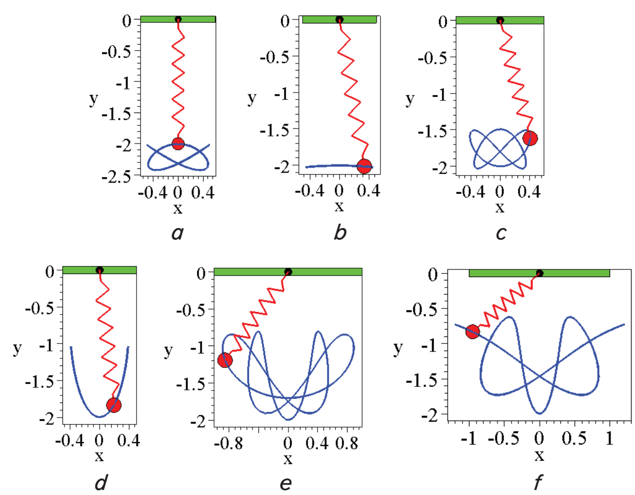


Fig. 3. Periodic paths of the spring load motion for: $k=7.99$ (*a*); $k=9.55$ (*b*); $k=12.67$ (*c*); $k=18.12$ (*d*); $k=22.96$ (*e*); $k=28.84$ (*f*)

Fig. 4 shows images of the phase paths of the generalized coordinate functions corresponding to periodic paths of Fig. 3. They were built in the coordinate phase planes {*u, Du*} and {*v, Dv*} which are shown together in the figure. Red color indicates the function $u(t)$ phase path and blue color indicates the function $v(t)$. phase path. Recall that the function $u(t)$ describes the spring length and the function $v(t)$ is the angle of deviation of the spring from vertical.

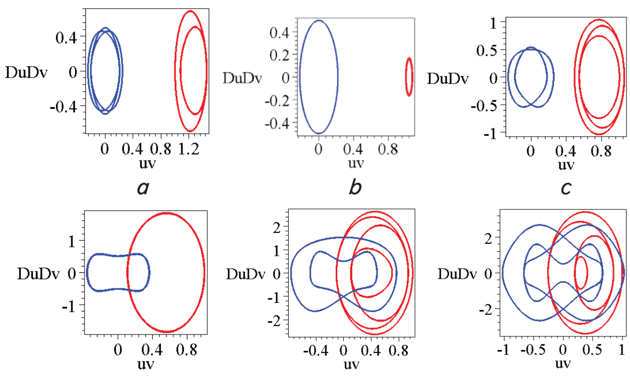


Fig. 4. Phase paths in phase planes $\{u, Du\}$ and $\{v, Dv\}$ for: $k=7.99$; $1<u(t)<1.4$; $-0.6<Du(t)<0.6$; $-0.3<v(t)<0.3$; $-0.45<Dv(t)<0.45$ (*a*); $k=9.55$; $0.9<u(t)<1$; $-0.2<Du(t)<0.2$; $-0.3<v(t)<0.3$; $-0.44<Dv(t)<0.44$ (*b*); $k=12.67$; $0.6<u(t)<0.9$; $-1.1<Du(t)<1.1$; $-0.35<v(t)<3.5$; $-0.6<Dv(t)<0.6$ (*c*); $k=18.12$; $0.2<u(t)<1$; $-1.8<Du(t)<1.8$; $-0.4<v(t)<0.4$; $-0.5<Dv(t)<0.5$ (*d*); $k=22.96$; $0<u(t)<1$; $-2.5<Du(t)<2.5$; $-0.8<v(t)<0.8$; $-1.5<Dv(t)<1.5$ (*e*); $k=28.84$; $0<u(t)<1$; $-3<Du(t)<3$; $-1<v(t)<1$; $-2.5<Dv(t)<2.5$ (*f*)

Hence, it is possible to determine ranges of change of generalized coordinates functions as well as rate of their change with the help of phase paths (Fig. 4).

$1<u(t)<1.4$; $-0{,}6<Du(t)<0{,}6$; $-0.3<v(t)<0.3$;
$-0.45<Dv(t)<0.45$ (Fig. 4, *a*);

$0.9<u(t)<1$; $-0.2<Du(t)<0.2$; $-0.3<v(t)<0.3$;
$-0.44<Dv(t)<0.44$ (Fig. 4, *b*);

$0.6<u(t)<0.9$; $-1.1<Du(t)<1.1$; $-0.35<v(t)<3.5$;
$-0.6<Dv(t)<0.6$ (Fig. 4, *c*);

$0.2<u(t)<1$; $-1.8<Du(t)<1.8$; $-0.4<v(t)<0.4$;
$-0.5<Dv(t)<0.5$ (Fig. 4, *d*);

$0<u(t)<1$; $-2.5<Du(t)<2.5$; $-0.8<v(t)<0.8$;
$-1.5<Dv(t)<1.5$ (Fig. 4, *e*);

$0<u(t)<1$; $-3<Du(t)<3$; $-1<v(t)<1$;
$-2.5<Dv(t)<2.5$ (Fig. 4, *f*).

At the next stage, find proportions between the coefficient of spring stiffness k and mass m which would provide *the same* (by geometric shape) paths of motion of loads (the spring length $h=1$ in the non-loaded state is known). For this purpose, it is necessary to express value of the load mass m as a function of the coefficient of stiffness k. Specify initial conditions for initiating oscillations by vertical position of the spring suspension $v(0)=0$ which was assigned an initial angular velocity $Dv(0)=0.5$. Let $u(0)=1$; $du(0)=0$.

Determine periodic path for the variable mass m by fixing value of the spring stiffness. Using the procedure given in [31–34], build the graph of change of the number of pixels in the image of the phase paths depending on mass m, for example, for the value of $k=18.12$ (Fig. 5). Locally minimum extreme values of the graph enable determination of critical values of mass m: 0.627; 0.788; 1; 1.43; 1.88; 2.24.

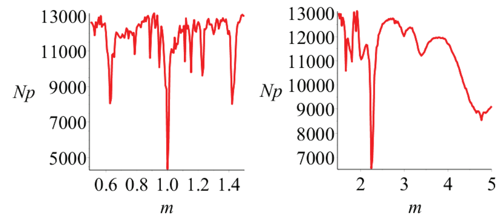


Fig. 5. Graphs of change of the number of pixels depending on m for $k=18.12$: within $0.5<m<1.5$ (*a*); within $1.5<m<5$ (*b*)

Fig. 6 shows instantaneous positions of the swinging spring with $k=18.12$ as well as periodic paths of the load motion for the calculated load masses. One can see that periodic paths in Fig. 6 are similar in their geometric forms to the paths in Fig. 3. This indicates existence of a certain regularity of «generation» of periodic paths.

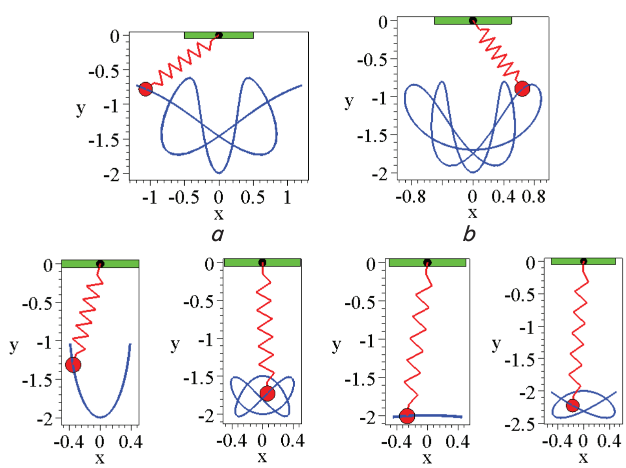


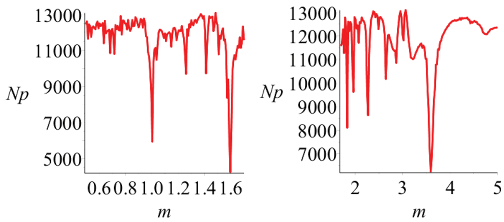
Fig. 6. Periodic paths of the spring load for: $m=0.627$ (*a*); $m=0.788$ (*b*); $m=1$ (*c*); $m=1.43$ (*d*); $m=1.88$ (*e*); $m=2.26$ (*f*)

Contents of Fig. 3 and Fig. 6 convinces that a number of geometric forms of periodic paths repeat in a certain sequence. To check this observation and generalize the technique, consider another variant of spring oscillation, for example, with stiffness $k=28.84$.

Fig. 7 shows the graph of change of the number of pixels in the image of phase paths depending on mass m for the value of $k=28.84$. Locally minimum extreme values of the graph enable determination of critical mass values m: 1; 1.26; 1.59; 2.28; 3; 3.6.

It turns out that the built periodic paths of load motion will look identical in their form to those shown in Fig. 6, however, at other values of mass: $m=1$ (*a*); $m=1.26$ (*b*); $m=1.59$ (*c*); $m=2.28$ (*d*); $m=3$ (*e*); $m=3.6$ (*f*). This observation facilitates making the table for classification of parameters m and k which would ensure existence of a periodic path of the swinging spring load motion ($h=1$).

Let us consider classification of periodic paths of motion of the spring load. To identify geometric form of periodic paths we will use the ratio of periods of vertical $T_y = 2\pi\sqrt{h/g}$ and horizontal $T_x = 2\pi\sqrt{m/k}$ oscillations of the load.



a *b*

Fig. 7. Graphs of change of the number of pixels depending on m for $k=28.84$: within $0.5<m<1.6$ (*a*); within $1.6<m<5$ (*b*)

Table 1 gives the value of ratio of horizontal oscillation periods to vertical ones depending on geometric shape of the motion paths with respect to periodic paths in Fig. 3, 6. Values of the load masses m for the corresponding values of the coefficient of stiffness k are also given. All values are in conditional units.

Table 1

Values of load masses for periodic paths corresponding to stiffness k

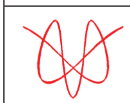
Values of stiffness coefficients → / Path form ↓	7.99	9.5	12.67	18.12	22.96	28.87	Values of the period ratios ↓
	0.276	0.33	0.44	0.63	0.8	1	1.71
	0.35	0.415	0.554	0.793	1	1.25	1.525
	0.44	0.23	0.7	1	1.26	1.58	1.36
	0.63	0.75	1	1.423	1.8	2.27	1.14
	0.83	1	1.32	1.9	2.4	3	0.986
	1	1.18	1.58	2.26	2.85	3.6	0.9

Table 1 enables finding of values load masses and forms of paths only for discrete values of k. In order to determine permissible value of mass m for arbitrary $10<k<35$ with the aim of obtaining a certain load path (for example, for verification, shown in Fig. 6, *a*), use interpolation formula for two points with coordinates (28.84, 1) and (18.12, 0.627). Computation gives function

$$m(k)=0.0348k-0.00348.$$

Other functions are obtained in a similar way:

$$m(k)=0.044k-0.00923;$$

$$m(k)=0.055k+0.00272;$$

$$m(k)=0.0792k-0.00675;$$

$$m(k)=0.104k-0.0131;$$

$$m(k)=0.127k-0.0588, \qquad (5)$$

which correspond to the load paths depicted in Fig. 6, *b–f*.

Consequently, to calculate mass m when it moves along a periodic path at a specified value of the coefficient of stiffness k, one of the formulas (5) must be used. At the same time, an opportunity appears not only to construct periodic paths but also *pre-select* one of the geometric forms of the paths shown in Fig. 6.

For example, to obtain periodic paths shown in Fig. 6, *a–f* for the coefficient of stiffness $k=22.96$, it is necessary to select mass values, respectively, $m=0.8$; $m=1$; $m=1.27$; $m=1.8$; $m=2.38$; $m=2.85$.

This illustrates the possibility of not only constructing a periodic path but also to choose one of the paths shown in Fig. 6. Thus, in a certain sense, the inverse problem of determining periodic paths of motion of the spring load was solved. Next, let us consider other classes of periodic paths.

4. 2. The class of periodic paths of motion of load of the spring whose axis oscillates together with the mathematical pendulum

Let us consider a kind of a swinging spring combined with a mathematical pendulum. Let the swinging spring axis be a mathematical pendulum of length R and a load of mass M (Fig. 8). Assume that masses m and M of the spring and the pendulum do not coincide. Determine the family of paths of movement of the swinging spring load in the vertical plane Oxy depending on the spring parameters.

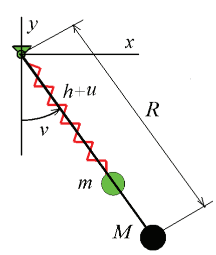


Fig. 8. Diagram of a swinging spring connected with a pendulum

Value of the angle formed by the swinging spring axis and the vertical axis Oy was taken as the first generalized coordinate function $v(t)$. The second generalized coordinate function $u(t)$ is associated with elongation of the spring in time; h denotes length of the swinging spring in the non-loaded state. Then virtual coordinates of the moving point load of the spring can be calculated by formulas (1):

Lagrangian systems:

$$L=0.5MR^2\left(\frac{dv}{dt}\right)^2+0.5m\left(\left(\frac{du}{dt}\right)^2+u^2\left(\frac{dv}{dt}\right)^2\right)+$$
$$+9.81(mu+MR)\cos v-0.5k(u-h)^2. \qquad (6)$$

To form a system of Lagrange differential equations of the second degree, use relation (3). As a result, the system of Lagrange equations of the second degree is obtained in the form:

$$m\frac{d^2u}{dt^2}-mu\left(\frac{dv}{dt}\right)^2-9.81m\cos v+k(u-h)=0; \qquad (7)$$

$$\left(MR^2+mu^2\right)\frac{d^2v}{dt^2}+2mu\frac{du}{dt}\frac{dv}{dt}+9.81\sin v(MR+mu)=0.$$

The problem statement. Determine value of mass M of the mathematical pendulum with length R which would provide periodic path of moving the load with mass m of the swinging spring with stiffness k and length h in non-loaded state. In initial position, the oscillating system is arranged vertically, that is, $v(0)=0$. Oscillations are initiated by means of an impulse applied to the spring load in direction of the axis Ox: $dv(0)=1.5$. The value 1.5 can be characterized as initial velocity of change in time of the angle value $v(t)$. Initial values for the parameter u of the spring elongation will be chosen in the form: $u(0)=1$; $du(0)=0$.

Applying algorithms and programs described in [31–34], solve the system of equations (7) by numerical Runge-Kutta method with initial conditions $v(0)=0$; $dv(0)=1.5$; $u(0)=2$; $du(0)=0$. To determine magnitude of critical value of M, the graph of saturation of the image of the phase path lines [34] can be used. Provide periodicity of the path of the swinging spring load for parameters $R=8$, $m=15$, $k=150$ and $h=2.5$ using the found value of M. Fig. 9 shows periodic paths depending on the mass M of the mathematical pendulum. Fig. 10 shows combined phase paths constructed in the coordinate phase planes $\{u, Du\}$ and $\{v, Dv\}$ in red and blue, respectively. To confirm value of the found critical value of M, the graph of saturation of the image of lines of the phase path [34] can be used. The Internet site [36] demonstrates computer animations of corresponding oscillations.

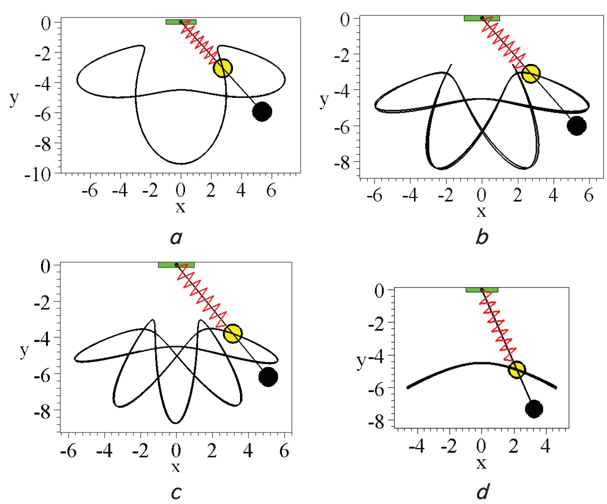


Fig. 9. Periodic paths of motion of the spring load for: $M=20.4$ (*a*); $M=7.17$ (*b*); $M=5.26$ (*c*); $M=2.19$ (*d*)

It is possible to determine ranges of change of generalized coordinates functions as well as rate of their change with the help of phase paths (Fig. 10).

$0.5<u(t)<7$; $-8.5<Du(t)<8.5$; $-1<v(t)<1$;

$-1<Dv(t)<1.5$ (Fig. 10, *a*);

$1<u(t)<7$; $-8.5<Du(t)<8.5$; $-1<v(t)<1$;

$-1<Dv(t)<1$ (Fig. 10, *b*);

$1<u(t)<6.1$; $-8.2<Du(t)<8.2$; $-1<v(t)<1$;

$-1<Dv(t)<1{,}5$ (Fig. 10, *c*);

$2<u(t)<5$; $-5<Du(t)<5$; $-1<v(t)<1$;

$-1.5<Dv(t)<1.5$ (Fig. 10, *d*).

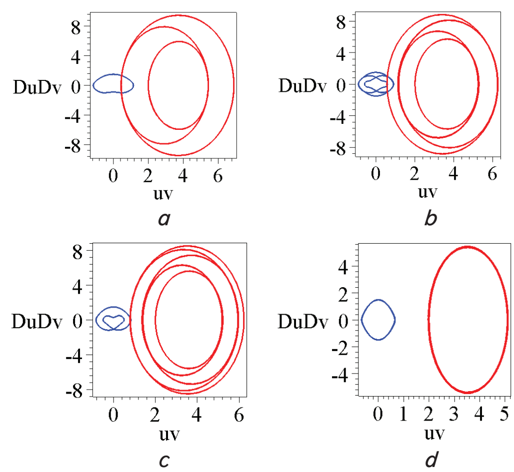


Fig. 10. Phase paths in planes $\{u, Du\}$ and $\{v, Dv\}$ for: $M=20.4$; $0.5<u(t)<7$; $-8.5<Du(t)<8.5$; $-1<v(t)<1$; $-1<Dv(t)<1.5$ (*a*); $M=7.17$; $1<u(t)<7$; $-8.5<Du(t)<8.5$; $-1<v(t)<1$; $-1<Dv(t)<1$ (*b*); $M=5.26$; $1<u(t)<6.1$; $-8.2<Du(t)<8.2$; $-1<v(t)<1$; $-1<Dv(t)<1{,}5$ (*c*); $M=2.19$; $2<u(t)<5$; $-5<Du(t)<5$; $-1<v(t)<1$; $-1.5<Dv(t)<1.5$ (*d*)

4. 3. The class of periodic paths of a joint movable load of two swinging springs

Let us consider an oscillation system formed of a pair of swinging springs. Construct a path of motion of a load common to these swinging springs in vertical plane Oxy (Fig. 11). Parameters include the load mass m, the same initial lengths h of springs in the non-loaded state, the same stiffness of the springs k and initial conditions for occurrence of oscillations. In addition, it is necessary to specify distance H between fasteners of the springs.

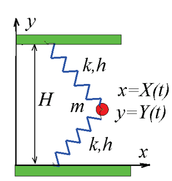


Fig. 11. Diagram of the system formed of two swinging springs

Take the values of Cartesian coordinates in the vertical plane Oxy as generalized coordinate functions $X(t)$ and $Y(t)$. We have Lagrangian:

$$L=0.5m\left(\left(\frac{dY}{dt}\right)^2+\left(\frac{dX}{dt}\right)^2\right)-0.5k\left(\sqrt{X^2+Y^2}-h\right)^2-$$

$$-0.5k\left(\sqrt{X^2+(Y-H)^2}-h\right)^2-9.81mY. \qquad (8)$$

To form a system of Lagrange differential equations of the second degree, use relation (3). As a result, a system of Lagrange equations of the second degree is obtained in the form:

$$mpq + 2pqku - kuh(q + p) = 0,$$

$$mpq\frac{d^2Y}{dt^2} + 2pqkY - kYqh -$$
$$- kpqH - kphY + kphH + 9.81mpq = 0,$$

where

$$p = \sqrt{X^2 + Y^2}; \;\; q = \sqrt{X^2 + Y^2 - 2YH + H^2}. \qquad (9)$$

The problem statement. Determine value of mass m which would provide a periodic path of movement of a joint load of a system of two swinging springs with coefficient of stiffness k and length h in the non-loaded state each.

Let load of the system of swinging springs have coordinates $X(0)=2$ and $Y(0)=3$ in initial position. Oscillations are initiated due to the energy of springs. That is, there will be no impulses applied to the spring load in direction of the axes: $dX(0)=0$ and $dX(0)=0$. Let $H=5$; $k=15$ and $h=2.5$.

Applying the algorithms and programs described in [31–34], solve the system of equations (9) by the numerical Runge-Kutta method with initial conditions $X(0)=2$ and $Y(0)=3$; $dX(0)=0$ and $dX(0)=0$. It is necessary to take value of mass m for the specified parameters k and h in order to provide periodicity of motion of the path of the swinging spring load. Fig. 12 shows an image of the class of periodic paths depending on the load mass m. Fig. 13 shows combined phase paths constructed in the $\{u, Du\}$ and $\{v, Dv\}$ planes shown in red and blue, respectively. To confirm value of the found critical value of M, the graph of saturation of the image of the phase path lines can be used [34]. The site [36] demonstrates computer animations of the corresponding oscillations.

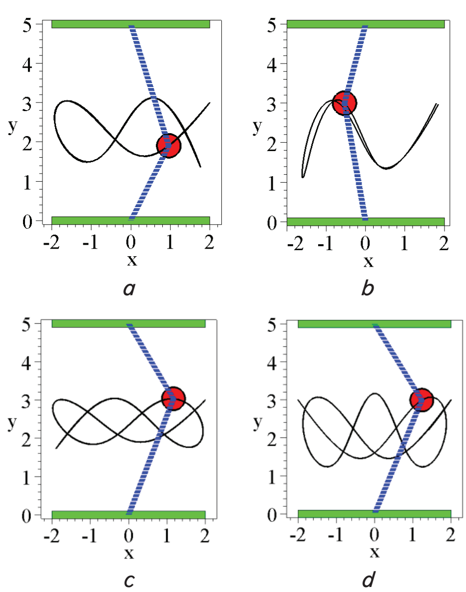


Fig. 12. Periodic paths of motion of the joint load of two swinging springs for: $m=0.53$ (*a*); $m=0.89$ (*b*); $m=0.2$ (*c*); $m=0.758$ (*d*)

With the help of phase paths, it is possible to determine ranges of change of functions of generalized coordinates as well as rate of this change (Fig. 13).

$2<u(t)<2$; $-5<Du(t)<5$; $1.5<v(t)<3$;
$-6<Dv(t)<6$ (Fig. 13, *a*);

$2<u(\mathrm{t})<2$; $-3<Du(t)<3$; $1.5<v(t)<3$;
$-5<Dv(t)<5$ (Fig. 13, *b*);

$2<u(t)<2$; $-8<Du(t)<8$; $1.8<v(t)<3$;
$-7<Dv(t)<7$ (Fig. 13, *c*);

$2<u(t)<2$; $-4<Du(t)<4$; $1.5<v(t)<3,2$;
$-6<Dv(t)<6$ (Fig. 13, *d*).

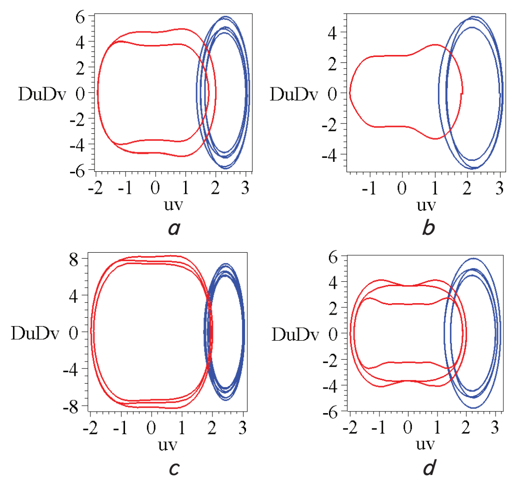


Fig. 13. Phase paths in planes $\{u, Du\}$ and $\{v, Dv\}$ for: $m=0.53$; $-2<u(t)<2$; $-5<Du(t)<5$; $1.5<v(t)<3$; $-6<Dv(t)<6$ (*a*); $m=0.89$; $-2<u(t)<2$; $-3<Du(t)<3$; $1.5<v(t)<3$; $-5<Dv(t)<5$ (*b*); $m=0.2$; $-2<u(t)<2$; $-8<Du(t)<8$; $1.8<v(t)<3$; $-7<Dv(t)<7$ (*c*); $m=0.758$; $-2<u(t)<2$; $-4<Du(t)<4$; $1.5<v(t)<3,2$; $-6<Dv(t)<6$ (*d*)

4. 4. The class of periodic paths of motion of a load attached to a spring suspended to a movable carriage

Let us apply a swinging spring as a mover when moving horizontally a carriage model installed on a «ramp». To do this, create an oscillatory system by suspending the swinging spring to the carriage bottom (Fig. 14). Show that to model translational motion of the carriage, it is necessary to ensure motion of the spring load in a periodic path. Also, it is necessary to apply impulse to the carriage in a direction of intended motion. Here is the way to determine path of movement of the load of the swinging spring suspended to a movable carriage along vertical axis Oxy. Parameters will be as follows: the carriage mass M, the load mass m, initial length of the spring in non-loaded state h, the spring stiffness k and initial conditions for occurrence of oscillations.

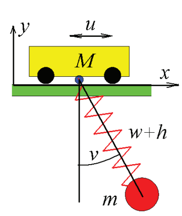


Fig. 14. Diagram of a swinging spring suspended to a movable carriage

Choose value of the carriage motion in horizontal direction as the first generalized coordinate function $u(t)$. Value of the angle between the swinging spring axis and vertical axis Oy will be the second generalized coordinate function $v(t)$. Relate the third generalized coordinate function $w(t)$ to the change of the spring length in time; denote length of the swinging spring in non-loaded state by h. Then virtual coordinates of the moving point load can be calculated according to the formulas:

$$x=u+(h+w)\sin v;\quad y=-(h+w)\cos v. \qquad (10)$$

Specify Lagrangian as difference between kinetic and potential energies:

$$L=0.5(M+m)\left(\frac{du}{dt}\right)^2-0.5k(w-h)^2+$$
$$+\left(\begin{array}{l}0.5m\left(\left(\frac{dw}{dt}\right)^2+w^2\left(\frac{dv}{dt}\right)^2+\right.\\ \left.+2\frac{du}{dt}\left(\frac{dw}{dt}\sin v+w\frac{dv}{dt}\cos v\right)\right)\end{array}\right)+9.81mw\cos v. \qquad (11)$$

To form a system of Lagrange differential equations of the second degree, apply relation (3). As a result, the system of Lagrange equations of the second degree is obtained in the form:

$$(m+M)\frac{d^2u}{dt^2}+\left(0.5m\frac{d^2w}{dt^2}\sin v+4\frac{dw}{dt}\frac{dv}{dt}\cos v+\right.$$
$$\left.+2w\frac{d^2v}{dt^2}\cos v-2w\left(\frac{dv}{dt}\right)^2\sin v\right)=0;$$

$$0.5m\left(4w\frac{dv}{dt}\frac{dw}{dt}+2w^2\frac{d^2v}{dt^2}+2w\frac{d^2u}{dt^2}\cos v+\right.$$
$$\left.+2\frac{du}{dt}\frac{dw}{dt}\cos v-2w\frac{du}{dt}\frac{dv}{dt}\sin v\right)-$$
$$-m\frac{du}{dt}\left(\frac{dw}{dt}\cos v-w\frac{dv}{dt}\sin v\right)+9.81mw\sin v=0; \qquad (12)$$

$$0.5m\left(2\frac{d^2w}{dt^2}+2\frac{d^2u}{dt^2}\sin v+2\frac{du}{dt}\frac{dv}{dt}\cos v\right)-$$
$$-0.5m\left(2w\left(\frac{dv}{dt}\right)^2+2\frac{du}{dt}\frac{dv}{dt}\cos v\right)-$$
$$-9.81m\cos v+k(w-h)=0.$$

The problem statement. Determine length h of non-loaded swinging spring which will oscillate to ensure translational motion of the carriage in horizontal direction provided that path of the load motion is periodic. Oscillations of the swinging spring suspended to the carriage as well as initial impulse supplied to the carriage in the direction of Ox axis will be movers of the motion process. Presence of such an impulse is necessary for translational motion of the carriage. With no impulse, the carriage will oscillate along the Ox axis relative to the initial position. Take the following parameters of the oscillatory system: the carriage mass $M=500$; the swinging spring load mass $m=86.8$; spring stiffness $k=750$.

Initial position of the swinging spring is horizontal, that is $v(0)=\pi/2$. Oscillations are initiated by pendular motions of the swinging spring and by means of the initial impulse $du(0)=1$ applied to the carriage in direction of the Ox axis. Choose initial values for the spring elongation parameter w in a form $w(0)=5$; $dw(0)=0$. That is, initial length of the swinging spring is equal to five conventional units.

In accordance with the procedure described in [31–34], first solve a system of equations (12) by Runge-Kutta numerical method at initial conditions $u(0)=1$; $du(0)=1$; $v(0)=\pi/2$; $dv(0)=0$; $w(0)=5$; $dw(0)=0$. Applying the method of projection focusing, select such value of parameter h which would ensure periodicity of the spring load path. To find critical value of h, construct integral curves in the phase spaces $\{u, Du, t\}$, $\{v, Dv, t\}$ and $\{w, Dw, t\}$ as well as phase paths in planes $\{u, Du\}$ and $\{v, Dv\}$. Using the phase paths, ranges of change of coordinate functions of parameters during oscillation of the swinging spring can be determined. To confirm critical value of h that was found, the graph of saturation of the image of the phase path lines [34] can be used.

After calculating h, this value must be substituted in the system of Lagrange equations of the second degree (12) and numerically solved by Runge-Kutta method with respect to the functions $u(t)$; $v(t)$ and $w(t)$. With the help of these solutions, an approximate image of the path of the swinging spring load motion is determined in the Oxy plane. Fig. 15 shows periodic paths of motion of swinging spring loads depending on the obtained values of h.

The Internet site [36] demonstrates computer animations of corresponding oscillations. The animations illustrate geometric models of the carriage motion to the right.

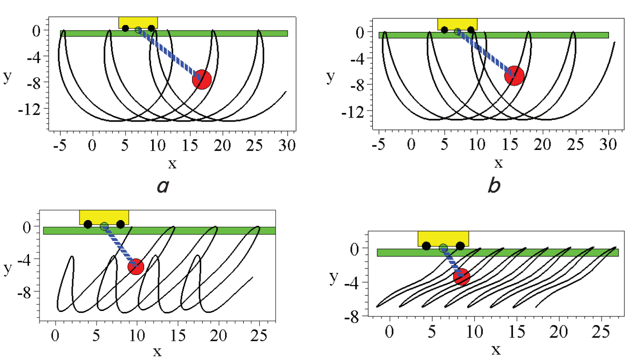


Fig. 15. Periodic paths of motion of the spring load for: $h=5.44$ (*a*); $h=5$ (*b*); $h=3.3$ (*c*); $h=2.08$ (*d*)

Fig. 16 shows phase paths in planes $\{u, Du\}$, $\{v, Dv\}$ and $\{w, Dw\}$ in red, blue and black, respectively.

Character of motion of geometric carriage model can be determined using the graph of derivative of the coordinate function $u(t)$ (Fig. 17).

With the help of phase paths, it is possible to determine ranges of change of generalized coordinate functions as well as rate of their change (Fig. 16).

$1<u(t)<12$; $-0.5<Du(t)<2.5$; $-1<v(t)<1$;
$-1<Dv(t)<1$; $3<w(t)<9$; $-3<Dw(t)<3$ (Fig. 16, *a*);

$1<u(t)<12$; $-0.5<Du(t)<2.5$; $-1<v(t)<1$;
$-1<Dv(t)<1$; $3<w(t)<9$; $-4<Dw(t)<4$ (Fig. 16, *b*);

$1<u(t)<11$; $0<Du(t)<0.5$; $-1<v(t)<2$; $-2<Dv(t)<2$;
$1<w(t)<8$; $-10<Dw(t)<10$ (Fig. 16, *c*);

$1<u(t)<12$; $0{,}5<Du(t)<2$; $-0.5<v(t)<3$;

$-5.5<Dv(t)<5.5$; $3<w(t)<7$; $-10<Dw(t)<10$ (Fig. 16, *d*).

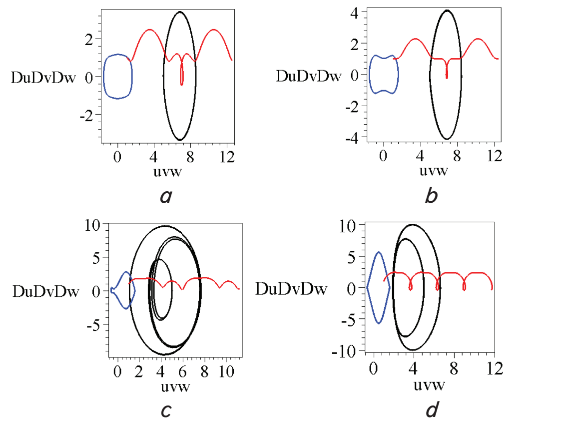


Fig. 16. Phase paths in planes $\{u, Du\}$, $\{u, Du\}$ and $\{w, Dw\}$ for: $h=5.44$; $1<u(t)<12$; $-0.5<Du(t)<2.5$; $-1<v(t)<1$; $-1<Dv(t)<1$; $3<w(t)<9$; $-3<Dw(t)<3$ (*a*); $h=5$; $1<u(t)<12$; $-0.5<Du(t)<2.5$; $-1<v(t)<1$; $-1<Dv(t)<1$; $3<w(t)<9$; $-4<Dw(t)<4$ (*b*); $h=3.3$; $1<u(t)<11$; $0<Du(t)<0.5$; $-1<v(t)<2$; $-2<Dv(t)<2$; $1<w(t)<8$; $-10<Dw(t)<10$ (*c*); $h=2.08$; $1<u(t)<12$; $0{,}5<Du(t)<2$; $-0.5<v(t)<3$; $-5.5<Dv(t)<5.5$; $3<w(t)<7$; $-10<Dw(t)<10$ (*d*)

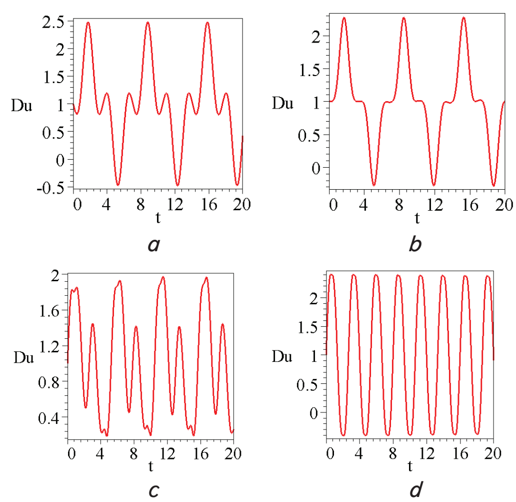


Fig. 17. Graphs of derivative of the coordinate function $u(t)$ for: $h=5.44$ (*a*); $h=5$ (*b*); $h=3.3$ (*c*); $h=2.08$ (*d*)

For completeness of classification, let us give other periodic paths of a load of a swinging spring suspended to a movable carriage.

The problem statement. Determine mass m of a swinging spring of length h in non-loaded state suspended to a movable carriage of mass M which would ensure periodic oscillatory character of motion of this carriage relative to its initial position. Choose parameters of the oscillatory system as follows: the carriage mass $M=10$; the spring stiffness $k=35$, the spring length $h=3$. Choose generalized coordinate functions and sequence of problem solving as in the previous example.

Let initial position of the swinging spring be horizontal, that is $v(0)=\pi/2$. Oscillations are only initiated by pendular motions of the swinging spring. Let us verify that the carriage will move only to the left or to the right and will not move translationally towards Ox axis without initial condition $du(0)=1$. In the previous example, the carriage translationally proceeded in a steady way to the right thanks to such an initial impulse. Initial values for of the spring elongation parameter w will be chosen as $w(0)=3$; $dw(0)=0$. That is, initial length of the swinging spring is equal to three conventional units.

Consequently, the problem is solved with initial conditions $u(0)=0$; $du(0)=0$; $v(0)=\pi/2$; $dv(0)=0$; $w(0)=3$; $dw(0)=0$. According to the procedure described in [31–34], determine possible masses m of the spring load that provide periodic paths: 5.06; 4.59; 3.85 and 1.91 (Fig. 18).

Fig. 19 shows phase paths in planes $\{u, Du\}$, $\{v, Dv\}$ and $\{w, Dw\}$ in red, blue, and black, respectively.

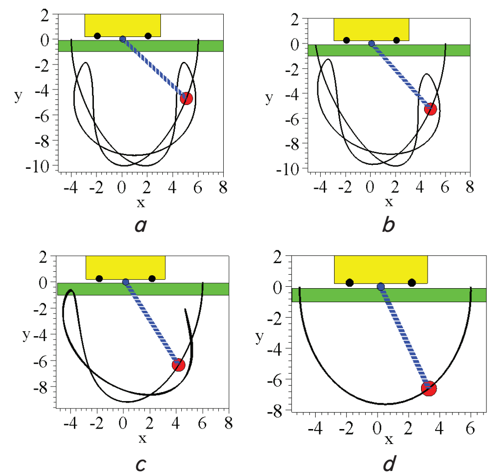


Fig. 18. Periodic paths of motion of a spring load for: $m_2=5.06$ (*a*); $m_2=4.59$ (*b*); $m_2=3.85$ (*c*); $m_2=1.91$ (*d*)

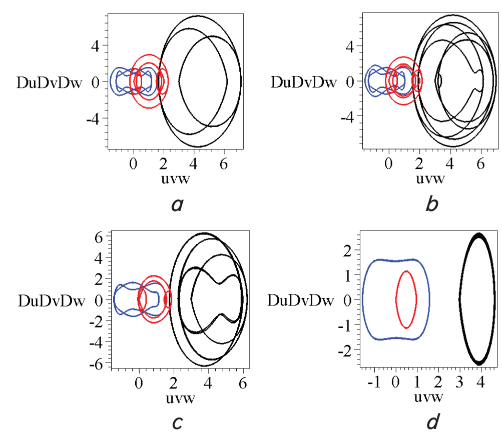


Fig. 19. Phase paths in planes $\{u, Du\}$, $\{v, Dv\}$ and $\{w, Dw\}$ for: $m_2=5.06$ (*a*); $m_2=4.59$ (*b*); $m_2=3.85$ (*c*); $m_2=1.91$ (*d*)

With the help of obtained phase paths, it is possible to determine ranges of change of generalized coordinate functions as well as rate of this change.

6. Discussion of classification and results of modeling the path of motion of the load of the swinging spring and is modifications

The developed classification and the compiled table of periodic paths make it possible to solve inverse problems, namely, add periodic curves as parameters in a graphic form to the list of numerical parameters of the swinging spring.

Values of the ratio of horizontal and vertical periods of oscillation of the swinging spring enable determination of numerical values of the parameters that would ensure existence of a periodic path of load motion preliminary chosen from the indicated six path forms.

Besides, the developed method for constructing periodic paths makes it possible to estimate their lengths by counting the number of pixels in the path image. That is, if necessary, a possibility should appear to specify a periodic path of a certain length which should be taken into account when studying dynamic systems in which energy «pumping» takes place between its parts.

The estimates of limits and speeds of change of pendulum angles as well as corresponding elongations and speeds of spring elongation given in the work allow one to investigate the swinging spring versions, for example, in a form of suspension to a movable carriage. Lack of a systems approach to modeling periodic paths of motion of the swinging spring load and the spring varieties hampered algorithmic implementation of similar solutions.

The results obtained can be explained by possibility of applying the Lagrange variational principle to calculation of mechanical oscillations of the swinging spring type. This has allowed us to use the Lagrange equation of the second degree to describe motion of the spring load.

The non-realized possibilities of geometric modeling in the study of oscillations of concrete swinging springs include consideration of their resonant state. A question arises: can the resonant state of the swinging spring manifest itself as a periodic path of its load motion? How this periodic path will look like? Answers to such questions are important as angular sway of a swinging spring is most effective due to this spring energy. Development of random transverse perturbation will last to a fixed value of amplitude since the spring energy reserves are finite. Stretching (or compression) of the swinging spring again occurs after reaching such an amplitude during oscillation of this spring. It is necessary to study the range of variation of parameters with a maximum corresponding to the ratio $mg/kh=1/4$, where m is the load mass, k is the spring stiffness, h is spring length in non-loaded state, g=9.81. It is necessary to check under what conditions this relation is fulfilled with acceptable accuracy and how it affects image of periodic paths of the spring load motion. It is necessary to determine number of possible periodic paths for a certain set of input parameters, as well as classify images of periodic paths and perform their gradation taking into account increase in their lengths.

It will be of interest to study from these positions nonlinear coupled systems with interacting subsystems on examples of engineering problems. Study of mechanical devices in which springs will affect paths of oscillation of their loads will be a step towards this goal. Some examples of such devices are given in this paper. It is still advisable to add mechanisms with springs and movable loads of the following schemes:

– a pendulum fixed to a vertical spring in a guide device;

– variants of a double pendulum, one of the elements of which is a swinging spring;

– a pendulum under a movable carriage whose position is influenced by a spring.

Difficulties in this direction of studies will arise when trying to determine the resonant state of a swinging spring included in such devices as well as in the case of study of oscillations of a spatial swinging spring.

7. Conclusions

1. It has been shown that there are at least six geometric forms of periodic paths of motion of the swinging spring load that correspond to specified coefficients of stiffness k and the value of mass m.

2. With the help of the ratio of horizontal and vertical periods of load oscillations, it was possible to determine six numbers, namely 1.71; 1.525; 1.36; 1.14; 0.986; 0.9 which correspond to geometric forms of periodic motion paths.

3. Classes of periodic paths of motion were found for:

a) the load of a swinging spring whose axis oscillates with the mathematical pendulum (for example, with the following parameters: $R=8$; $m=15$; $k=150$; $h=2.5$; $M=2.19$ and initial conditions: $v(0)=0$; $dv(0)=1.5$; $u(0)=2$; $du(0)=0$);

b) the load positioned at the common point of attachment of two swinging springs (for example, with the following parameters: $H=5$; $k=15$; $h=2.5$; $m=0.758$ and initial conditions: $X(0)=2$ and $Y(0)=3$; $dX(0)=0$ and $\mathrm{d}Y(0)=0$);

c) the load of the swinging spring suspended to a movable carriage (for example, with the following parameters: $M=500$; $m=86.8$; $k=750$; $h=2.08$ and initial conditions: $u(0)=1$; $du(0)=1$; $v(0)=\pi/2$; $dv(0)=0$; $w(0)=5$; $dw(0)=0$).

4. For all considered variants, phase paths of functions of generalized coordinates of the swinging spring and its varieties were constructed and the range of change of their values and velocities of load motion was estimated.

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