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UDC 62-752+62-755: 621.634
DOI: 10.15587/1729-4061.2019.169181

A PROCEDURE OF STUDYING STATIONARY MOTIONS OF A ROTOR WITH ATTACHED BODIES (AUTO-BALANCER) USING A FLAT MODEL AS AN EXAMPLE

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Модернізований енергетичний метод дослідження динаміки роторів. Метод застосовний для роторів на ізотропних пружно-в'язких опорах, коли до ротора приєднані тіла, на які при відносному русі діють пружні і в'язкі сили. Метод призначений для пошуку, визначення умов існування і оцінки стійкості стаціонарних рухів роторної системи. На стаціонарних рухах відносні руху приєднаних тіл припиняються, і система обертається як одне ціле навколо осі обертання, утвореної опорами.

Ефективність методу проілюстрована на прикладі плоскої моделі ротора з автобалансиром з багатьма вантажами у вигляді куль, роликів і маятників.

Встановлено, що як при наявності, так і відсутності демпфірування в опорах, при достатній балансувальній ємності автобалансира система має сім'ю основних рухів (на них ротор збалансований).

При відсутності демпфірування в опорах система має:

– при наявності неврівноваженості ротора – ізольовані побічні рухи (на них ротор незбалансований), в яких центри мас вантажів відхилені в бік неврівноваженості або в протилежний бік;

– при відсутності неврівноваженості ротора – однопараметричні сім'ї побічних рухів, в яких центри мас вантажів лежать на одній прямій.

При наявності демпфірування в опорах:

– при наявності неврівноваженості ротора система має ізольовані побічні рухи, в яких центри мас вантажів лежать на одній прямій, і пряма утворює з вектором неврівноваженості кут, що залежить від швидкості обертання ротора;

– при відсутності неврівноваженості ротора побічних рухів не існує.

При відсутності демпфірування в опорах побічні рухи і області їх існування не залежать від кутової швидкості обертання ротора, а при наявності – залежать.

Як при наявності, так і при відсутності демпфірування в опорах:

– на дорезонансних швидкостях обертання ротора стійким може бути тільки той побічний рух, на якому сумарна неврівноваженість ротора і вантажів найбільша;

– на зарезонансних швидкостях обертання ротора може бути стійка тільки сім'я основних рухів

Ключові слова: ротор, ізотропна опора, автобалансир, стаціонарний рух, стійкість руху, рівняння усталеного руху

1. Introduction

Passive auto-balancers are used to balance high-speed rotors [1–3]. The motion of such systems sets in over time. Loads (balls, rollers, pendulums, etc.) balance the rotor at the so-called main steady motions but not at the secondary ones. From a mathematical point of view, for the auto-balancer to be operable, it is necessary and sufficient that the main

Received date: 11.05.2019 Accepted date: 29.05.2019 Published date: 26.06.2019

motions were stable and the secondary ones unstable. Therefore, the analytical theory of passive auto-balancers searches for all possible steady motions of systems and studies their stability [1–17].

In cases of isotropic supports, when using a coordinate system rotating synchronously with the rotor, equations of motions and a part of steady motions are stationary. Stability is studied using the theory of stability of stationary solutions of systems of nonlinear autonomous differential equations [2–5, 8–13, 16–17].

The search for and study of stability of steady motions of a rotor - auto-balancer system is a complex mathematical problem [1–17]. The problem gets much more complicated for auto-balancers with many loads when damping in the rotor supports is taken into account, at a spatial rotor motion, when balancing the rotor with several auto-balancers, etc.

The energy method [5, 10] is most effective in searching for all stationary motions, in determining conditions of their existence and assessing their stability. The method was set forth and applied to rotors with auto-balancers mounted on isotropic elastic supports (without taking into account damping in supports). In conventional auto-balancers, viscous resistance forces act on loads during relative motion.

Damping takes place in supports of actual rotor systems. Devices in which viscous and elastic forces act on the loads can balance the rotor and damp or suppress its oscillations [16, 17]. Therefore, it is important to extend the energy method to the rotors mounted on isotropic viscous-elastic supports in the case when other bodies on which elastic and viscous forces act at relative motion are attached to the rotor. It is important to demonstrate effectiveness of the method on a concrete example.

2. Literature review and problem statement

Design and principle of operation of ring, ball, pendulum (classical) auto-balancers are described in [1]. The issues of designing classical auto-balancers are considered and extensive patent information on designs of auto-balancers of the above-mentioned type is given in [2]. So-called non-classical auto-balancers with loads in the form of bodies of certain shapes having a fixed point on the longitudinal axis of the rotor were studied in [3] along with classical auto-balancers.

Within the framework of the general theory of passive auto-balancers, all possible stationary motions of systems are sought for and their stability is assessed. Load motions relative to the rotor cease at stationary motions and the system rotates as a single whole around the axis formed by supports. The system is balanced at the so-called main motions but not at the secondary motions. In order that the auto-balancers were operable, it is necessary that the main motions were stable and the secondary motions unstable at working speed of the rotor.

Let us consider the results obtained in the framework of a flat model of a rotor mounted on isotropic bearings and balanced with a two-ball (two-pendulum) auto-balancer.

Ring, pendulum, and ball auto-balancers are described in [1]. A rotor mounted on isotropic elastic supports is considered. It was shown that in the case of an auto-balancer having sufficient balancing capacity, the rotor always has one isolated main motion and three isolated secondary motions. Based on the rotor self-centering phenomenon, it was concluded that solely the secondary motion at which balls deflect to the heavy side of the rotor is stable at sub-resonant rotor speeds and the main motion is stable at super-resonant speeds.

An analytical theory based on the theory of stability of stationary motions of nonlinear autonomous systems was developed in [4]. Generalized coordinates describing rotor motion in a coordinate system rotating together with the rotor and generalized coordinates describing motion of balls relative to the rotor were used. Differential equations of the system motion were derived. All stationary motions were found. Their stability has been studied using the first Lyapunov's method. The results obtained in [1] were confirmed in general. However, it was found that damping in supports narrows the domain of existence of secondary motions. It has been established under some simplifying assumptions that the main motion becomes asymptotically stable at velocities slightly exceeding the resonant frequency of the rotor rotation. Labor-intensiveness of such an approach to creation of a general theory should be noted. Thus, it is necessary to derive differential equations of the disturbed motion and a characteristic equation and study its roots for each stationary motion. However, this approach is the most accurate since it allows one to find necessary and sufficient conditions of stability of stationary motions. Implementation of the approach becomes much more complicated in the case of auto-balancers with many loads, at a spatial motion of rotors, when balancing the rotor with the help of several auto-balancers, etc.

An analytical theory for a rotor mounted on isotropic elastic supports was built in [5] based on the energy approach. Generalized coordinates describing rotor motion in a coordinate system rotating together with it and generalized coordinates describing motion of balls relative to the rotor were used. A generalized potential was found for the system. The potential was studied for the conditional extremum: it was assumed that some equations of steady motion corresponding to the generalized coordinates of the rotor are executed. As a result, conclusions on stability of stationary motions made in [1] were confirmed. The smallest laboriousness of the approach should be pointed out. Differential equations of the system motion are not derived but a generalized potential is sought in the approach. The conditions necessary for stability of steady motion are determined proceeding from the condition of minimum of the generalized potential at a steady motion. In a number of cases relevant to practice, the obtained necessary conditions for stability are close to sufficient ones. There is no general description of an approach suitable for rotors mounted on isotropic supports and the procedure of accounting for damping in supports was not described in this paper.

A method of synchronization of dynamic systems was used in [6] to elaborate a general theory. A rotor mounted on isotropic elastic supports was considered. The results obtained in [1, 5] were confirmed. The approach is less time consuming than that used in [4]. In contrast to the earlier described approaches, the synchronization method makes it possible to analytically analyze stability of main motions in the case of anisotropic supports [7]. However, stability is studied in a smallness (according to Lyapunov) using a small parameter. This reduces accuracy of determining boundaries of stability domains. The approach gets much more laborious in the presence of damping in supports.

Stability of the main motion was analytically studied in [8] most completely. The approach described in [4] was applied. A characteristic equation corresponding to the main motion was obtained. Its roots were decomposed in powers of a small parameter at various smallness ratios between parameters. Dependence of boundaries of the stability domains on imbalance and internal and external forces of viscous resistance was found. It was also found that there are one or three critical speeds in the vicinity of the resonant speed of the rotor. The main motion is stable: in the case of one critical speed (when it is exceeded in the case of three critical speeds) between the first and the second and above the third critical speed. It is essential that when the small parameter is zero, all critical speeds coincide with the resonant speed of the rotor. This allows one to conclude that the method of synchronization of dynamic systems and the energy method of critical speeds (upon their transition, the motion stability changes to instability or vice versa) were found in a «zero» approximation.

Let us consider the analytical results obtained in the framework of a flat model of the rotor mounted on isotropic elastic-viscous supports in the case of auto-balancers with many identical loads (balls, rollers, pendulums).

Stability of a family of main motions was studied analytically in [9]. The approach used in [4, 8] was applied. The zeros in the characteristic equation were attributed to a multiparameter family. The study of nonzero roots in the characteristic equation has found boundaries of the stability domains. It has been established that the number of loads does not exert an essential effect on these boundaries.

The approach considered in [4, 8] was applied in [3]. A method for studying stability of families of main motions proceeding from generalized rotor coordinates and unbalance parameters was proposed. Main and secondary motions of the system and conditions of their existence were found. A characteristic equation was derived for each motion (a family of motions) and its roots were studied. It has been established that when the family of main motions is stable, secondary motions are unstable or do not exist at all. It should be noted that despite the modernization, the approach remains considerably laborious. Its application becomes much more complicated in the case of different loads, spatial motion of the rotor, when balancing the rotor with several auto-balancers, etc.

The energy approach previously used in [5] was generalized in [10] for rotors mounted on isotropic elastic supports. Main assumptions for the rotor system were formulated and main stages of the approach were described. Effectiveness of the approach is illustrated by an example of a rotor with a fixed point balanced with a passive auto-balancer. The modernized approach does not take into account damping in supports. Nevertheless, this is the least time-consuming method for creating an analytical theory of stability of stationary motions of rotors with auto-balancers.

An attempt was made in [11] to construct a bifurcation theory of stationary motions of a multi-ball auto-balancer within the framework of a flat model of a rotor mounted on isotropic supports. However, to construct a (complete) bifurcation theory, it is necessary to know all possible steady motions of the system and the conditions for their existence [12]. Then, if we take angular velocity of the rotor as a bifurcation parameter, we can determine from the bifurcation points which steady motions will lose and which ones will acquire stability with an increase in the rotor angular velocity. For example, such a theory was built for an isolated system consisting of a carrying body and two pendulums [13] since only stationary steady motions are possible in such a system. The rotor with auto-balancers also has non-stationary steady motions caused by:

– sticking (lagging behind the rotor) of balls [3] or pendulums [14];

– excitation of parametric oscillations of loads in the vicinity of a relative equilibrium position [15].

Therefore, it is impossible to create a bifurcation theory of rotors with auto-balancers according to steady motions alone. However, the study of stationary motions is a necessary stage in construction of such a theory.

Damping in supports is present in actual rotor systems and is used. The bodies attached to the rotor can be oscillation dampers, dynamic oscillation dampers, auto-balancers, parts of a composite rotor, etc. During relative body motion, they can be affected by elastic and viscous forces such as in a ball-rod auto-balancer with springs [17] or in a ball auto-balancer with springs [18]. Therefore, it is important to extend the energy approach to such rotor systems. It is also relevant to illustrate effectiveness of the approach using a concrete example.

3. The aim and objectives of the study

The study aim is to determine features of application of the energy method of studying stationary motions to the systems consisting of a rotor mounted on isotropic elastic-viscous supports and bodies attached to it and subjected to viscous and elastic forces during relative motion. This will make it possible to assess operability of the devices attached to the rotor (auto-balancers, dampers, dynamic oscillation dampers, etc.) with minimum labor input.

To achieve this aim, it is necessary to solve the following tasks:

1) to determine main stages of application of the energy method of studying stationary motions to the considered rotor systems;

2) using this method, to study stationary motions within the framework of a flat model of a rotor mounted on isotropic elastic-viscous supports and balanced with an auto-balancer having many corrective loads attached, for what:

– to construct a model and find stationary motions of the rotor system and the conditions for their existence;

– to find the conditions necessary for stability of stationary motions of the rotor system;

– to assess the effect of damping in supports on the system dynamics.

4. The methods used in studying stationary motions of the rotor with attached bodies

Holonomic mechanical systems with stationary restraints are considered. Differential equations of motion of such systems can be obtained using the Lagrange equations of the second kind in the form [18]:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}+\frac{\partial D}{\partial \dot{q}_j}=0, \quad / j=\overline{1,N}/, \tag{1}$$

where t is time; $L=T-V$ is the Lagrange function in which T is kinetic energy and V is potential energy of the system; q_j is the generalized coordinate, $\dot{q}_j$ is the generalized velocity number j; N is the number of degrees of freedom in the system; $D=\frac{1}{2}\sum_{i=1}^{n}\beta_i v_i^2$ is the dissipative function in which n is the number of material points in the system; β_i is the coefficient of viscous friction forces acting on the i point; v_i is the modulus of velocity of the i point.

In the general case, kinetic energy and dissipative function will not explicitly depend on time and will have free, linear and quadratic terms relative to generalized velocities:

$$T = T_0 + T_1 + T_2; \quad D = D_0 + D_1 + D_2. \tag{2}$$

The generalized coordinates will be constant at stationary steady motions:

$$q_j = \text{const}_j, \quad /j = \overline{1,N}/. \tag{3}$$

In this case, the generalized velocities are zero and:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial L_0}{\partial q_j}, \quad L_0 = T_0 - V, \quad \frac{\partial D}{\partial \dot{q}} = \frac{\partial D_1}{\partial \dot{q}}.$$

Introduce the generalized potential:

$$\Pi = V - T_0. \tag{4}$$

Then, the equations of steady motions take the form:

$$\frac{\partial \Pi}{\partial q_j} + \frac{\partial D_1}{\partial \dot{q}_j} = 0, \quad /j = \overline{1,N}/. \tag{5}$$

These equations are used to search for stationary steady motions (3) of the system.

According to the Lagrange-Dirichlet theorem, a necessary condition for stability of some stationary steady motions (3) consists in that the generalized potential has at least a non-isolated minimum at it.

The theory is defined concretely for the mechanical systems that include a rotor and bodies attached to it.

5. The study of stationary motions of the rotor with attached bodies

5. 1. The energy method of searching for and assessing the stability of stationary motions of the rotor with attached bodies

Let us consider a holonomic mechanical system. The system includes a rotor mounted on isotropic elastic-viscous supports. The rotor rotates at a constant angular velocity ω. Bodies are attached to the rotor. Newton's resistance forces and linear elastic forces act on the bodies in their relative motion.

Denote generalized coordinates of the rotor as $z_i, /i = \overline{1,n_r}/$, where n_r is the number of degrees of freedom of the rotor. Denote generalized coordinates of the attached bodies as ψ_j, $/j = \overline{1,n_b}/$, where n_b is the number of degrees of freedom of the attached bodies. Note that $n_r + n_b = N$. Then, the generalized coordinates will be constant at stationary motions:

$$z_i = \text{const}_i, \quad /i = \overline{1,n_r}/; \quad \psi_j = \text{const}_{j+n_r}, \quad /j = \overline{1,n_b}/. \tag{8}$$

The equations of stationary motions will be divided into two groups.

$$\frac{\partial \Pi}{\partial z_j} + \frac{\partial D_1}{\partial \dot{z}_j} = 0, \; /j = \overline{1,n_r}/; \quad \frac{\partial \Pi}{\partial \psi_j} + \frac{\partial D_1}{\partial \dot{\psi}_j} = 0, \; /j = \overline{1,n_b}/. \tag{9}$$

Assume that motions of the system are divided into fast and slow. At fast motions, the rotor takes a position corresponding to the current total imbalance (imbalance from the attached bodies and imbalance of the rotor itself). At slow motions, the attached bodies slowly take the position of their relative balance. Restrain the rotor motions,

$$\frac{\partial \Pi}{\partial z_j} + \frac{\partial D_1}{\partial \dot{z}_j} = f_j(z_1, ..., z_{n_r}, \psi_1, ..., \psi_{n_b}) = 0, \quad /j = \overline{1,n_r}/. \tag{10}$$

Under the restraints, the rotor instantly takes the position corresponding to total imbalance.

Solve the equations of constraints (10) relative to the generalized coordinates of the rotor and obtain:

$$z_i = Z_i(\psi_1, ..., \psi_{n_b}), \quad /i = \overline{1,n_r}/. \tag{11}$$

Elimination of generalized rotor coordinates from the generalized potential and the linear part of the dissipative function by using equations (11) gives the following:

$$\Pi^* = \Pi^*(\psi_1, ..., \psi_{n_b}), \quad D_1^* = D_1^*(\psi_1, ..., \psi_{n_b}, \dot{\psi}_1, ..., \dot{\psi}_{n_b}). \tag{12}$$

The equations of steady motions of bodies (in the new system with imposed constraints):

$$\frac{\partial \Pi^*}{\partial \psi_j} + \frac{\partial D_1^*}{\partial \dot{\psi}_j} = 0, \quad /j = \overline{1,n_b}/. \tag{13}$$

For stability of some stationary motion (8) of the rotor system, it is necessary that the transformed reduced potential (12) at this motion had at least a non-isolated minimum.

The method is applied in the following sequence:

1) physical-mathematical model of the rotor with attached bodies is described;

2) generalized potential and dissipative function are derived;

3) equations of stationary motions are set up; all possible stationary motions are sought and conditions for their existence determined;

4) using the equations of stationary motion of the rotor, generalized coordinates of the rotor (or total imbalances) are excluded from the generalized potential;

5) necessary stability conditions for each stationary motion are determined from the condition of minimum of the transformed generalized potential.

5. 2. Application of the method in the framework of the flat model of the rotor and the auto-balancer with many corrective loads

5. 2. 1. Description of the flat model of the rotor and the auto-balancer

Diagrams explaining the system motion are shown in Fig. 1. To describe the system motion, the following is used:

– fixed axes, Ξ, H, (Fig. 1, *a*) coming from the fixed center of rotation, the *K* point;

– moving *X*, *Y* axes coming from the *K* point and rotating synchronously with the rotor with a constant angular velocity, ω;

– moving axes, X_O, Y_O coming from the rotor center (*O* point) and parallel to the *X*, *Y* axes.

The rotor motion is defined as a sum of two motions (Fig. 1, *b*): a rotational motion together with the *X* and *Y* axes and a translational motion together with the X_O, Y_O axes. Translational motion of the rotor and position of its center of mass determine the *x, y* coordinates.

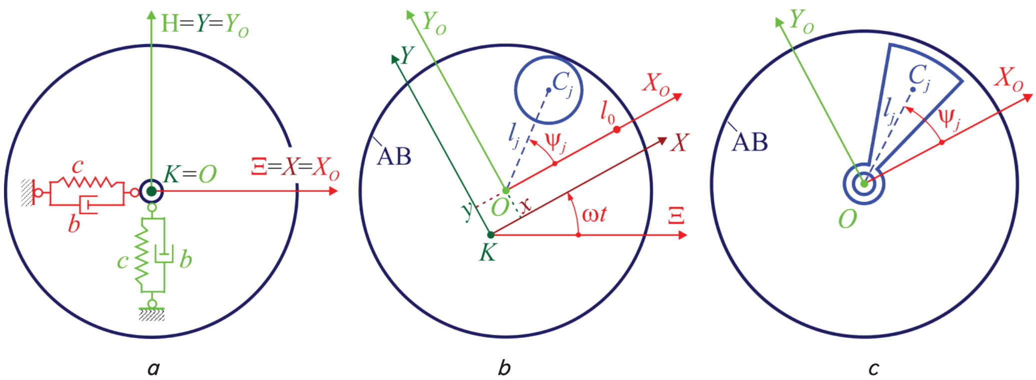

Fig. 1. A flat model of a rotor and an auto-balancer: a rotor mounted on isotropic elastic-viscous supports (*a*); kinematics of motion of a rotor, an unbalanced mass and a ball or a roller (*b*); kinematics of motion of a pendulum (*c*)

The rotor is held by elastic-viscous isotropic supports with the coefficient of stiffness, *c*, and the coefficient of viscosity, *b*.

The static imbalance of the rotor is caused by the point mass m_0 located at a distance l_0 from the longitudinal axis of the rotor (the *O* point). Assume without loss of generality that the point mass is located on the *X* axis.

The auto-balancer consists of n_b identical loads: pendulums, balls or rollers. The mass of the auto-balancer body is referred to the mass of the rotor. As is customary in the theory of passive auto-balancers, assume that the loads on the track do not interfere with each other's motion [1–14]. Neglect the effect of gravity on motion of the loads. Mass of one load is m_j. The center of mass of the load number *j* moves along a circle of radius l_j with its center located on the longitudinal axis of the rotor (Fig. 1, *b*). Position of the load relative to the rotor sets angle ψ_j between the *X* axis and the radius of the center of mass of the load, $/\, j=\overline{1,n_b}\,/$. Motion of the load number *j* relative to the body of the auto-balancer (rotor) is hampered by some force of viscous resistance $b_j l_j \,|\dot{\psi}_j|$, where b_j is the coefficient of force of viscous resistance.

The following are expressions for the system mass, the total imbalance of the rotor in the projections on the moving axes *X* and *Y* and the resonant frequency of the rotor rotation, respectively:

$$M_\Sigma = M + \sum\nolimits_{j=0}^{n_b} m_j,$$

$$s_x = m_0 l_0 + \sum\nolimits_{j=1}^{n_b} m_j l_j \cos\psi_j,$$

$$s_y = \sum\nolimits_{j=1}^{n_b} m_j l_j \sin\psi_j,$$

$$\omega_0 = \sqrt{c / M_\Sigma}. \tag{14}$$

The following expressions are projections of velocity of the center of mass of the rotor on the *X* and *Y* axes, respectively,

$$v_{Ox} = \dot{x} - \omega y, \; v_{Oy} = \dot{y} + \omega x. \tag{15}$$

Assume that the balancing capacity of the auto-balancer is enough to balance static unbalance of the rotor $(\sum\nolimits_{j=1}^{n_b} m_i l_i \geq m_0 l_0)$. It can be said that all loads are different if $\forall i, j \in \{1,2,\ldots,n_b\}$ $m_i l_i \neq m_j l_j$ and all loads are the same if $\forall i, j \in \{1,2,\ldots,n_b\}$ $m_i l_i = m_j l_j$.

5. 2. 2. The generalized potential and the dissipative function

Potential energy for the considered system:

$$\Pi = \frac{1}{2} c (x^2 + y^2). \tag{16}$$

Relative motions of loads cease at steady motions. The system behaves like an absolutely rigid body rotating at a constant angular velocity, ω. Its kinetic energy does not contain generalized velocities and can be represented as:

$$T_0 = I_K \omega^2 / 2, \tag{17}$$

where I_K is the axial moment of inertia of the system relative to the axis of rotation.

In turn, it is the sum of the following axial moments of inertia relative to the *K* point:

– of the rotor: $I_K^{(r)} = I_O^{(r)} + M(x^2 + y^2)$;

– of the point mass causing imbalance:

$$I_K^{(0)} = m_0[(x + l_0)^2 + y^2] = m_0 l_0^2 + m_0(x^2 + y^2) + 2 m_0 l_0 x =$$
$$= I_O^{(0)} + m_0(x^2 + y^2) + 2 m_0 l_0 x;$$

– of the load number *j*:

$$I_K^{(j)} = I_{C_j}^{(j)} + m_j[(x + l_j \cos\psi_j)^2 + (y + l_j \sin\psi_j)^2] =$$
$$= I_{C_j}^{(j)} + m_j l_j^2 + m_j(x^2 + y^2) + 2 m_j l_j (x\cos\psi_j + y\sin\psi_j) =$$
$$= I_O^{(j)} + m_j(x^2 + y^2) + 2 m_j l_j (x\cos\psi_j + y\sin\psi_j).$$

Thus,

$$I_K = I_K^{(r)} + \sum\nolimits_{j=0}^{n_b} I_K^{(j)} = I_O^{(r)} + \sum\nolimits_{j=0}^{n_b} I_O^{(j)} + M_\Sigma(x^2 + y^2) +$$
$$+ 2 x s_x + 2 y s_y = I_O + M_\Sigma(x^2 + y^2) + 2 x s_x + 2 y s_y, \tag{18}$$

where $I_O = I_O^{(r)} + \sum\nolimits_{j=0}^{n_b} I_O^{(j)}$ is the axial moment of inertia of the system relative to the longitudinal axis of the rotor passing through the *O* point.

Then the generalized potential (3) takes the form:

$$\Pi = V - T_0 = \frac{1}{2}(c - M_\Sigma \omega^2)(x^2 + y^2) -$$
$$- I_O \frac{\omega^2}{2} - \omega^2 (x s_x + y s_y). \tag{19}$$

This function generalizes analogue of the potential energy obtained in [5, 6] and extends it to the case of auto-balancers with an arbitrary number of different loads: pendulums, rollers, balls.

The dissipative function:

$$D=\frac{1}{2}bv_C^2+\frac{1}{2}\sum_{j=1}^{n_b}b_ju_j^2=$$

$$=\frac{1}{2}b[(\dot{x}-\omega y)^2+(\dot{y}+\omega x)^2]+\frac{1}{2}\sum_{j=1}^{n_b}b_jl_j^2\dot{\psi}_j^2. \quad (20)$$

The component which is linear relative to the generalized velocities:

$$D_1=b\omega(-\dot{x}y+\dot{y}x). \quad (21)$$

5. 2. 3. Stationary motions of the system

The equations of stationary motions (9) for the system in question take the form:

$$\frac{\partial \Pi}{\partial x}+\frac{\partial D_1}{\partial \dot{x}}=\left(c-M_\Sigma\omega^2\right)x-\omega^2 s_x-b\omega y=0,$$

$$\frac{\partial \Pi}{\partial y}+\frac{\partial D_1}{\partial \dot{y}}=\left(c-M_\Sigma\omega^2\right)y-\omega^2 s_y+b\omega x=0,$$

$$\frac{\partial \Pi}{\partial \psi_j}+\frac{\partial D_1}{\partial \dot{\psi}_j}=m_jl_j\omega^2(x\sin\psi_j-y\cos\psi_j)=0, /j=\overline{1,n_b}/. \quad (22)$$

This is a system of (n_b+2) nonlinear algebraic equations relative to (n_b+2) unknowns $x, y, \psi_j, /j=\overline{1,n_b}/$.

Stationary motions are solutions of the system of algebraic equations (22).

The system may have the following stationary motions:

– a family of main motions (at $n\geq 3$) at which the auto-balancer balances static unbalance of the rotor and there is no deviation of the longitudinal axis of the rotor from the axis of rotation:

$$x=0,\quad y=0,$$

$$s_x=\sum\nolimits_{j=1}^{n_b}m_jl_j\cos\psi_j+m_0l_0=0,$$

$$s_y=\sum\nolimits_{j=1}^{n_b}m_jl_j\sin\psi_j=0; \quad (23)$$

– secondary motions at which the auto-balancer does not balance static imbalance of the rotor and there is a deviation of the longitudinal axis of the rotor from the axis of rotation.

Let us find secondary motions. The following is found from the first two equations (22):

$$x=\Delta_1/\Delta,\quad y=\Delta_2/\Delta, \quad (24)$$

where

$$\Delta=(M_\Sigma\omega^2-c)^2+b^2\omega^2,$$

$$\Delta_1=-\omega^2[(M_\Sigma\omega^2-c)s_x-b\omega s_y],$$

$$\Delta_2=-\omega^2[(M_\Sigma\omega^2-c)s_y+b\omega s_x]. \quad (25)$$

The following is obtained from the last n_b equations in (22):

$$x\sin\psi_j-y\cos\psi_j=0 \Leftrightarrow \Delta_1\sin\psi_j-\Delta_2\cos\psi_j=0,$$

$$/j=\overline{1,n_b}/. \quad (26)$$

Equations (26) can simultaneously be satisfied provided there is such a certain θ angle that if:

$$\psi_j=\theta+k_j\cdot\pi,\quad k_j\in\{0,1\},\quad /j=\overline{1,n_b}/, \quad (27)$$

then $\theta=\arctan(y/x)=\arctan(\Delta_2/\Delta_1)$.

Notice that:

– a concrete secondary motion is characterized by the n_b-bit binary number $k_1k_2,...,k_{n_b}$ and the θ angle corresponding to the motion;

– at this secondary motion, the centers of mass of the loads lie on the same U axis which forms the θ angle with the X axis;

– when $k_j=0$, then load j is deflected in positive direction of the U axis and when $k_j=1$, it is deflected in the negative direction;

– the binary number $k_1k_2,...,k_{n_b}$ can take 2^{n_b} values from 00,..., 0 to 11..., 1;

– 0, 1, 2,... values of the θ angle can correspond to one binary number $k_1k_2,...,k_{n_b}$;

– different secondary motions correspond to one binary number $k_1k_2,...,k_{n_b}$ and two different values θ_1, θ_2 of the θ angle provided that $\tan\theta_1\neq\tan\theta_2$.

Find all possible values of the q angle corresponding to the binary number $k_1k_2,...,k_{n_b}$ and the conditions for existence of corresponding secondary motions.

Introduce a decimal number k corresponding to the binary number $k_1k_2,...,k_{n_b}$:

$$k=\sum\nolimits_{i=0}^{n_b-1}k_j2^i,\quad k\in\{0,1,\ldots,2^{n_b}-1\}. \quad (28)$$

Values of k_j are determined by the decimal number k according to the following recurrent formulas, $k_{n_b-1}=$ $=\mathrm{int}(k/2^{n_b-1})$:

$$k_{n_b-j}=\mathrm{int}\left[\left(k-\sum\nolimits_{i=1}^{j-1}k_{n_b-i}2^{n_b-i}\right)\Big/2^{n_b-j}\right], \quad (29)$$

where $\mathrm{int}(x)$ is the integer part of x.

Substitute (27) into (14) and obtain:

$$s_x(k)=s_{AB}(k)\cos\theta+m_0l_0,\quad s_y(k)=s_{AB}(k)\sin\theta, \quad (30)$$

where

$$s_{AB}(k)=\sum\nolimits_{j=1}^{n_b}(-1)^{k_j}m_jl_j \quad (31)$$

is a projection on the U axis of the total imbalance caused by the loads.

Transform the second equation in (26). Substitute Δ_1, Δ_2 from (25) into it and obtain:

$$-[(M_\Sigma\omega^2-c)s_x-b\omega s_y]\sin\theta+$$

$$+[(M_\Sigma\omega^2-c)s_y+b\omega s_x]\cos\theta=0.$$

Substitute s_x, s_y from (30) and obtain the following after transformations:

$$-m_0 l_0 (M_\Sigma \omega^2 - c)\sin\theta + b\omega s_{AB}(k) + b\omega m_0 l_0 \cos\theta = 0. \quad (32)$$

Values of the θ angle are determined from this equation (for any $\omega > 0$).

1) **The case of absence of damping in supports.** In the absence of damping in supports, $b=0$ and equation (32) takes the form:

$$-m_0 l_0 (M_\Sigma \omega^2 - c)\sin\theta = 0. \quad (33)$$

If $m_0 l_0 \neq 0$, then find $\theta = \{0, \pi, 2\pi, ...\}$ from (32). However, there is only one essentially different value:

$$\theta = 0. \quad (34)$$

Thus, the U axis coincides with the X axis directed towards the mass causing imbalance. When $k_j = 0$, the center of mass of the load j is deflected to the heavy side of the rotor (along the X axis) and when $k_j = 1$, then it is deflected to the light side (against the X axis).

If all loads are different, then the system has 2^{n_b} different secondary motions. If all loads are the same, then there will be only $(n+1)$ fundamentally different secondary motions.

At the first secondary motion, all loads are deflected in the direction of the vector of static imbalance of the rotor and, consequently, total imbalance of the rotor and loads is the greatest:

$$s_{\max} = \sum_{j=0}^{n_b} m_j l_j. \quad (35)$$

If $m_0 l_0 = 0$, then the θ angle becomes an undefined parameter. Assume that $\theta \in [0, \pi)$. In this case, 2^{n_b} one-parameter families of secondary motions are obtained. The U axis forms any angle $\theta \in [0, \pi)$ with the X axis in each family. A part of loads in the family number k is directed along the U axis and a part against it which is determined by the numbers k_j in the binary number corresponding to the decimal k.

Note that the θ angle does not depend on angular velocity of the rotor in absence of damping in supports.

2) **The case of presence of damping in supports**. Let us study behavior of the q angle in the presence of damping forces in supports. When the rotor speed ω tends to zero from above ($\omega \to +0$), equation (32) takes the form:

$$m_0 l_0 c \sin\theta = 0. \quad (36)$$

There is only one essentially different value of the θ angle: $\theta = 0$. Classify secondary motions as if there were no damping in supports, only at $\omega \to +0$.

Using trigonometric identities:

$$\sin\theta = \frac{2u}{1+u^2}, \quad \cos\theta = \frac{1-u^2}{1+u^2}, \quad u = \tan(\theta/2), \quad (37)$$

reduce equation (32) to the following form:

$$\omega b[s_{AB}(k) - m_0 l_0]u^2 + $$
$$+ 2m_0 l_0 (c - M_\Sigma \omega^2)u + \omega b[s_{AB}(k) + m_0 l_0] = 0. \quad (38)$$

Denote it through:

$$D(k) = m_0^2 l_0^2 (M_\Sigma \omega^2 - c)^2 +$$
$$+ \omega^2 b^2 [m_0^2 l_0^2 - s_{AB}^2(k)] = m_0^2 l_0^2 M_\Sigma^2 \omega^4 -$$
$$- \{2c m_0^2 l_0^2 M_\Sigma + b^2[s_{AB}^2(k) - m_0^2 l_0^2]\}\omega^2 + c^2 m_0^2 l_0^2. \quad (39)$$

Then the roots of equation (38):

$$u_{1/2}(k,\omega) = \frac{m_0 l_0 (M_\Sigma \omega^2 - c) \pm \sqrt{D(k)}}{\omega b[s_{AB}(k) - m_0 l_0]}. \quad (40)$$

Two angles correspond to two roots $u_{1/2}$:

$$\theta_{1/2}(k,\omega) = 2\arctan[u_{1/2}(k,\omega)] =$$
$$= 2\arctan\left\{\frac{m_0 l_0 (M_\Sigma \omega^2 - c) \pm \sqrt{D(k)}}{\omega b[s_{AB}(k) - m_0 l_0]}\right\}. \quad (41)$$

It can be seen from (41) that in the presence of damping in supports, the θ angle depends on angular velocity of the rotor.

For small ω:

$$u_1(k,\omega) \approx -\frac{b[s_{AB}(k) + m_0 l_0]}{2c m_0 l_0}\omega,$$

$$u_2(k,\omega) \approx -\frac{2c m_0 l_0}{\omega b[s_{AB}(k) - m_0 l_0]};$$

$$\theta_1(k,\omega) \approx -\frac{b[s_{AB}(k) + m_0 l_0]}{c m_0 l_0}\omega,$$

$$\theta_2(k,\omega) \approx -\pi \cdot \operatorname{sgn}[s_{AB}(k) - m_0 l_0] + \frac{b[s_{AB}(k) - m_0 l_0]}{c m_0 l_0}\omega.$$

Thus, only one value of θ tends to 0 when $\omega \to +0$:

$$\theta(k,\omega) = \theta_1(k,\omega) =$$
$$= 2\arctan\left\{\frac{m_0 l_0 (M_\Sigma \omega^2 - c) + \sqrt{D(k)}}{\omega b[s_{AB}(k) - m_0 l_0]}\right\}. \quad (42)$$

Note that if $m_0 l_0 \neq 0$, then:

– at the resonant rotor speed:

$$\theta(k,\omega_0) = 2 \cdot \arctan\frac{\sqrt{m_0^2 l_0^2 - s_{AB}^2(k)}}{s_{AB}(k) - m_0 l_0}; \quad (43)$$

– at high super-resonant rotor speeds:

$$\omega \gg \omega_0: \ \theta(k,\omega) \approx \pi \cdot \operatorname{sgn}[s_{AB}(k) - m_0 l_0] -$$
$$- \frac{b[s_{AB}(k) - m_0 l_0]}{m_0 l_0 M_\Sigma \omega}. \quad (44)$$

a) **The case when** $0 \leq |s_{AB}(k)| < m_0 l_0$. In this case, loads at the secondary motion cause a total imbalance less than the rotor imbalance. It can be seen from (39) that $D(k) > 0$ for any rotor speed. Therefore, there is an angle $\theta(k,\omega)$ for any speed. Since:

$$\theta(k,+0) = -0;$$

$$\theta(k,\omega_0) = -2\cdot\arctan\frac{\sqrt{m_0^2 l_0^2 - s_{AB}^2(k)}}{m_0 l_0 - s_{AB}(k)} < 0;$$

$$\theta(k,\omega\to+\infty)\to-\pi+0,$$

then the U axis on which centers of mass of the loads lie, lags behind the X axis on which the unbalanced mass lies. The lag grows with an increase in ω. Dependence of the $\theta(k,\omega)$ angle on ω is shown in Fig. 2. Calculations were performed at the following values of parameters:

$$n_b = 3;\ \ m_0 = 0.05\ \text{kg};\ \ l_i = 0.1\ \text{m},$$

$$i = 0,1,\ldots,n_b;\ \ m_j = 0.03\ \text{kg},\ \ j = 1,\ldots,n_b;$$

$$n_b = 3;\ \ c = 10000\ \text{N/m};\ \ M_\Sigma = 4\ \text{kg};$$

$$M = M_\Sigma - \sum\nolimits_{j=0}^{n_b} m_j;\ \ b = 40\ \text{N}\cdot\text{s/m};$$

$$\omega_0 = \sqrt{c/M_\Sigma} = 50\ \text{rad/s}.$$

Note that at the stationary motion number $(2^{n_b}-1-k)$ at $\omega=+0$, the centers of mass of the loads are deflected opposite to their deviations at the motion number k. At these motions, $s_{AB}(k) = -s_{AB}(2^{n_b}-1-k)$.

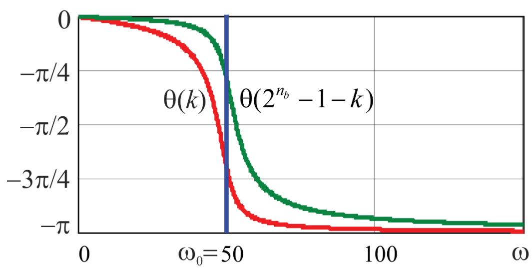

Fig. 2. Dependence of angles $\theta(k)$, $\theta(2^{n_b}-1-k)$ on angular velocity ω of the rotor, $k=1$

b) **The case when** $|s_{AB}(k)| > m_0 l_0 > 0$. In this case, the loads at the secondary motion create a total imbalance greater than imbalance of the rotor. There are two positive real roots in the equation $D(k)=0$.

$$\omega_{1/2}(k) = \left\langle \frac{c}{M_\Sigma} + \frac{b^2}{2M_\Sigma^2}\left[\frac{s_{AB}^2(k)}{m_0^2 l_0^2} - 1\right] \mp \frac{b}{\sqrt{2}M_\Sigma} \times \left\langle \left\{\frac{2c}{M_\Sigma} + \frac{b^2}{2M_\Sigma^2}\left[\frac{s_{AB}^2(k)}{m_0^2 l_0^2} - 1\right]\right\}\left[\frac{s_{AB}^2(k)}{m_0^2 l_0^2} - 1\right]\right\rangle^{\frac{1}{2}} \right\rangle^{\frac{1}{2}}. \quad (45)$$

Wherein:

$$0 < \omega_1(k) < \omega_0 < \omega_2(k),$$

and secondary motions determined by angles $\theta_{1/2}(\omega,k)$ exist at such rotor speeds:

$$\omega\in(0,\ \omega_1(k))\cup(\omega_2(k),\ +\infty).$$

If all loads are the same, then there will be only $2(n_b+1)$ fundamentally different secondary motions.

It can be seen from the form of (45) that critical speeds $\omega_{1/2}(k)$ depend on $s_{AB}^2(k)$. Therefore, motions k and $2^{n_b}-1-k$ have the same critical speeds and two motions merge and disappear at the $\omega_1(k)$ point and originate and split at the $\omega_2(k)$ point (Fig. 3).

Calculations were performed for the same values of parameters as in the previous case for $k=0$.

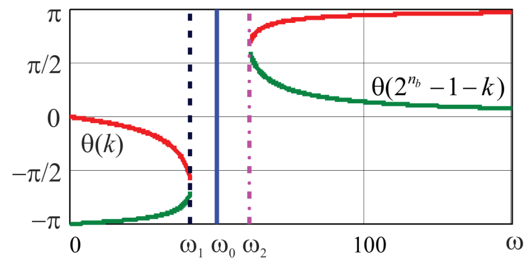

Fig. 3. Dependence of angles $\theta(k)$, $\theta(2^{n_b}-1-k)$ on angular velocity of the rotor, ω, $k=0$

c) **The case when** $m_0 l_0 = 0$ **and** $s_{AB}(k)\neq 0$. In this case, equation (32) has no solutions. Therefore, in the presence of damping in supports, there are no one-parameter families of secondary motions.

5. 2. 4. Reduced potential when imposing restraints

Let us exclude generalized coordinates of the rotor from the reduced potential. The following is found from the first two equations in (22):

$$x = \frac{\omega^2[(c - M_\Sigma\omega^2)s_x + b\omega s_y]}{(c - M_\Sigma\omega^2)^2 + b^2\omega^2},$$

$$y = \frac{\omega^2[(c - M_\Sigma\omega^2)s_y - b\omega s_x]}{(c - M_\Sigma\omega^2)^2 + b^2\omega^2}. \quad (46)$$

Substitute (46) into (19) to obtain the transformed generalized potential in the form:

$$\Pi = \frac{\omega^4(M_\Sigma\omega^2 - c)s^2}{2[(c - M_\Sigma\omega^2)^2 + b^2\omega^2]} - I_O\frac{\omega^2}{2}, \quad (47)$$

where $s = \sqrt{s_x^2 + s_y^2}$ is the module of total static imbalance of the rotor and loads.

Exclude the rotor imbalances from the reduced potential. The following is found from the first two equations in (22):

$$s_x = -[(M_\Sigma\omega^2 - c)x + b\omega y]/\omega^2,$$

$$s_y = -[(M_\Sigma\omega^2 - c)y - b\omega x]/\omega^2. \quad (48)$$

Substitute (48) into (19) to obtain the transformed generalized potential in the form:

$$\Pi = \frac{(M_\Sigma\omega^2 - c)(x^2 + y^2)}{2} - I_O\frac{\omega^2}{2}. \quad (49)$$

5. 2. 5. Assessing stability of stationary motions

The following conclusions on stability of existing stationary motions can be drawn from the form of (47):

1) s has a minimum at an isolated main motion or at a family of main motions and this motion or family of motions can be stable at super-resonant rotor speeds $(\omega > \omega_0)$;

2) s has a maximum at the secondary motion corresponding to the highest total static imbalance of the rotor and the loads and this motion can be stable at the sub-resonant rotor speeds $(\omega < \omega_p)$;

3) s has a non-extremal value (neither minimum nor maximum value) at the remaining secondary motions and these motions are unstable at any speeds of the rotor.

6. Discussion of the results obtained in the study of stationary motions of the rotor with attached bodies

In the modernized method, damping in supports is taken into account through the linear part of the dissipative function and elastic forces acting on the attached bodies are taken into account through the potential energy.

Thanks to the modernization, the method has become applicable to the study of stationary motions and in the cases when the attached bodies are parts of a composite rotor and form dampers or suppressors of oscillations, etc.

The method still makes it possible to find and assess stability of stationary motions of the rotor systems under consideration without setting up differential equations of motion. This makes the method the least time consuming.

Effectiveness of the method and the main stages of its application were illustrated in the framework of a flat model of a rotor mounted on isotropic supports with a number of attached loads in the form of balls, rollers and pendulums.

As a result of analytical studies, it was confirmed that when there are no isotropic elastic supports (no viscosity), the rotor has a single critical speed which coincides with the natural frequency. Auto-balancing can only occur at supercritical rotor speeds.

The method enables taking into account the effect of damping in supports on stationary motions. It was established that this damping:

– does not affect existence and domain of stability of the main motions;

– affects both secondary motions proper and the domains of their existence.

In the absence of damping in supports, secondary motions do not depend on the angular velocity of the rotor.

In the presence of damping in supports, both secondary motions and the domains of their existence depend on the angular velocity of the rotor. The angular velocities at which bifurcation of secondary motions occurs are also the velocities at which non-stationary steady motions can arise.

The method has drawbacks inherent to approximate methods designed for studying stability of motion according to Lyapunov. The method gives approximate boundaries of the domains of stability of main and secondary motions. Also, it does not make it possible to study non-stationary steady motions of the system and transient processes.

In the future, it is planned to obtain (by means of application of the energy method) conditions for the onset of auto-balancing for rotors both with different kinematics and different attached bodies.

7. Conclusions

1. The described energy method is applicable to rotors mounted on isotropic elastic-viscous supports when other bodies are attached to the rotor and affected by viscous and elastic forces during relative motion. The method is aimed at searching for stationary motions of the rotor system, determining conditions of their existence and assessing their stability. Peculiarity of the method consists in the fact that stationary motions of the system determine the generalized potential and the linear part of the dissipative function. Stability of motions is assessed through their study for the conditional extremum of the generalized potential. The equations of stationary motion set up for the generalized coordinates of the rotor serve as conditions.

2. Effectiveness of the method was demonstrated by the example of a flat model of a rotor mounted on isotropic supports with many loads in the form of balls, rollers or pendulums.

2. 1. It has been established that the system has a multiparametric family of main motions, both with and without damping in supports at a sufficient balancing capacity of the auto-balancer.

In the absence of damping in supports, the system has:

– isolated secondary motions at which centers of mass of the loads are deflected to the side of imbalance or in the opposite direction in the presence of imbalance in the rotor;

– one-parameter families of secondary motions in which centers of mass of the loads lie on one straight line in the absence of the rotor imbalance.

In the presence of damping in supports, the system has:

– isolated secondary motions at which centers of mass of the loads lie on one straight line and this straight line forms an angle with the imbalance vector depending on rotor speed in the presence of the rotor imbalance;

– secondary motions do not exist in the absence of the rotor imbalance.

2. 2. It has been established that both in the presence and in the absence of damping in supports at super-resonant rotor speeds, only the family of main motions can be stable.

In the absence of damping in supports at sub-resonant rotor speeds, only the secondary motion at which all corrective loads are deflected to the heavy side of the rotor can be stable.

In the presence of damping in supports at sub-resonant rotor speeds:

– all existing stationary motions are unstable in the absence of the rotor imbalance;

– only the secondary motion at which total imbalance of the rotor and the loads is greatest can be stable, however, if such a motion exists in the presence of the rotor imbalance.

2. 3. It was established that damping in supports:

– does not affect conditions of existence of a family of main motions;

– affects both the secondary motions proper and the conditions of their existence.

In the absence of damping in supports, the secondary motions do not depend on the rotor speed but they depend on it in the presence of damping.

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