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CALCULATION OF STRESS CONCENTRATIONS IN ORTHOTROPIC CYLINDRICAL SHELLS WITH HOLES ON THE BASIS OF A VARIATIONAL METHOD

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Запропоновано варіаційний чисельно-аналітичний метод (названий RVR-методом) розрахунку міцності та жорсткості статично навантажених нетонких ортотропних оболонкових конструкцій, послаблених отворами (концентраторами напружень) довільних форм і розмірів. Теоретично обґрунтований новий метод заснований на варіаційному принципі Рейсснера і методі І. М. Векуа (методі розкладання шуканих функцій у ряди Фур'є по ортогональним поліномах Лежандра щодо координати уздовж постійної товщини оболонки). При цьому використання в запропонованому RVR-методі загальних рівнянь тривимірних задач лінійної теорії пружності дозволяє визначити повний напружено-деформований стан пружної оболонки (зокрема, пластини) з отворами. В той же час за допомогою R-функцій на аналітичному рівні враховується геометрична інформація крайових задач для багатозв'язних областей і будуються структури розв'язків, які точно задовольняють різним варіантам граничних умов. Застосування при дослідженні змішаних варіаційних задач програмно здійснюваного алгоритму двосторонньої інтегральної оцінки точності наближених розв'язків дозволяє автоматизувати пошук такої кількості апроксимацій, при якому процес збіжності розв'язків набуває стійкого характеру.

Для ортотропного й ізотропного матеріалів можливості RVR-методу показані в чисельних прикладах розв'язання відповідних крайових задач розрахунку концентрації напружень в циліндричній оболонці з еліптичним або прямокутним отвором при осьовому навантаженні. Обговорено результати виконаних досліджень і особливості, що характерні для нового методу, який може знайти ефективне застосування при проектуванні відповідальних пластинчастих і оболонкових елементів конструкцій в різних галузях сучасної техніки

Ключові слова: ортотропна оболонка з отворами, концентрація напружень, принцип Рейсснера, теорія R-функцій

1. Introduction

Elastic shells, weakened by holes (openings), are widely used in modern engineering practice as the most crucial structural elements the strength and rigidity of which often depend on the performance and reliability of the structure as a whole.

Estimation of stress concentration near the holes in non-thin shells involves calculating their stress-strained state based on the solutions of the corresponding boundary problems of the three-dimensional theory of elasticity. The solution of such problems in spatial formulation is usually associated with significant mathematical and computational difficulties that must be overcome in the process of performing specific calculations. As is known, the effectiveness of numerical methods is determined primarily by the possibility of obtaining reliable results. Therefore, the improvement of existing and the development of new methods for investigating anisotropic shells of arbitrary thickness weakened by holes is still an urgent and practically significant scientific problem in the mechanics of a deformable solid matter.

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In this regard, it is primarily essential to use modern computers to create reliable, fairly universal and algorithmically simple methods for calculating non-thin shell elements of structures with openings.

2. Literature review and problem statement

Understanding the significant effect of holes (stress concentrators) on the bearing capacity of shell structures is very important when designing them. In [1], it is shown that in a linear placement of elastic plates, the stress concentration factors do not depend on the size of the holes. However, when calculating these coefficients in shells, it is necessary to take into account the relative dimensions of the holes compared to the dimensions of the shell. The stress distribution near the holes in the elastic shells has been thoroughly studied on the basis of the classical Kirchhoff–Love theory. In many practical cases, this theory gives results that are consistent with experimental data and are the initial approximation for three-dimensional problems. Thus, studies of the stress-strained state of thin shells with holes based on the use of experimental data and analytical methods are presented in [2]. Besides, several variational methods, which are becoming increasingly widespread in the construction of direct approximate algorithms for solving boundary problems of the theory of shells, are described in detail in the work by K. Washizu [3]. However, the application of the considered methods to calculate stress concentration in shells (especially non-thin) with holes of complex shapes can lead to unreliable results.

The limitations of exact analytical solutions to the problems of the mechanics of a deformable body have led to the intensive development of approximate methods. These methods, as a rule, are based on the direct integration of the corresponding differential equations, on the use of variational approaches, and also on discrete methods, among which the most common is the finite element method (FEM). In particular, to analyse the stress-strained state of thin shells, a comparison is made in [4] of the effectiveness of algorithms for using stiffness matrices of finite elements of different dimensions. In this case, using the example of calculating a cylinder clamped along the ends, it is shown that the two-dimensional formulation in the calculations of thin shells is adequate and allows obtaining acceptable results with optimal expenditure of machine time.

In [5], on the basis of FEM, a scaling technique is developed for static and dynamic analysis of cylindrical shells, which is used to provide an accurate representation of the shell boundaries within the framework of the three-dimensional linear theory of elasticity. In [6], the main attention is paid to obtaining a new finite element of a thin-walled shell, the practical application of which is promised to be presented in further publications. Moreover, the numerous examples that are presented in [7] illustrate difficulties and uncertainties associated with the regular analysis of buckling of thin-walled elements with holes. Thus, the direct use of FEM in solving a number of problems in the mechanics of elastic shells is associated not only with obvious achievements but also with certain problems.

For the mechanics of a deformable solid body, the problems associated with solving three-dimensional formulations of boundary value problems for anisotropic shells of arbitrary thickness weakened by apertures are still relevant. A detailed review and comparative analysis of the diverse, often contradictory, variants of the refined theories of shells known in scientific literature is given in the monograph [8]. The same work presents a new variational numerical-analytical method (called the RVR method) for calculating the strength and stiffness of statically loaded non-thin orthotropic shells with openings of arbitrary shapes and sizes.

The theoretically grounded method is based on the E. Reissner principle [9–11], the I. N. Vekua method [12] (the method of decomposing the desired functions in Fourier series in Legendre polynomials relative to the coordinate along the shell thickness). In this case, the use of general equations of three-dimensional problems of the linear theory of elasticity in the proposed RVR method [13] helps determine the full stress-strained state of an elastic shell (in particular, a plate) with holes.

Success in the calculations of shells with holes is determined not only by the capabilities of the adopted refined shell model but also by the level of implementing the method used in the study of specific structural elements in various fields of technology. The effectiveness and capabilities of the proposed RVR method are confirmed by solving a three-dimensional formulation of a number of complex applied problems for various types of static loads applied to shell objects. Thus, calculations of a cylindrical structure under the action of a local load and centrifugal loads are presented, respectively, in [14] and [15], and calculations of a spherical structure loaded with internal pressure are given in [16].

However, the less studied problem of using effective methods for determining the stress-strained state of non-thin orthotropic shells, weakened by holes of arbitrary shapes and sizes, still requires solving. Moreover, it is important to note that in the case of calculating the stress concentration in such shells, it is necessary to use the basic relations of the three-dimensional theory of elasticity.

3. The aim and objectives of the study

The aim of the study is to determine the level of stress concentration in statically loaded cylindrical elements of structures with holes, using the numerical-analytical RVR method.

To achieve the aim, the following objectives were set and done:

– to build such structures of solutions that exactly satisfy all the boundary conditions of the studied elastic region of a shell with holes of arbitrary shape;

– to use the RVR method for obtaining numerical results to assess the effect of anisotropy of the shell material on the stress concentration on the contour of an elliptical or rectangular hole in an orthotropic cylinder.

4. Materials and methods for the study of stress concentration in orthotropic cylindrical shells with holes

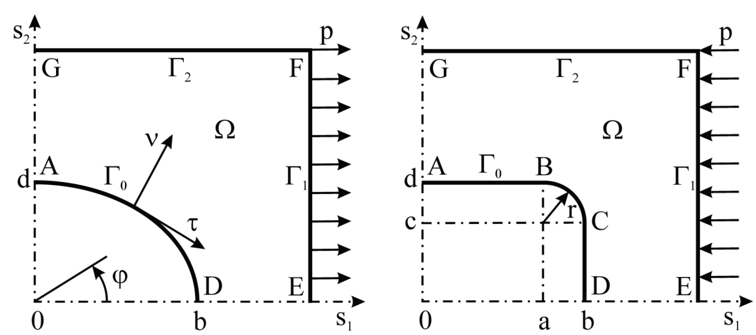
4. 1. Obtaining an analytical expression for the Reissner variational equation for orthotropic cylindrical shells

In order to increase the accuracy of solving boundary value problems, it is advisable to determine independently the parameters of the stressed and deformed states when constructing refined models of the shell, which can be implemented using the Reissner variational principle. An alternative approach with respect to the classical Lagrange and Castigliano functionals $\mathbf{I}_L$ and $\mathbf{I}_C$ is associated with the Reissner functional $\mathbf{I}_R$ with independent approximation of the displacement vector of an arbitrary point of the region **u** and the stress tensor σ. It should be noted that by virtue of the independence of displacements and stresses, the Reissner variational equation $\mathbf{I}_R=0$ leads to a system of first-order differential equations for the unknown quantities. However, the equations corresponding to the classical variational formulations have a higher differential order, require the implementation of time-consuming mathematical operations when solving them, and significantly complicate the structures of solutions that exactly satisfy the boundary conditions of the problem.

In computational engineering practice, more and more attention is paid to mixed variational formulations, which are devoid of the well-known deficiencies inherent in the classical Lagrange and Castigliano functionals and are based mainly on the Reissner functional. The numerical implementation of such a variational statement was significantly hampered by difficulties in estimating the accuracy of solutions caused by the absence of an extremum at the stationarity point $\mathbf{I}_R$. This problem was solved by the theorem proved in [8, 17]: «...the sequences of the Ritz method coincide with the exact solution of the boundary value problem formulated on the Reussner principle if the structures of the solutions exactly satisfy all the boundary conditions».

Therefore, using the R-functions theory [18, 19], the proposed RVR method at the analytical level takes into account the geometric information of the studied boundary-value problems for multiply connected domains and constructs solution structures that precisely satisfy diverse variants of the boundary conditions.

For a particular analytical representation of the Reissner variational equation (stationary conditions for the functional $\mathbf{I}_R$), let us consider the problem of the stress-strained state of an elastic length $2L$ and a constant thickness h weakened by a hole. We introduce an orthogonal curvilinear coordinate system $\{s_1, s_2, z\}$ with the origin at the centre of the hole in the middle surface Ω_s of the radius R of the cylinder under study ($-L \le s_1 \le L$, and s_2 is the arc length of the parallel circle of the cylinder). In this case, the coordinate line z ($-h/2 \le z \le h/2$) is perpendicular to the surface Ω_s ($z=0$), and the coordinate lines s_1 and s_2 coincide with the elastic-equivalent directions of the orthotropy of the cylinder material. With the symmetry of the hole shape and the load with respect to the planes $s_1=0$ and $s_2=0$, the calculation of the shell is reduced to the study in terms of the elastic region Ω (at $L=$OE), which is periodic along the s_2 line (Fig. 1). This area is a cylindrical panel ADEFG with an elliptical hole (Fig. 1, *a*) with semi-axes $r_1=$OD and $r_2=$OA, or a panel ABCDEFG with a rectangular hole (Fig. 1, *b*) with rounded corners of the radius r.



a *b*

Fig. 1. The calculated periodic elastic area of a shell: *a* — with an elliptical hole; *b* — with a rectangular hole

The hole dimensions are considered arbitrary, since in the proposed RVR method [8], there are no limitations for the value of $\mu = r_0/\sqrt{Rh}$, where $r_0=(b+d)/2$), for typically used and widely published scientific methods of calculating multiply connected shells [2].

In the coordinate system $\{s_1, s_2, z\}$ for orthotropic cylindrical shells, we represent the Reissner variational equation in the following form:

$$\left.\begin{aligned}
&\iiint_\Omega \Bigg\{ -\left[\frac{\partial\sigma_{11}}{\partial s_1} + \frac{1}{\chi}\frac{\partial\sigma_{12}}{\partial s_2} + \frac{\partial\sigma_{13}}{\partial z} + \frac{1}{R\chi}\sigma_{13}\right]\delta u_1 - \\
&-\left[\frac{\partial\sigma_{12}}{\partial s_1} + \frac{1}{\chi}\frac{\partial\sigma_{22}}{\partial s_2} + \frac{\partial\sigma_{23}}{\partial z} + \frac{2}{R\chi}\sigma_{23}\right]\delta u_2 - \\
&-\left[\frac{\partial\sigma_{13}}{\partial s_1} + \frac{1}{\chi}\frac{\partial\sigma_{23}}{\partial s_2} + \frac{\partial\sigma_{33}}{\partial z} + \frac{\sigma_{33}-\sigma_{22}}{R\chi}\right]\delta u_3 + \\
&+\left[\frac{\partial u_1}{\partial s_1} - \frac{1}{E_1}\left(\sigma_{11} - \nu_{21}\sigma_{22} - \nu_{31}\sigma_{33}\right)\right]\delta\sigma_{11} + \\
&+\left[\frac{1}{\chi}\frac{\partial u_2}{\partial s_2} + \frac{1}{R\chi}u_3 - \frac{1}{E_2}\left(\sigma_{22} - \nu_{12}\sigma_{11} - \nu_{32}\sigma_{33}\right)\right]\delta\sigma_{22} + \\
&+\left[\frac{\partial u_3}{\partial z} - \frac{1}{E_3}\left(\sigma_{33} - \nu_{13}\sigma_{11} - \nu_{23}\sigma_{22}\right)\right]\delta\sigma_{33} + \\
&+\left[\frac{\partial u_2}{\partial s_1} + \frac{1}{\chi}\frac{\partial u_1}{\partial s_2} - \frac{\sigma_{12}}{G_{12}}\right]\delta\sigma_{12} + \left[\frac{\partial u_3}{\partial s_1} + \frac{\partial u_1}{\partial z} - \frac{\sigma_{13}}{G_{13}}\right]\delta\sigma_{13} + \\
&+\left[\frac{1}{\chi}\left(\frac{\partial u_3}{\partial s_2} - \frac{1}{R}u_2\right) + \frac{\partial u_2}{\partial z} - \frac{\sigma_{23}}{G_{23}}\right]\delta\sigma_{23}\Bigg\}\chi\,ds_1 ds_2 dz = 0,
\end{aligned}\right\} \quad (1)$$

where ν_{12}, ν_{21}, ν_{13}, ν_{31}, ν_{23}, and ν_{32} are Poisson's coefficients; E_1, E_2, and E_3 are Young's moduli in the main directions of the shell orthotropy; and G_{12}, G_{13}, and G_{23} are shear moduli:

$$\chi = 1 + z/R;\ \nu_{12}/E_1 = \nu_{21}/E_2;$$
$$\nu_{13}/E_1 = \nu_{31}/E_3;\ \nu_{23}/E_2 = \nu_{32}/E_3. \quad (2)$$

In this case, in equation (1), the factor χ is necessary because non-thin shells are considered, and contour integrals are absent because the structure of the solutions exactly satisfies all specified boundary conditions of the problems when using the RVR method.

4. 2. Construction of solutions structures for boundary value problems under study

Suppose that axial forces P with intensity $p=P/(2\pi Rh)$ are applied to the end surfaces Γ_1 (Fig. 1) of the cylinder, whereas the surface Γ_0 and the front surfaces of the cylinder are free from external forces and moments. The boundary conditions formulated through the components of the vector **u** and the tensor σ have the form:

$$\left.\begin{aligned}
&\sigma_{\nu\nu}=0,\ \sigma_{\nu\tau}=0,\ \sigma_{\nu n}=0 \text{ on } \Gamma_0;\\
&\sigma_{11}=p,\ \sigma_{12}=0,\ \sigma_{13}=0 \text{ on } \Gamma_1 \text{ (at } |x|=L);\\
&\sigma_{13}=0,\ \sigma_{23}=0,\ \sigma_{33}=0 \text{ at } |z|=h/2.
\end{aligned}\right\} \quad (3)$$

In addition, on the marginal surface Γ_2, determined on the coordinate line s_2 by the distance $\pi R/n$ (where n is the number of holes along the guide of the cylinder), the periodicity conditions must be met as follows:

$$u_2=0,\ \ \sigma_{12}=0,\ \ \sigma_{23}=0\ \text{ on }\ \Gamma_2. \qquad (4)$$

Let us present the desired components of displacements u_i and stresses σ_{ij}, which are independently varied in the Reussner functional $\mathbf{I}_R$ and exactly satisfy boundary conditions (3) and (4), in the form of finite series:

$$\left.\begin{aligned}
&u_1=\sum_{i=0}^{m_1}\sum_{j=0}^{n_1}\sum_{k=0}^{l_1-1}u_1^{ijk}S_i(s_1)C_j(s_2)P_k(\zeta);\\
&u_2=\sum_{i=0}^{m_2}\sum_{j=0}^{n_2}\sum_{k=0}^{l_2-1}u_2^{ijk}C_i(s_1)S_j(s_2)P_k(\zeta);\\
&u_3=\sum_{i=0}^{m_3}\sum_{j=0}^{n_3}\sum_{k=0}^{l_3-1}u_3^{ijk}C_i(s_1)C_j(s_2)P_k(\zeta);\\
&\sigma_{11}=\chi^{-1}\left[\begin{aligned}&\omega_4 q+f_2^2\omega_5 T_1+\\&+\omega_0\omega_1\sum_{i=0}^{m_{11}}\sum_{j=0}^{n_{11}}\sum_{k=0}^{l_{11}-1}\sigma_{11}^{ijk}C_i(s_1)C_j(s_2)P_k(\zeta)\end{aligned}\right];\\
&\sigma_{22}=f_1^2\omega_5 T_1+\omega_0\sum_{i=0}^{m_{22}}\sum_{j=0}^{n_{22}}\sum_{k=0}^{l_{22}-1}\sigma_{22}^{ijk}C_i(s_1)C_j(s_2)P_k(\zeta);\\
&\sigma_{12}=\chi^{-1}\left[\begin{aligned}&-f_1f_2\omega_5T_1+\\&+\omega_0\omega_1\sum_{i=0}^{m_{12}}\sum_{j=0}^{n_{12}}\sum_{k=0}^{l_{12}-1}\sigma_{12}^{ijk}S_i(s_1)S_j(s_2)P_k(\zeta)\end{aligned}\right];\\
&\sigma_{13}=\chi^{-1}\left[\begin{aligned}&f_2\omega_5T_2+\\&+\omega_0\omega_1\omega_3\sum_{i=0}^{m_{13}}\sum_{j=0}^{n_{13}}\sum_{k=0}^{l_{13}-1}\sigma_{13}^{ijk}S_i(s_1)C_j(s_2)P_k(\zeta)\end{aligned}\right];\\
&\sigma_{23}=-f_1\omega_5T_2+\omega_0\omega_3\sum_{i=0}^{m_{23}}\sum_{j=0}^{n_{23}}\sum_{k=0}^{l_{23}-1}\sigma_{23}^{ijk}C_i(s_1)S_j(s_2)P_k(\zeta);\\
&\sigma_{33}=\chi^{-1}\omega_3\sum_{i=0}^{m_{33}}\sum_{j=0}^{n_{33}}\sum_{k=0}^{l_{33}-1}\sigma_{33}^{ijk}C_i(s_1)C_j(s_2)P_k(\zeta);\\
&T_g=\sum_{i=0}^{t_1}\sum_{j=0}^{t_2}\sum_{k=0}^{2-g}T_g^{ijk}C_i(s_1)C_j(s_2)P_k(\zeta)\quad (g=1,2).
\end{aligned}\right\}\qquad (5)$$

In this case, the valid analytical expressions are the following:

$$\left.\begin{aligned}
&\omega_1=1-\left(\frac{s_1}{L}\right)^2;\ \ \omega_2=1-\left(\frac{s_2 n}{\pi R}\right)^2;\ \ \omega_3=1-\zeta^2;\\
&\omega_4=\frac{\omega_0}{\omega_0+\omega_1};\quad \omega_5=\frac{\omega_1\omega_2}{\omega_0+\omega_1\omega_2};\quad \chi=1+\frac{h\zeta}{2R},
\end{aligned}\right\}\qquad (6)$$

where u_p^{ijk}, σ_{ps}^{ijk}, T_g^{ijk} (p, s=1,2,3; g=1, 2) are the desired values; $C_i(s_1)$, $C_j(s_2)$ and $S_i(s_1)$, $S_j(s_2)$ are the even and odd approximating functions of the coordinates s_1 and s_2; $P_k(\zeta)$ denotes the Legendre polynomials ($\zeta=2z/h$); and $n_i=$ $=[k(m_i+1)+p](t_i+1)+r+1$; $n_i\to n_{ij}$.

It is noteworthy that $q=p$ when the cylinder is stretched, and $q=-p$ when the cylinder is compressed. In this case, f_1 and f_2 are the values of the direction cosines of the normal ν to the hole contour (Fig. 1, *a*) and, according to the R-functions theory [18, 19], they are calculated by the formulae:

$$\begin{aligned}
&f_1=\left(\partial\omega_0^*/\partial s_1\right)\Big|_{\Gamma_0};\ \ f_2=\left(\partial\omega_0^*/\partial s_2\right)\Big|_{\Gamma_0};\\
&\omega_0^*=\omega_0\Big/\sqrt{\omega_0^2+\left|grad\,\omega_0\right|^2}
\end{aligned}\qquad (7)$$

and are necessary for the fulfilment of conditions (3) on Γ_0, since the formulae connecting the stresses in the main and inclined areas of the elastic shell are of the form:

$$\left.\begin{aligned}
&\sigma_{\nu\nu}=f_1^2\sigma_{11}+2f_1f_2\sigma_{12}+f_2^2\sigma_{22};\\
&\sigma_{\nu n}=-f_1\sigma_{13}-f_2\sigma_{23};\\
&\sigma_{\nu\tau}=f_1f_2(\sigma_{22}-\sigma_{11})+\left(f_1^2-f_2^2\right)\sigma_{12};\\
&\sigma_{\tau\tau}=f_2^2\sigma_{11}-2f_1f_2\sigma_{12}+f_1^2\sigma_{22}.
\end{aligned}\right\}\qquad (8)$$

The function Ω_0 for an elliptical hole is determined by the formula:

$$\omega_0=(s_1/r_1)^2+(s_2/r_2)^2-1. \qquad (9)$$

For a cylinder weakened by a through (n=2) rectangular orifice with rounded corners (Fig. 1*b*), we introduce the notation (with i, j=2):

$$\left.\begin{aligned}
&\omega_{01}=s_1^2/a^2-1;\quad \omega_{02}=s_1^2/b^2-1;\\
&\omega_{03}=s_2^2/c^2-1;\quad \omega_{04}=s_2^2/d^2-1;\\
&\omega_{ij}=\left[s_1+(-1)^i a\right]^2/r^2+\left[s_2+(-1)^j c\right]^2/r^2-1.
\end{aligned}\right\}\qquad (10)$$

The equations on the boundary surfaces Γ_g (g=0, 1, 2) of the studied region Ω (Fig. 1, *b*) have the form $\omega_g\big|_{\Gamma_g}=0$ and are determined by the functions Ω_1, Ω_2, (6), and Ω_0:

$$\begin{aligned}
&\omega_0=\left((\omega_{01}\wedge_0\omega_{03})\vee_0(\omega_{02}\vee_0\omega_{04})\right)\wedge_0\\
&\wedge_0(\omega_{11}\vee_0\omega_{12}\vee_0\omega_{21}\vee_0\omega_{22}),
\end{aligned}\qquad (11)$$

where $\wedge_0$ and $\vee_0$ are symbols of R-operations of the R-functions theory [18, 19].

In the structures of (5), the numbers l_i, l_{ij} (i, j=1,2) of the approximations of the desired displacements and stresses through the thickness of the construction determine the selected shear model of the shell when specifying a combination of values (l_i, l_3, l_{ij}, l_{i3}, l_{33}). In this case, l_i is the number of terms retained in the expansion in the transverse coordinate z of the tangential displacements u_i; l_3 is for the normal displacement u_3; l_{ij} is for the tangential stresses σ_{ij}; l_{i3} is for the transverse tangential stresses σ_{i3}; and l_{33} is for the transverse normal stress σ_{33}.

If the combination of the parameters (l_i, l_3, l_{ij}, l_{i3}, l_{33}) is specified, it allows the software complex to switch automatically to various refined two-dimensional theories that estimate the stress-strained state of the shells with a given accuracy. At the same time, such a procedure can be used as a basis for creating a consistent and logically coherent classification (as to the order degree of approximations) of the variety of shear models of shells existing in the scientific literature.

In particular, in the case of $l_i=l_{ij}$, the value of $N=l_i-1$ can be considered as a kind of parameter characterizing the order of the Nth approximation (in the terminology of I. V. Vekua [12]) of the shell theory under consideration. In the proposed RVR method, the method of reduction of three-dimensional problems of the theory of shells with the algorithm of regular refinement of the shear model of the shell is used. For example, the variant (2, 1, 2, 1, 0) corresponds to the theory of thin shells with finite shear stiffness like Tymoshenko's [13], and (4, 3, 4, 1, 2) relates to the applied theory of shells [20].

5. The research results of the formulated boundary value problems

After substituting the structure of solutions (5) into variational equation (1) and integrating the triple integrals numerically, the boundary problem is reduced to solving a system of linear algebraic equations for the desired constants u_p^{ijk}, σ_{ps}^{ijk} and T_g^{ijk}. The found values are used to determine all the characteristics of the stress-strained state of the calculated elastic shell region.

In this case, the matrix of the system of equations has a tape structure, and the tape width depends on the order of indexation of the desired components of the displacement vector **u** and the stress tensor σ.

The numerical implementation of the problem for a shell with an elliptical hole (with $r_1/r_2=0.5$) was performed for cylinders of different materials in the case of using the third approximation shift model (4, 3, 4, 3, 2) of the applied theory [20]. For an isotropic cylindrical shell (at $E_i=26.18$ MPa; $\nu_{ij}=0.3$), Table 1 presents (in the denominator, the results of [2]), the values of the coefficients of the concentration of membrane $k_1=\sigma_{\tau\tau}^{k=0}/p$ and maximum bending $k_2=\sigma_{\tau\tau}^{k=1}/p$ stresses in the thickness. In this case, the numerical values of k_1 and k_2 were obtained for the most loaded point A ($\varphi=\pi/2$) of the contour of a large hole ($\mu=2.5$) (Fig. 1, *a*).

Table 1

The values of the stress concentration factors k_1 and k_2

r_1/r_2	0.5	1.5^{-1}	1.3^{-1}	1	1.3	1.5	2
k_1	$\frac{8.619}{8.667}$	$\frac{7.275}{7.090}$	$\frac{6.448}{6.376}$	5.400	$\frac{4.556}{4.337}$	$\frac{4.148}{3.965}$	$\frac{3.503}{3.455}$
k_2	$\frac{-2.522}{-2.517}$	$\frac{-1.921}{-1.992}$	$\frac{-1.731}{-1.754}$	−1.125	$\frac{-1.006}{-1.081}$	$\frac{-0.855}{-0.959}$	$\frac{-0.757}{-0.796}$

For an orthotropic shell (with $\mu=1.667$; $\nu=0.4$), the graphs of the distribution of the coefficients k_1 and k_2 are shown in Fig. 2 for different values of the ratio of the elastic moduli in the axial E_1 and circumferential E_2 directions of the cylinder.

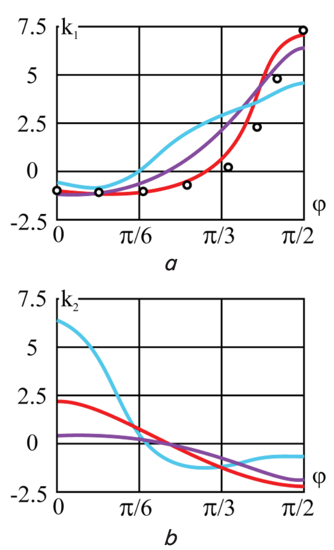


Fig. 2. The distribution of the coefficients k_1 and k_2 in the area $0\le\varphi\le\pi/2$: *a* – with an elliptical hole; *b* – with a rectangular hole; ── at $E_2/E_1=0.2$; ── at $E_2/E_1=1$; ── at $E_2/E_1=1.5$

Fig. 3 shows the results of the study of the effect of the anisotropy of the shell material on the stress concentration at the points of the contour of the hole.

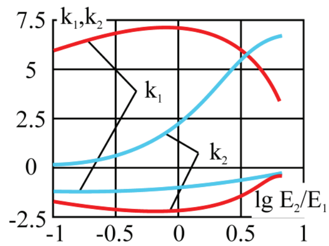


Fig. 3. The effect of the anisotropy on the values of the coefficients k_1 and k_2 on the contour Γ_0 of the elliptical hole (Fig. 1, *a*): ── in point *A* (with $\varphi=\pi/2$); ── in point *D* (with $\varphi=0$)

The numerical implementation of the boundary value problem was also performed for a non-thin cylinder ($h/R=1/3$) weakened by two ($n=2$) through-going rectangular holes (Fig. 1, *b*) with the following geometrical parameters (in mm): $h=9$; $R=27$; $L=50$; $\psi=\pi/4$; $b=12.5$; and $r=3$. The calculations were performed both for an isotropic material ($E_i=26.18$ MPa; $\nu_{ij}=0.5$) and for orthotropic fiberglass with elastic characteristics: $\nu_{12}=0.15$; $\nu_{23}=0.31$; $\nu_{31}=0.08$; $E_1=1.79\ E_0$; $E_2=1.31\ E_0$; $E_3=0.43\ E_0$; $G_{12}=0.28\ E_0$; $G_{23}=G_{31}=0.24\ E_0$; $E_0=9.81$ MPa. Fig. 4 shows the distribution of the reduced displacements $u=u_3E_3/pR$ and stresses $\sigma=\sigma_{\tau\tau}/\sigma_{st}$ ($\sigma_{st}=-p$) on the contour Γ_0 of a rectangular hole with $z=-h/2$ and $z=h/2$ (with u^-, σ^- and u^+, σ^+, respectively).

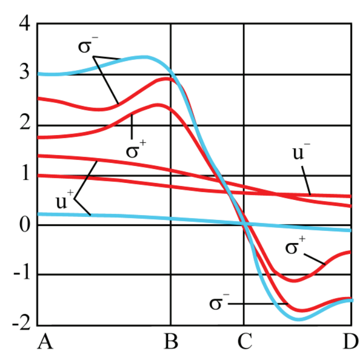


Fig. 4. The graphs of the distribution of the desired values on the contour Γ_0 (Fig. 1, *b*): ── for an isotropic shell; ── for an orthotropic shell

It is noteworthy that the presented graphs were obtained in the case of applying a refined fourth-order approximation shift model, first proposed in [21].

6. Discussion of the results of calculating the stress concentration in a cylindrical shell with a hole under axial load

In the study of the considered boundary-value problems, a software-implemented algorithm of the a posteriori integral estimation of the accuracy of approximate solutions of mixed variational problems was used [22]. This enabled us to automate the search in structure (5) of such a number of approximations in which the process of convergence of solutions is stable and the final results become reliable. Presented in the numerators of Table 1, the numerical results of the calculation of a shell with a large elliptical hole (μ=2.5; r_1/r_2=0.5) practically do not differ from those in the denominators of the results of [2]. In particular, Fig. 2, *a* shows a satisfactory agreement of the theoretical values of k_1 (red line) with the experimental data from [23] (indicated by circles) for an isotropic (with E_2/E_1=1) cylinder.

From the graphical results (Fig. 3) of the study of the influence of the anisotropy of the shell material on the stress concentration value, it can be seen that a change in the E_2/E_1 value over the entire considered interval significantly affects the stress state of the elastic shell.

When calculating a non-thin orthotropic shell with two rectangular holes, in follows from the graphs of Fig. 4 that the maximum stresses σ_{max} occur in the vicinity of point B (Fig. 1) of the angular zone of the hole. In particular, it concerns an isotropic cylinder with $\sigma^{-}_{max}=2.872$ at $z=-h/2$ and $\sigma^{+}_{max}=2.382$ at $z=h/2$ (experimental data from [24]: 2.9 and 2.4, respectively).

One of the advantages of the proposed method is also the possibility to use it for studying shells that are heterogeneous in thickness and made of composite materials, which provides wide opportunities for improving the existing critical structures of various purposes. In addition, the RVR method used in this study also allows solving three-dimensional problems in the theory of multiply connected anisotropic shells of arbitrary Gaussian curvature.

Regarding the limitations of this study, we should note that the proposed RVR method is used for statically loaded elastic shells with holes. However, in the future, there is the possibility of using this method to calculate the stress concentrations in non-thin shells weakened by holes during their dynamic loading.

The shells studied by the RVR method are widely used in many fields of modern engineering, including aerospace engineering, shipbuilding, and automotive.

7. Conclusions

1. On the basis of the Reissner principle, a variational formulation of three-dimensional boundary value problems of the statics of elastic shells of arbitrary thickness is formulated and an analytical expression is presented in a mixed Reiussner variational equation for the orthotropic cylindrical shell under study.

Structures of solutions are constructed to satisfy all the boundary conditions of the studied elastic region of a shell that is weakened by holes of arbitrary shapes. On the basis of the mathematical apparatus of the R-functions theory [18, 19], the study has specified the functions that determine the equations for the boundary surface of an orifice of a complex shape (in particular, a rectangular orifice with curves).

2. Tabular and graphical data of numerical calculations were obtained, which is of interest for engineering practice in evaluating the influence of the degree of anisotropy of the shell material on the stress concentration on the hole contour in an orthotropic cylinder. The reliability of the results was established by comparing them with numerical and experimental data known in the scientific literature [22, 23]. The analysis of the obtained results confirms the efficiency of using the RVR method [8] when solving complex three-dimensional boundary value problems for elastic shells with holes the stress concentration near which can significantly affect the bearing capacity of the related constructions.

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