Запропоновано метод розв'язання задачі дрібно-нелінійної оптимізації. Показано, що саме до такої математичної моделі зводяться численні задачі управління запасами, раціонального розподілу обмежених ресурсів, відшукання оптимальних шляхів на графі, раціональної організації перевезень, управління динамічними системами та інші задачі у випадках, коли вихідні дані задачі описані в термінах теорії ймовірностей або нечіткої математики. Проведено аналіз відомих методів вирішення задач дрібно-нелінійної оптимізації. Найбільш продуктивний з них заснований на ітераційної процедурі послідовного поліпшення початкового рішення задачі. При цьому на кожному кроці вирішується задача математичного програмування. Метод сходиться, якщо область допустимих рішень компактна. Очевидний недолік методу – неконтрольована швидкість збіжності. В роботі запропонований метод розв'язання задачі, ідея якого перегукується з відомим методом дрібно-лінійної оптимізації. Запропонована технологія перетворює вихідну задачу з дрібно-раціональним критерієм до звичайної задачі математичного програмування. Головне достоїнство методу і його відмінність від відомих полягає в тому, що метод реалізується з використанням однокрокової процедури отримання рішення. При цьому розмірність задачі не є обмежуючим фактором. Вимоги до математичної моделі задачі, які звужують область можливих додатків розробленої методики:

1) компоненти цільової функції повинні бути сепарабельними функціями;

2) показники ступеня всіх нелінійних доданків компонентних функцій повинні бути однаковими.

Інша важлива перевага методу полягає в можливості його використання для вирішення задачі безумовної та умовної оптимізації. Розглянуто приклади

Ключові слова: оптимізація дрібно-нелінійної функції, лінійні обмеження, однокрокова процедура, точне розв'язання

CONSTRUCTION OF THE FRACTIONAL-NONLINEAR OPTIMIZATION METHOD

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1. Introduction

Numerous practical problems are reduced to optimizing a nonlinear fractional functional in the form:

$$F(x) = \frac{\varphi(X)}{g(X)}, \quad X = (x_1, x_2, ..., x_n) \in R_n, \quad (1)$$

where $\varphi(X)$, $g(X)$ are the arbitrary functions, and $g(X)$ does not change its sign throughout the entire region of determination. It is such a mathematical model to which, in particular, inventory management tasks, on the rational allocation of limited resources, routing, etc. are reduced under conditions of uncertainty when a problem's parameters are defined in terms of a probability theory or fuzzy mathematics. A general scheme for solving such problems is as follows. Assume, for example, that the problem's parameters are random variables. It is clear that this uncertainty passes in transit into the problem's objective function. In this case, a conventional approach to solving the problem is optimization of the average [1–3]. The obvious disadvantage of the solution thus derived is the danger of obtaining a result that would grossly deviate from an optimum in some specific situations, which may occur by accident during operation of the analyzed object. In this regard, a more appropriate approach is to modernize the criterion, which should be wisely chosen as the probability of obtaining a value for winning that is not below the assigned threshold. The solution that is derived in this case is better than the solution, optimal on average, for the following reasons. This solution is warranting and managed owing to the possibility of a rational choice of the threshold value. We shall show below that under the simplest and natural assumptions concerning the nature of uncertainty in the initial data the maximization of a source probability leads to the optimization of a fractional-quadratic functional in the form (1).

It is clear that solving this problem is possible when using direct zero-order optimization methods (for example, by a Nelder-Mead method) [4, 5]. However, the extremely slow convergence of these methods, which manifests itself defiantly in the problems of high dimensionality, makes it difficult to implement them. On the other hand, the use of more powerful methods of optimization of the first and second orders [6–8] is complicated due to the need to compute the gradient vector and the Hessian-matrix for functionals in the form (1). In this connection, it is a relevant task to construct a fast and accurate method of fractional-nonlinear optimization.

2. Literature review and problem statement

The issue on the nonlinear optimization with a fractional criterion in the form (1) has been addressed by a large number of studies. The common characteristic of these works is that they solve a fractional-linear problem, that is the criterion to be optimized (1) is not linear in structure only, while the functions $\Phi(X), g(X)$ that form it are linear. In this case, the problem is solved by reducing a fractional criterion to normal. Current results from the modification and strengthening of the method extend the scope of its application in some important areas (solving integer fractional-linear problems [9, 10], the problems in which the objective function coefficients are set over an interval [11, 12] or fuzzily assigned [11–15].

Those problems are much more difficult in which the functional to be optimized (1) is not linear not only structurally, but also due to the nonlinearity of functions that form it.

Known methods for solving such a problem employ different variants for transforming the fractional criterion (1) into normal [16–21]. This concept is implemented as follows. Assume $g(X)>0$. Choose an arbitrary vector $X^{(0)}=(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)})$ from the function determination domain (1). If the original problem is to maximize the criterion (1) and the selected vector $X^{(0)}$ is not a solution to this problem, then there must be some other vector $X^{(1)}$ for which:

$$F\left(X^{(1)}\right)-F\left(X^{(0)}\right)>0. \tag{2}$$

Then, considering (1), record:

$$\Phi\left(X^{(1)}\right)=\left[F\left(X^{(1)}\right)-F\left(X^{(0)}\right)\right]g\left(X^{(1)}\right)=$$
$$=\varphi\left(X^{(1)}\right)-g\left(X^{(1)}\right)\cdot F\left(X^{(0)}\right)>0. \tag{3}$$

Hence is a simple technique to solve the original problem: one must maximize the criterion $\Phi(X^{(1)})$. If inequality (3), derived in this case for plan $X^{(1)}$, is satisfied, the plan $X^{(1)}$ is better the plan $X^{(0)}$, otherwise the plan $X^{(0)}$ is optimal. Thus, the original problem is reduced to an iterative procedure of finding a sequence of vectors:

$$\left\{X^{(0)}, X^{*(1)}, X^{*(2)}, ..., X^{*(k)}, X^{*(k+1)}, ...\right\}, \tag{4}$$

for which the recurrence ratio holds:

$$\varphi\left(X^{*(k+1)}\right)-g\left(X^{*(k+1)}\right)\cdot F\left(X^{*(k)}\right)=$$
$$=\max_{x^{(k+1)}}\left\{\varphi\left(X^{(k+1)}\right)-g\left(X^{(k+1)}\right)\cdot F\left(X^{*(k)}\right)\right\}. \tag{5}$$

In this case, it is clear that the problems obtained at every step of this procedure are easier than the original one. In a general case, it is a difficult task to prove the convergence of a sequence of vectors (4) to vector X^* that maximizes (1). However, for a series of specific tasks the problem can be solved, for example, if the set of permissible solutions to the problem is finite.

Assume the problem implies finding a vector X that maximizes the non-linear-fractional function:

$$F\left(x_1, x_2\right)=\frac{\varphi\left(x_1, x_2\right)}{g\left(x_1, x_2\right)}=\frac{x_1^2+5x_2^2}{2x_1^2+x_2^2}, \tag{6}$$

and which satisfies the constraint:

$$x_1+2x_2=3. \tag{7}$$

In accordance with the methodology described, we shall assign any initial vector $X^{(0)}$ that satisfies (7), for example, $X^{(0)}=(1; 1)$. In this case, $F(X^{(0)})=2$. Next we introduce the function:

$$\Phi\left(x_1, x_2\right)=\varphi\left(x_1, x_2\right)-g\left(x_1, x_2\right)F\left(X^{(0)}\right)=$$
$$=x_1^2+5x_2^2-2\left(2x_1^2+x_2^2\right)=-3x_1^2+3x_2^2. \tag{8}$$

We shall find vector (x_1, x_2) that maximizes (8) and satisfies (7) using the uncertain Lagrange multipliers method. Introduce:

$$L\left(x_1, x_2\right)=-3x_1^2+3x_2^2-\lambda\left(x_1+2x_2-3\right). \tag{9}$$

Next:

$$\frac{dL\left(x_1, x_2\right)}{dx_1}=-6x_1-\lambda=0, \quad x_1=-\frac{1}{6}\lambda;$$

$$\frac{dL\left(x_1, x_2\right)}{dx_2}=6x_2-2\lambda=0, \quad x_2=\frac{1}{3}\lambda.$$

Substitute the derived expressions for x_1 and x_2 in (7) and find λ:

$$-\frac{1}{6}\lambda+\frac{2}{3}\lambda=\frac{\lambda}{2}=3.$$

Hence $\lambda=6$ and

$$x_1^{(1)}=-\frac{1}{6}\cdot 6=-1, \quad x_2^{(1)}=-\frac{1}{3}\cdot 6=2.$$

Check that the inequality $F(X^{(1)})>F(X^{(0)})$ holds for the derived vector $X^{(1)}=(-1; 2)$. Since:

$$F\left(X^{(1)}\right)=\frac{1+20}{2+4}=\frac{7}{2}>F\left(X^{(0)}\right)=2,$$

the procedure should continue.

Forming the following function anew:

$$\Phi\left(x_1, x_2\right)=\varphi\left(x_1, x_2\right)-g\left(x_1, x_2\right)F\left(X^{(1)}\right)=$$
$$=x_1^2+5x_2^2-\frac{7}{2}\left(2x_1^2+x_2^2\right)=-6x_1^2+\frac{3}{2}x_2^2,$$

as well as the new Lagrangian function:

$$L\left(x_1, x_2\right)=-6x_1^2+\frac{3}{2}x_2^2-\lambda\left(x_1+2x_2-3\right).$$

Next:

$$\frac{dL\left(x_1, x_2\right)}{dx_1}=-12x_1-\lambda=0, \quad x_1=-\frac{1}{12}\lambda;$$

$$\frac{dL\left(x_1, x_2\right)}{dx_2}=3x_2-2\lambda=0, \quad x_2=\frac{2}{3}\lambda.$$

Find λ and the values for vector $X^{(2)}$ components. We obtain:

$$-\frac{1}{12}\lambda + \frac{4}{3}\lambda = \frac{5}{4}\lambda = 3; \quad \lambda = \frac{12}{5}; \quad x_1^{(2)} = -\frac{1}{5}, \quad x_2^{(2)} = \frac{8}{5}.$$

Since:

$$F\left(X^{(2)}\right) = \frac{\dfrac{1}{25} + 5 \cdot \dfrac{64}{25}}{2 \cdot \dfrac{1}{25} + \dfrac{64}{25}} = 4.863 > F\left(X^{(1)}\right) = 3.5,$$

the procedure continues. At the next step, we obtain:

$$\Phi(x_1, x_2) = x_1^2 + 5x_2^2 - 4.863\left(2x_1^2 + x_2^2\right) =$$
$$= -8.726x_1^2 + 0.137x_2^2;$$

$$L(x_1, x_2) = -8.726x_1^2 + 0.137x_2^2 - \lambda(x_1 + 2x_2);$$

$$\frac{dL(x_1, x_2)}{dx_1} = -17.452x_1 - \lambda = 0, \quad x_1 = -\frac{1}{17.452}\lambda;$$

$$\frac{dL(x_1, x_2)}{dx_2} = 0.274x_2 - 2\lambda = 0, \quad x_2 = \frac{2}{0.274}\lambda = 7.3\lambda;$$

$$\left(-\frac{1}{17.452} + 2 \cdot 7.3\right)\lambda = 14.54\lambda = 3; \quad \lambda = \frac{3}{14.54};$$

$$x_2^{(3)} = -\frac{3}{17.452 \cdot 14.54} = 0.012; \quad x_2^{(3)} = 1.506;$$

$$F\left(X^{(3)}\right) = \frac{(-0.012)^2 + 5 \cdot (1.506)^2}{2(-0.012)^2 + (1.506)^2} = \frac{11.3403}{1.5002} =$$
$$= 4.996 > F\left(X^{(3)}\right).$$

Because the value for $F(x_1, x_2)$ continues to grow, the procedure should continue. Solving it further shows that the values for variables x_1, x_2, $F(x_1, x_2)$ asymptotically approach, respectively, $x_1^* = 0$, $x_2^* = 1.5$, $F(x_1^*, x_2^*) = 5$. In this particular example, the computational procedure is simple and the rate of its convergence is quite high, however, this gives no reasons for the statement of general optimistic conclusions. This fundamental circumstance renders relevance to the task on devising an alternative finite-step method to solve the fractional nonlinear optimization problems, which defines the purpose of the current study.

3. The aim and objectives of the study

The aim of this study is to construct a single-step method to solve the fractional nonlinear optimization problem, which would make it possible to obtain a result in a single step.

To accomplish the aim, the following tasks have been set:
– to transform the original model of a fractional nonlinear optimization problem to the form typical for conventional problems of mathematical programming;
– to devise a computational procedure to solve the problem of mathematical programming, derived in this case, in a single step.

4. Construction of a single-step method of fractional-nonlinear optimization

Let us consider a possibility to build a single-step optimization method for the nonlinear fractional functional.

Here us a specific and practical task (the rational allocation of limited resource), which comes down to optimizing the fractional-quadratic functional. We introduce:

b – value for the resource utilized when making a single piece of product of the j-th type, $j = 1, 2, ..., n$;

x_j – planned number of manufactured pieces of product of the j-th type, $j = 1, 2, ..., n$;

c_j – profit earned when selling a piece of product of the j-th type.

Then

$$L(x) = \sum_{j=1}^{n} c_j x_j$$

is the total profit when implementing production schedule $X = (x_1, x_2, ..., x_n)$.

The task on the rational resource allocation implies finding plan X, which maximizes the total profit $L(x)$ and satisfies the constraints:

$$\sum_{j=1}^{n} a_j x_j = b, \tag{10}$$

$$x_j \geq 0, \quad j = 1, 2, ..., n. \tag{11}$$

This problem is trivial and its obvious solution takes the form:

$$X = \left\{x_j = 0, j \neq j_0\right\}, \quad j_0 = \max_j \left\{\frac{c_j}{a_j}\right\}, \quad x_{j_0} = \frac{b}{a_{j_0}}.$$

However, the task is greatly complicated if the profit earned when selling a product is random. Assume that the random profit from selling a piece of product of the j-th type is distributed in line with the normal law with density:

$$f(c_j) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left\{-\frac{(c_j - m_j)^2}{2\sigma_j^2}\right\}, \quad j = 1, 2, ..., n,$$

where m_j is the mathematical expectation of a random profit from the sale of a piece of product of the j-th type; σ_j^2 is the variance of a random profit from selling a product piece of the j-th type.

Then the total profit $L(x)$ from implementing plan $X = (x_1, x_2, ..., x_n)$ is a random variable with a distribution density:

$$f(L(x)) = \frac{1}{\sqrt{2\pi}\sigma_\Sigma} \exp\left\{-\frac{(L - m_\Sigma)^2}{2\sigma_\Sigma^2}\right\},$$

where

$$m_\Sigma = \sum_{j=1}^{n} m_j x_j, \quad \sigma_\Sigma^2 = \sum_{j=1}^{n} \sigma_j^2 x_j^2.$$

We can now compute a probability of that a random value for the total profit exceeds a certain preset threshold L_Π, which is equal to:

$$P\left(L(x) > L_{\Pi}\right) =$$

$$= \int_{L_{\Pi}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_{\Sigma}} \exp\left\{-\frac{\left(L - m_{\Sigma}\right)^2}{2\sigma_{\Sigma}^2}\right\} dL = \int_{\frac{L_{\Pi}-m_{\Sigma}}{\sigma_{\Sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du. \quad (12)$$

In this case, the problem on the rational distribution of a resource can be restated as follows: it is required to find plan X that maximizes (12) and satisfies constraints (10), (11). We shall transform the lower limit in integral (12):

$$\frac{L_{\Pi} - m_{\Sigma}}{\sigma_{\Sigma}} = \frac{L_{\Pi} - \sum_{j=1}^{n} m_j x_j}{\left(\sum_{j=1}^{n} \sigma_j^2 x_j^2\right)^{\frac{1}{2}}} = \frac{\frac{1}{\sum_{j=1}^{n} a_j x_j} L_{\Pi} \sum_{j=1}^{n} a_j x_j - \sum_{j=1}^{n} m_j x_j}{\left(\sum_{j=1}^{n} \sigma_j^2 x_j^2\right)^{\frac{1}{2}}} =$$

$$= \frac{\sum_{j=1}^{n} \frac{L_{\Pi}}{b} a_j x_j - \sum_{j=1}^{n} m_j x_j}{\left(\sum_{j=1}^{n} \sigma_j^2 x_j^2\right)^{\frac{1}{2}}} = \frac{\sum_{j=1}^{n} \left(\frac{L_{\Pi}}{b} a_j - m_j\right) x_j}{\left(\sum_{j=1}^{n} \sigma_j^2 x_j^2\right)^{\frac{1}{2}}}.$$

It is clear that the maximization of integral (12) is enabled by minimizing the derived value for its lower limit.

We introduce:

$$d_j = m_j - \frac{L_{\Pi}}{b} a_j, \quad d_j > 0, \quad j = 1, 2, ..., n.$$

Now, the rational resource allocation problem is reduced to the following: it is required to find plan X that maximizes:

$$\frac{\sum_{j=1}^{n} d_j x_j}{\left(\sum_{j=1}^{n} \sigma_j^2 x_j^2\right)^{\frac{1}{2}}} \quad (13)$$

and satisfies (10), (11). The maximization problem (13) is equivalent to the minimization problem:

$$F(x) = \frac{\sum_{j=1}^{n} \sigma_j^2 x_j^2}{\left(\sum_{j=1}^{n} d_j x_j\right)^2}. \quad (14)$$

Thus, the problem is stated as follows: it is required to find a non-negative vector X that minimizes (14) and satisfies constraints (10). In a general case, when allocating a multidimensional resource, constraint (10) takes the following form:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, 2, ..., m. \quad (15)$$

Therefore, we obtained a fractional nonlinear optimization problem.

To solve the problem obtained, we introduce new variables:

$$y_0 = \frac{1}{\sum_{j=1}^{n} d_j x_j}, \quad y_j = y_0 x_j, \quad x_j = \frac{y_j}{y_0}, \quad j = 1, 2, ..., n.$$

In this case:

$$y_0 \sum_{j=1}^{n} d_j x_j = \sum_{j=1}^{n} d_j y_j = 1, \quad (16)$$

$$\sum_{j=1}^{n} a_{ij} y_j = b_i y_0, \quad i = 1, 2, ..., m. \quad (17)$$

Then the objective function (14) is transformed to the form:

$$R(y) = \left[\sum_{j=1}^{n} \sigma_j^2 \left(\frac{y_j}{y_0}\right)^2\right] y_0^2 = \left(\sum_{j=1}^{n} \sigma_j^2 y_j^2\right). \quad (18)$$

The original task is now stated as follows: it is required to find vector $(y_0, y_0, ..., y_n)$, which minimizes (18) and satisfies constraints (16), (17).

We shall obtain a solution by applying the uncertain Lagrange multipliers method.

We introduce:

$$\Phi(Y) = \sum_{j=1}^{n} \sigma_j^2 y_j^2 - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} y_j - b_i\right) - \lambda_{m+1} \left(\sum_{j=1}^{n} d_j y_j - 1\right).$$

Next:

$$\frac{d\Phi(Y)}{dy_j} = 2\sigma_j^2 y_j - \sum_{i=1}^{m} \lambda_i a_{ij} - \lambda_{m+1} \cdot d_j = 0, \quad j = 1, 2, ..., n,$$

whence

$$y_j = \frac{1}{2\sigma_j^2} \left(\sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+1} \cdot d_j\right), \quad j = 1, 2, ..., n. \quad (19)$$

To define a set of $(\lambda_1, \lambda_2, ..., \lambda_{m+1})$, the derived expressions for y_i will be substituted in (16), (17) and we shall solve the resulting system of linear algebraic equations:

$$\begin{cases} \sum_{j=1}^{n} a_{ij} \cdot \frac{1}{2\sigma_j^2} \left(\sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+1} \cdot d_j\right) = b_i y_0, \quad i = 1, 2, ..., m, \\ \sum_{j=1}^{n} d_j \cdot \frac{1}{2\sigma_j^2} \left(\sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+1} \cdot d_j\right) = 1. \end{cases} \quad (20)$$

The system of linear algebraic equations is solved by any known method. For the simplest particular case of a single-dimensional resource allocation we obtain $m = 1$ and $a_{ij} = a_j$, $b_i = b$; system (20) is simplified to the form:

$$\begin{cases} \sum_{j=1}^{n} y_j a_j = \sum_{i=1}^{n} \frac{a_j}{2\sigma_j^2} (\lambda_1 a_j + \lambda_2 \cdot d_j) = by_0, \\ \sum_{j=1}^{n} y_j d_j \cdot \sum_{i=1}^{n} \frac{a_j}{2\sigma_j^2} (\lambda_1 a_j + \lambda_2 \cdot d_j) = 1, \end{cases}$$

or

$$\begin{cases} \lambda_1 \sum_{i=1}^{n} \frac{a_j^2}{2\sigma_j^2} + \lambda_2 \sum_{i=1}^{n} \frac{a_j d_j}{2\sigma_j^2} = by_0, \\ \lambda_1 \sum_{i=1}^{n} \frac{a_j d_j}{2\sigma_j^2} + \lambda_2 \sum_{i=1}^{n} \frac{d_j^2}{2\sigma_j^2} = 1, \end{cases}$$

or

$$\lambda_1 C_{11} + \lambda_2 C_{12} = by_0, \quad \lambda_1 C_{12} + \lambda_2 C_{22} = 1.$$

Hence:

$$\lambda_1 = \frac{y_0 b C_{22} - C_{12}}{C_{11}C_{22} - C_{12}^2}, \quad \lambda_2 = \frac{C_{11} - C_{12}y_0 b}{C_{11}C_{22} - C_{12}^2}. \tag{21}$$

Substituting (21) in (19), we obtain:

$$y_j = \frac{y_0 b C_{22} - C_{12}}{C_{11}C_{22} - C_{12}^2} \cdot \frac{a_j}{2C_j} + \frac{C_{11} - C_{12}y_0 b}{C_{11}C_{22} - C_{12}^2} \cdot \frac{d_j}{2\sigma_j^2}, \; j = 1, 2, ..., n. \tag{22}$$

Now, by substituting (22) in (17), we shall find y_0, then we shall define the desired solution:

$$x_j = \frac{y_j}{y_0}, \quad j = 1, 2, ..., n.$$

Thus, the introduced transformation that converts the non-linear-fractional criterion (14) to the form typical of mathematical programming problems, has made it possible to obtain a single-step solution to the original problem.

Consider an example. Assume:

$$F(x_1, x_2) = \frac{x_1^2 + 2x_2^2}{(2x_1 + x_2)^2}, \tag{23}$$

$$x_1 + 3x_2 = 1.$$

We give a solution to the example using the proposed technique without detailed explanation.

$$y_0 = \frac{1}{2x_1 + x_2}, \quad y_j = y_0 x_j, \quad x_j = \frac{y_j}{y_0}, \quad j = 1, 2.$$

In this case:

$$F(y_1, y_2) = y_1^2 + 2y_2^2,$$

$$\begin{cases} y_1 + 3y_2 = y_0, \\ 2y_1 + y_2 = 1. \end{cases} \tag{24}$$

Next:

$$\Phi(Y) = y_1^2 + 2y_2^2 - \lambda_1(y_1 + 3y_2 - y_0) - \lambda_2(2y_1 + y_2 - 1);$$

$$\frac{d\Phi(Y)}{dy_1} = 2y_1 - \lambda_1 - 2\lambda_2 = 0, \quad y_1 = \frac{\lambda_1 + 2\lambda_2}{2};$$

$$\frac{d\Phi(Y)}{dy_2} = 4y_2 - 3\lambda_1 - \lambda_2 = 0, \quad y_2 = \frac{\lambda_2 + 2\lambda_1}{4}. \tag{25}$$

To find λ_1 and λ_2, substitute (25) in (24):

$$\begin{cases} \dfrac{\lambda_1 + 2\lambda_2}{2} + 3 \cdot \dfrac{\lambda_2 + 3\lambda_1}{4} = \dfrac{11}{4}\lambda_1 + \dfrac{7}{4}\lambda_2 = y_0, \\ \lambda_1 + 2\lambda_2 + \dfrac{\lambda_2 + 3\lambda_1}{4} = \dfrac{7}{4}\lambda_1 + \dfrac{9}{4}\lambda_2 = 1. \end{cases}$$

Hence:

$$\lambda_1 = \frac{-14 + 18y_0}{25}, \quad \lambda_2 = \frac{22 - 14y_0}{25}. \tag{26}$$

Now, by substituting (26) in (25), we find y_1, y_2:

$$y_1 = \frac{1}{5}(3 - y_0); \quad y_2 = \frac{1}{25}(16 - 17y_0). \tag{27}$$

Finally, by substituting (27) in the second equation of system (24), we find y_0.

$$\frac{2}{5}(3 - y_0) + \frac{1}{25}(16 - 17y_0) = \frac{1}{25}(46 - 27y_0) = 1,$$

hence $y_0 = 7/9$.

In this case:

$$y_1 = \frac{1}{5}\left(3 - \frac{7}{9}\right) = \frac{4}{9}, \quad y_2 = \frac{1}{25}\left(16 - 17 \cdot \frac{7}{9}\right) = \frac{1}{9}.$$

Then:

$$x_1 = \frac{y_1}{y_0} = \frac{4}{9} \cdot \frac{9}{7} = \frac{4}{7}, \quad x_2 = \frac{1}{9} \cdot \frac{9}{7} = \frac{1}{7}.$$

The problem is solved.

Consider another example. We return to the problem considered above (6), (7) and solve it by the proposed method of non-linear-fractional programming. Thus, the problem is stated as follows: it is required to find vector $X = (x_1, x_2)$, which maximizes the objective function (6) and satisfies constraint (7). We introduce:

$$y_0^2 = \frac{1}{2x_1^2 + x_2^2}, \quad y_j = y_0 x_j, \quad x_j = \frac{y_j}{y_0}, \quad j = 1, 2, ..., n.$$

In this case:

$$F(y_1, y_2) = y_0^2\left(\frac{y_1^2}{y_0^2} + 5\frac{y_2^2}{y_0^2}\right) = y_1^2 + 5y_2^2, \quad \frac{y_1}{y_0} + 2\frac{y_2}{y_0} = 3.$$

The problem now takes the form: it is required to find (y_1, y_1), which maximize:

$$F(Y) = y_1^2 + 5y_2^2$$

and satisfy the system of constraints:

$$\begin{cases} y_1 + 2y_2 = 3y_0, \\ 2y_1^2 + y_2^2 = 1. \end{cases} \tag{28}$$

We introduce the Lagrangian function:

$$\Phi(Y) = y_1^2 + 5y_2^2 - \lambda_1(y_1 + 2y_2 - 3y_0) - \lambda_2(2y_1^2 + y_2^2 - 1).$$

Next:

$$\frac{d\Phi(Y)}{dy_1} = 2y_1 - \lambda_1 - 4\lambda_2 y_1 = 0, \quad y_1 = \frac{\lambda_1}{2(1 - 2\lambda_2)};$$

$$\frac{d\Phi(Y)}{dy_2} = 10y_2 - 2\lambda_1 - 2\lambda_2 y_2 = 0, \quad y_2 = \frac{\lambda_1}{5 - \lambda_2}. \tag{29}$$

Substitute (29) in the equation of system (28):

$$\begin{cases} \dfrac{\lambda_1}{2(1-2\lambda_2)}+2\cdot\dfrac{\lambda_1}{5-\lambda_2}=3y_0, \\[3mm] \dfrac{\lambda_1^2}{2(1-2\lambda_2)^2}+\dfrac{\lambda_1^2}{(5-\lambda_2)^2}=1. \end{cases}$$

Hence:

$$\begin{cases} \lambda_1(5-\lambda_2)+4\lambda_1(1-2\lambda_2)=6y_0(1-2\lambda_2)(5-\lambda_2), \\[2mm] \lambda_1^2(5-\lambda_2)^2+2\lambda_1^2(1-2\lambda_2)^2=2(1-2\lambda_2)^2(5-\lambda_2)^2. \end{cases}$$

The solution to this system is: $\lambda_1=0$, $\lambda_2=5$, $y_1=0$. To calculate y_2 and y_0, we shall use equation (28). In this case, we obtain $y_1=1$, $y_0=3/2$. Then, the desired solution to the problem takes the form:

$$x_1^*=y_1/y_0=0, \quad x_2^*=y_2/y_0=3/2.$$

A value for the objective function at optimum set $\left(x_1^*, x_2^*\right)$ is equal to $F(0;3/2)=5$. This solution naturally coincides with the one obtained earlier.

It should be noted that the described technique of the fractional-non-linear optimization is applicable for solving problems with an arbitrary power of variables in the numerator and denominator of the optimized functional.

5. Discussion of results of constructing a method to solve a fractional nonlinear optimization problem

We have proposed a single-step method for solving a fractional nonlinear optimization problem. In contrast to those known [15–19], the proposed method has the following fundamental features:

– a possibility to derive a solution to the problem of arbitrary dimensionality without using a labor-intensive iterative procedure, whose rate of convergence cannot be estimated even approximately;

– the order of nonlinearity of the components of an objective function is not limited. To implement the proposed method for solving a problem, the model must meet the following requirements: functions $\Phi(X)$ and $g(X)$ must be separable; the order of nonlinearity of all terms for the numerator and denominator of the optimized function must be the same.

If these requirements are met, the method implements a single-step procedure for solving a problem.

Directions for the further research are associated with the development of techniques for extending the method to cases when the parameters for the problem's objective function and constraints are described in terms of fuzzy [22] or inaccurate [23, 24] mathematics. Possible ways to overcome the problems that emerge in this case are proposed in [25–27].

6. Conclusions

1. We have proposed a method for solving the optimization problem of nonlinear-fractional functional in the presence of linear or nonlinear constraints. The method is based on the introduced special transformation of an original fractional-nonlinear structure of the optimized criterion to the form typical for standard problems of mathematical programming of arbitrary dimensionality.

2. The fundamental merit of the proposed method is that the method makes it possible to obtain the desired solution in a single step of the computational procedure that employs the standard methods of mathematical programming.

References

1. Yudin, D. B. (1974). Matematicheskie metody upravleniya v usloviyah nepolnoy informatsii. Zadachi i metody stohasticheskogo programmirovaniya. Moscow: Sovetskoe radio, 392.

2. Bazaraa, M. S., Shetty, C. M. (1979). Nonlinear Programming: Theory and Algorithms. Wiley, 312.

3. Larichev, O. I. (2002). Teoriya i metody prinyatiya resheniy. Moscow: Logos, 392.

4. Karmanov, V. G. (1980). Matematicheskoe programmirovanie. Moscow, 256.

5. Himmelblau, D. (1972). Applied Nonlinear Programming. New York: McGraw-Hill, 498.

6. Zangwill, W. I. (1969). Nonlinear Programming: A Unified Approach. Prentice-Hall, 356.

7. Rekleytis, G., Reyvindran, A., Regsdel, K. (1989). Optimizatsiya v tekhnike. Moscow: MIR, 349.

8. Zuhovitskiy, S. I., Avdeeva, L. I. (1967). Lineynoe i vypukloe programmirovanie. Moscow: «Nauka», 460.

9. Malaguti, E., Monaci, M., Paronuzzi, P., Pferschy, U. (2019). Integer optimization with penalized fractional values: The Knapsack case. European Journal of Operational Research, 273 (3), 874–888. doi: https://doi.org/10.1016/j.ejor.2018.09.020

10. Yue, D., You, F. (2014). Reformulation-linearization Method for Global Optimization of Mixed Integer Linear Fractional Programming Problems with Application on Sustainable Batch Scheduling. 24th European Symposium on Computer Aided Process Engineering, 949–954. doi: https://doi.org/10.1016/b978-0-444-63456-6.50159-9

11. Borza, M., Rambely, A. S., Saraj, M. (2012). Solving Linear Fractional Programming Problems with Interval Coefficients in the Objective Function. A New Approach. Applied Mathematical Sciences, 6 (69), 3443–3452.

12. Khalifa, H. (2018). On Solutions of Linear Fractional Programming Problems with Rough-interval Coefficients in The Objective Function. The Journal of Fuzzy Mathematics, 26 (2), 415–422.

13. Das, S. K., Mandal, T., Edalatpanah, S. A. (2017). A new approach for solving fully fuzzy linear fractional programming problems using the multi-objective linear programming. RAIRO – Operations Research, 51 (1), 285–297. doi: https://doi.org/10.1051/ro/2016022

14. Deb, M. (2018). A Study of Fully Fuzzy Linear Fractional Programming Problems by Signed Distance Ranking Technique. Advances in Data Mining and Database Management, 73–115. doi: https://doi.org/10.4018/978-1-5225-5091-4.ch004

15. Arya, R., Singh, P., Bhati, D. (2018). A fuzzy based branch and bound approach for multi-objective linear fractional (MOLF) optimization problems. Journal of Computational Science, 24, 54–64. doi: https://doi.org/10.1016/j.jocs.2017.12.011

16. Arya, R., Singh, P. (2019). Fuzzy efficient iterative method for multi-objective linear fractional programming problems. Mathematics and Computers in Simulation, 160, 39–54. doi: https://doi.org/10.1016/j.matcom.2018.11.013

17. Raskin, L. G. (1976). Analiz slozhnyh sistem i elementy teorii upravleniya. Moscow: Sovetskoe radio, 344.

18. Wolf, H. (1986). Solving special nonlinear fractional programming problems via parametric linear programming. European Journal of Operational Research, 23 (3), 396–400. doi: https://doi.org/10.1016/0377-2217(86)90305-x

19. Saxena, P., Jain, R. (2017). Duality for a nonlinear fractional programming under fuzzy environment with parabolic concave membership functions. International Journal of ADVANCED AND APPLIED SCIENCES, 4 (1), 131–136. doi: https://doi.org/10.21833/ijaas.2017.01.019

20. Jeflea, A. (2003). A parametric study for solving nonlinear fractional problems. An. St. Univ. Ovidius Constanta, 11 (2), 87–92.

21. Bhurjee, A. K., Panda, G. (2013). Nonlinear fractional programming problem with inexact parameter. Journal of Applied Mathematics & Informatics, 31 (5-6), 853–867. doi: https://doi.org/10.14317/jami.2013.853

22. Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8 (3), 338–353. doi: https://doi.org/10.1016/s0019-9958(65)90241-x

23. Pawlak, Z. (1985). Rough sets and fuzzy sets. Fuzzy Sets and Systems, 17 (1), 99–102. doi: https://doi.org/10.1016/s0165-0114(85)80029-4

24. Pawlak, Z. (1991). Rough Sets: Theoretical Aspects of Reasoning about Data. Dordrecht: Kluwer Academic Publisher, 284. doi: https://doi.org/10.1007/978-94-011-3534-4

25. Raskin, L., Sira, O. (2016). Method of solving fuzzy problems of mathematical programming. Eastern-European Journal of Enterprise Technologies, 5 (4 (83)), 23–28. doi: https://doi.org/10.15587/1729-4061.2016.81292

26. Raskin, L., Sira, O. (2016). Fuzzy models of rough mathematics. Eastern-European Journal of Enterprise Technologies, 6 (4 (84)), 53–60. doi: https://doi.org/10.15587/1729-4061.2016.86739

27. Raskin, L., Sira, O., Ivanchykhin, Y. (2017). Models and methods of regression analysis under conditions of fuzzy initial data. Eastern-European Journal of Enterprise Technologies, 4 (4 (88)), 12–19. doi: https://doi.org/10.15587/1729-4061.2017.107536