Дослідження присвячено розв'язанню задач, пов'язаних з моделюванням поверхонь другого порядку (квадрик), у визначник яких включено дотичні конуси. Всі дослідження виконано за парадигмою використання конструктивних методів створення алгоритмів. Це обумовлено тим, що існує можливість спиратись на значну кількість базових геометричних задач, реалізованих у САПР.

Задача моделювання квадрик за дотичними конусами є актуальною, тому що існує принаймні два важливих її застосування. Перше – це побудова поверхонь за її лінією обрису на перспективних зображеннях. При цьому, точка зору та перспективна лінія обрису задають обгортаючий конус, який у випадку квадрик, збігається з дотичним. Такі задачі розв'язуються в контексті задач технічної естетики та архітектурного проектування.

Друге застосування відбувається у задачах побудови квадрик, що спряжені по заданим кривим, або у задачах спряження двох квадрик третьою. Задача спряження поверхонь має широке практичне значення, що підтверджується зацікавленістю нею користувачів та розробників систем комп'ютерного моделювання.

У рамках дослідження акумульовано існуючі теоретичні геометричні властивості для моделювання квадрик, у визначник яких включено дотичні конуси та встановлено низку нових геометричних властивостей.

Розроблено спосіб, за яким задаючи лінію контакту на одному конусі є можливість знайти лінію контакту на другому конусі, а також знайти центр вписаної у ці два конуси квадрики. Запропоновано також альтернативний спосіб моделювання описаних поверхонь. За цим способом перерізи всіх квадрик, дотичних до двох конусів, є вписаними у чотирикутники, вершини яких належать лініям перетину заданих конусів. На основі конструктивних геометричних досліджень розроблено алгоритми для комп'ютерної реалізації задач моделювання об'єктів за лініями обрисів на їх перспективних зображеннях.

Отримані результати дослідження у вигляді теоретичних викладок та прикладів їх застосування показують дієздатність запропонованих алгоритмів. Описаний підхід до розв'язання поставлених задач дозволяє розширити можливості існуючих комп'ютерних систем при їх застосуванні в роботі конструкторів і значно спростити процес створення реальних об'єктів

Ключові слова: квадрика, дотичні конуси, визначник поверхні, лінія контакту, перспективне зображення, спряження поверхонь

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STUDY OF THE PROBLEM ON CONSTRUCTING QUADRICS AT THE ASSIGNED TANGENT CONES

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1. Introduction

The problem of modeling of second-order surfaces (quadrics) by the assigned tangent cones is relevant, because there are at least two its important applications.

The first one is the construction of surfaces by its contour line on perspective images. In this case, the point of view and the perspective contour line assign the enveloping cone, which in the case of quadrics, coincides with the tangent one. When constructing different types of surfaces, the rules, by which one or more of enveloping cones are assigned, are determined by the statement of the problem and the form of the modeled surface itself. Such problems are solved in the context of the problems of technical aesthetics and architectural design. In the case of quadrics, there can be one, two or three enveloping cones. It is known that three cones determine the quadric unambiguously, but these cones cannot be assigned arbitrarily. Two cones, tangent to the quadric, cannot be assigned arbitrarily, because according to the theorem of G. Monge, these cones must have two common tangent planes.

The second application occurs in the problems of the construction of quadrics that are conjugate by assigned curves, or in the problems of conjugation of two quadrics by the third one. The problem of conjugation of surfaces is of broad practical significance as evidenced by the interest of users and developers of computer modeling systems. The unresolved task is the implementation of the stated problem if all conjugate surfaces are quadrics. In general, the systems of 3D computer modeling «hardly support» the geometry of modeling the quadrics, which stimulates the development of the algorithms and creation of applications for operating these geometric objects.

That is why the issues and theoretical research related to the structural and parametric analysis of the quadric construction by two tangent cones and conjugation of two quadrics by the third one remain relevant.

2. Literature review and problem statement

The modern paradigm of computer geometric modeling of surfaces is based on two approaches. The first approach provides the possibility of constructing as dense frame of points as desired, according to kinematic schemes or by the schemes of incidences. The second uses the parametric equations of curves and surfaces, which include interpolation points and control points.

However, some studies are performed in a traditional analytical form and their computer implementation makes it possible to get useful for practice results. Thus, in papers [1, 2], the authors conduct a detailed parametric analysis of the existing possibilities for the construction of quadrics by the assigned number of points, and the impact of the given conditions on the form and the properties of the modeled quadrics.

In paper [3], it is proposed to model quadrics using the apparatus of geometric algebra. Geometric algebra can be understood as a totality of tools for constructing and converting geometric objects. Geometric algebra makes it possible to choose these tools using intuition, but in any case requires a very large amount of computations. There are different implementations of geometric algebra. The authors of [3] stated and solved the problem of the construction of algebra with the optimum volume of computations. It allowed carrying out the process of modeling quadrics and solving differential-geometric and positional problems on them.

Article [4] considers the use of quadrics as primitives. It was shown that the bodies limited by quadrics provide a wide set for constructing models of three-dimensional objects. The regularized operators that guarantee topological isolation of objects are discussed and classified.

The problem of topological correctness of the result of intersection of two quadrics generally attracts the attention of many researchers. In article [5], a thorough analysis of the results in this direction was performed. The conclusion was made that although geometrical approaches have sustainability of results, they are more limited. Instead, article [5] proposed its own approach. The main aim is to obtain a topologically correct line of intersection of two quadrics. The approach is based on the analysis of the projection (contour lines) of two intersecting surfaces. This approach is largely structural.

In the 1990s, a new branch of geometry, named by the authors as discrete differential geometry, was founded [6]. Its distinctive feature is the possibility of obtaining the main theorems of differential geometry for objects of any dimension, based on structural and logical considerations with minimal involvement of analytical dependences. The basis for interpretations of geometric constructions was the concept of discrete congruence, derived in paper [7]. Another important result of this work is the fact that discrete congruence, formed based of the four-dimensional single parametric cube, is limited to a two-parametric set of quadrics. This suggests that the quadrics have not lost their significance not only in the practical [8], but also in theoretical-geometric sense.

Instead, applied geometry traditionally solves the problems of modeling curves and surfaces by constructive methods [9].

The situation with conics can be considered to be solved. Since there are systems (for example, SolidWorks [10]), which provide ample possibilities for their construction. In addition, a number of authors created an application to the system AutoCAD, which, on a purely constructive basis, makes it possible to perform the construction of conics under any conditions: actual, non-proper, imaginary and their combinations, and to solve complex geometric problems. This approach is presented in papers [11, 12].

Nowadays, scientists in up-to-date papers often return to the constructive method of analysis and synthesis of algorithms. This is due to the emergence of computer modeling, which significantly spread the tools for research implementation. Thus, the author of research [13] proposed and implemented the method for constructing a quadric by nine points. In research [14], the lines of intersection of quadrics are explored. The author of paper [15] applied the constructive method for studying the properties of parametrically assigned curves of higher orders. Paper [16] considers the issue of conjugation of quadrics. In the presented research, the author is limited to quadric cones.

However, none of the noted papers considers a tangent cone as a component of the surface determinant.

The construction of quadrics by tangent cones is based on the fact that the line of the contact of a cone that is tangent to the quadric is a conic. This result was also obtained by the founder of descriptive geometry G. Monge in paper [17] and is the basis of the methods for the construction of quadrics by tangent cones.

The problem has two statements. The first one is the reconstruction of the quadric by its image. Such studies are presented in paper [18]. In his research, the author analyzes the light reflection from the surface of the quadric. This makes it possible to determine unambiguously the quadric even not knowing the centers of enveloping cones. The second is modeling the surfaces in the environment of the graphical computer system. In this case, a designer assigns the desired contour lines from the desired points of view. This approach was implemented in [19]. The author solves the problem of computer modeling of spherical design objects on perspective images according to contour lines. However, other quadrics are not considered.

In paper [20], there is a parametric analysis of the problem of construction of the rotation quadric by the tangent cone. It was shown that in this case the tangent cone determines two-parametric set of surfaces. In the study conducted in paper [21], the author analyzed the inclusion of the contour line in the determinant of a quadric of the general form and it was shown that in this case the tangent cone determines a four-parameter set of quadrics, when the contact line is not assigned. When the contact line is assigned, a one-parameter set of quadrics is determined.

The conducted analysis revealed that there is a reason to believe that the development of computer technologies has given impetus to the development of new methods of modeling quadrics by assigned engineering conditions. This development goes in two directions: analytical and constructive. Significant results were obtained in both directions.

However, there are no fundamental studies, which make it possible to include in the process of modeling the conditions of tangency to cones. That is why these studies are carried out within the framework of the problem of computer modeling of quadrics under conditions of tangency.

3. The aim and objectives of the study

The aim of this research is to develop theoretical and algorithmic means for solving the problem of modeling the quadrics, the determinant of which includes tangent cones, by the methods of computer graphics.

To accomplish the aim, the following tasks have been set:

– to perform theoretical research on the possibility of modeling the quadric, that is tangent to two cones, within the framework of solution of the problem of construction of the quadric surface by the contour line on its perspective images;

– using the constructive methods, to develop the algorithms for computer realization of the problems of modeling objects by the contour lines on their perspective images;

– to give examples of practical application of the research findings.

4. The object, subject, and methods of research on the possibility of modeling quadrics, the determinant of which includes tangent cones

The object of the presented study is quadrics, and the subject is the research into conditions, under which these quadrics can be constructed. At the same time, the main chosen conditions are the conditions of tangency.

It is commonly known that geometry methods are divided into two categories – analytical and constructive. Constructive geometry is based on the constructive and logical considerations at the assigned totality of axioms and theorems. Historically, geometry emerged as a constructive science. With the advent of analytical methods, constructive methods have somewhat yielded to them in popularity. However, today almost all branches of geometry, such as topology, computational geometry, discrete differential geometry, not to mention projective geometry, successfully use constructive methods.

The paradigm of using constructive methods in the applied branch of geometric research has become relevant due to the fact that in the systems of computer-aided design, a large number of rather complex geometric problems are solved in the automatic mode. That is why constructive geometric research in the applied branch can operate geometric properties (theorems or axioms) and the tools, implemented in computer systems. This makes it possible to create effective algorithms, even without using the analytical support methods. The conducted studies rely on this paradigm.

5. Theoretical research on the possibility of constructing quadrics, the determinant of which includes tangent cones

5. 1. The property about the centers of conical cross-sections

Assigned: quadric cone S and straight-line p of arbitrary position.

A bundle of planes, incident to an arbitrary straight line, crosses the quadric cone by the conics, the centers of which belong to the same plane.

Consider two cases.

Case 1. Straight line p does not intersect cone S. The design in Fig. 1, *a* corresponds to this case. To prove the property, draw arbitrary plane Π_0 through straight line p. This plane crosses the cone in certain conic s_0. Point P is the pole of straight line p relative to curve s_0. It was constructed according to the known algorithm: points 1 and 2 are arbitrary points of straight line p, straight lines d and k are the polars of these points, their intersection point P is the pole of straight line p. Draw straight line p', parallel to assigned straight line p, through point P. Find its pole $P' \in p$.

According to one of the definitions [22], the polar is a straight line that includes all points, harmoniously conjugated with the pole relative to the points of intersection with the conic. This is satisfied for all straight lines belonging to the pole. Thus, on straight line d, complex ratio of four points $[A' B' PD] = -1$, that is the points are harmoniously conjugated. However, if one of the conjugated points is infinitely distant, the point that is paired to it divides the segment of another pair in half [22]. That is, for straight line $p' \parallel p$, point D is infinitely distant, and point P is the middle of segment AB. However, the center of the conic cross-section belongs to the straight line that runs through the pole and the middle of the chord of the polar of this pole [21].

Arbitrary plane Π_i, belonging to the bundle of planes p, crosses the cone by conic s_i, its chord $A_iB_i = \Pi_i \wedge SAB$. Planes Π_i and SAB belong to the bundles of planes with parallel axes $p' \parallel p$, that is, these planes are perpendicular to the same plane. All lines of pair-wise intersection of the planes of these bundles will be perpendicular to this plane. That is why $A_iB_i \parallel AB$, and triangles ABS and A_iB_iS are similar. It follows from this that point P_i is the middle of chord A_iB_i of straight line p_i. The straight lines tangent to conic s_i in points A_i and B_i, belong to planes SAP' and SBP', that are tangent to the cone and belong to plane Π_i. These three planes intersect in one point P', which is the pole for chord A_iB_i. Straight line $P'P_i$ goes through the middle of chord A_iB_i, and the center of curve s_i lies on it That is, the center of curve s_i belongs to plane SPP'. This is true for any plane Π_i.

Property 1. The centers of all cross-sections of the quadric come are a bundle of planes belonging to a plane, incident to: the vertex of the cone; the pole of the straight line that assigns the axis of cross-section planes and the pole of the straight line that is parallel to the axis of the bundle and passes through the pole of this axis.

Case 2. Straight line p crosses the cone with vertex S. The structure in Fig. 1, *b* corresponds to this case.

Accept that straight line p crosses the cone in points A and B. The planes tangent to cone $P'SA$ and $P'SB$ at these points form a dihedral angle with edge $P'S$. By assigning arbitrary points P_i' on straight line $P'S$, we will separate planes $P_i'AB$. These planes cross the dihedral angle by straight lines $P_i'A$ and $P_i'B$, which will be tangent to the cross-sections.

That is, P_i' with be the poles of chord AB – the common polar of all cross-sections. Designate the middle of chord AB through P. Then the centers of cross-sections will belong to straight lines $P_i'P$, which form plane $P'PS$, like in case 1.

Property 2. The centers of all cross-sections of the quadric cone are a bundle of planes with axis p, which crosses the cone and belong to the same plane. This plane passes through the middle of the chord, which connects the points of intersection of straight line p with the cone, and through the edge the dihedral angle formed by the tangent planes at the point of intersection of straight line p with the cone.

If in both cases we replace straight line p with the straight line that is parallel to it, the position of straight line p' will change, but its direction will not. Due to it, the positions of points P and P' will change, but these two points, along with point S, will later determine the same plane.

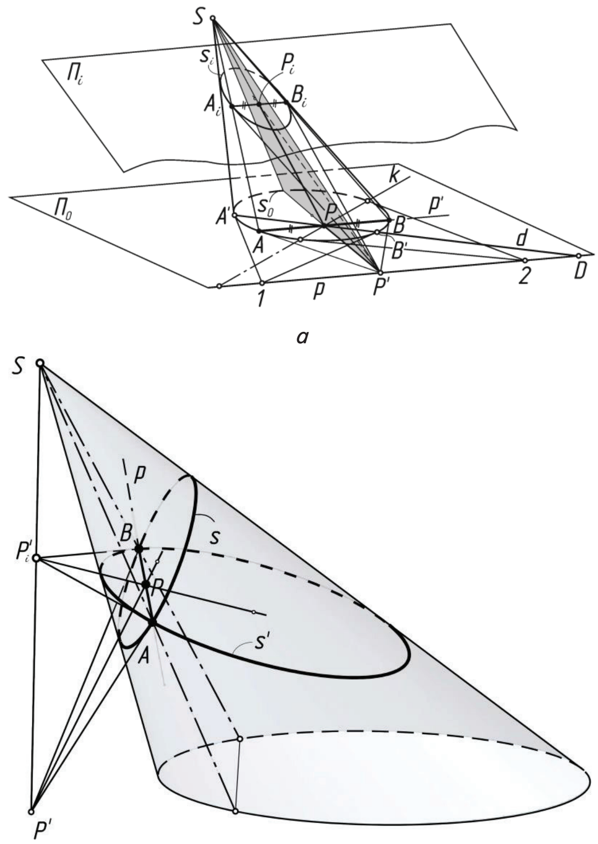


Fig. 1. Property about the centers of conic cross-sections in bundles of planes: *a* — axis of the bundle does not cross the cone; *b* — axis of the bundle crosses the cone

Generalization. We see that both properties have a lot in common and can be combined into one, specifically: all planes that are parallel to the same straight line or belong to it, determine on the assigned cone the cross-sections, the centers of which belong to the same plane. This plane will be called the plane of centers.

In addition, one more generalization follows from these properties: all the cones, the vertices of which belong to one straight line and have a common chord, are intersected by the planes that are incident to this chord or parallel to it, by the conics, the centers of which belong to one plane.

Comments.

1. If the subset of these planes is parallel not only to straight line p and at the same time is parallel to one of the geriatrics of the cone, this straight line will cross the cone by non-central curve – a parabola. In this case, all the planes that are parallel to the same straight line p and to the same geriatrics of the cone, have a common plane, to which belong the diameters of the parabola that are conjugate with chords A_iB_i. The proof of this will not be used in this study and is not given.

2. For the subset of the planes, which are incident to the vertices of the cone and cross the cone by two actual straight lines, the problem becomes degenerated, as the vertex of cone S is the center of all these cross-sections.

5. 2. The variant of construction of the geometric structure of modeling of quadric by tangent cones

In Fig. 2, *a*, two perspective planes of projections (images) Π' and Π'' and two centers of projecting S_1 and S_2, respectively, are assigned. The quadric curves $d_1 \subset \Pi'$ and $d_2 \subset \Pi''$ must be determined, so that along with vertices S_1 and S_2 they should form the cones that have two common tangent planes. Points C_1 and C_2 are the points of intersection of planes Π' and Π'' with the straight line that is assigned by points S_1 and S_2.

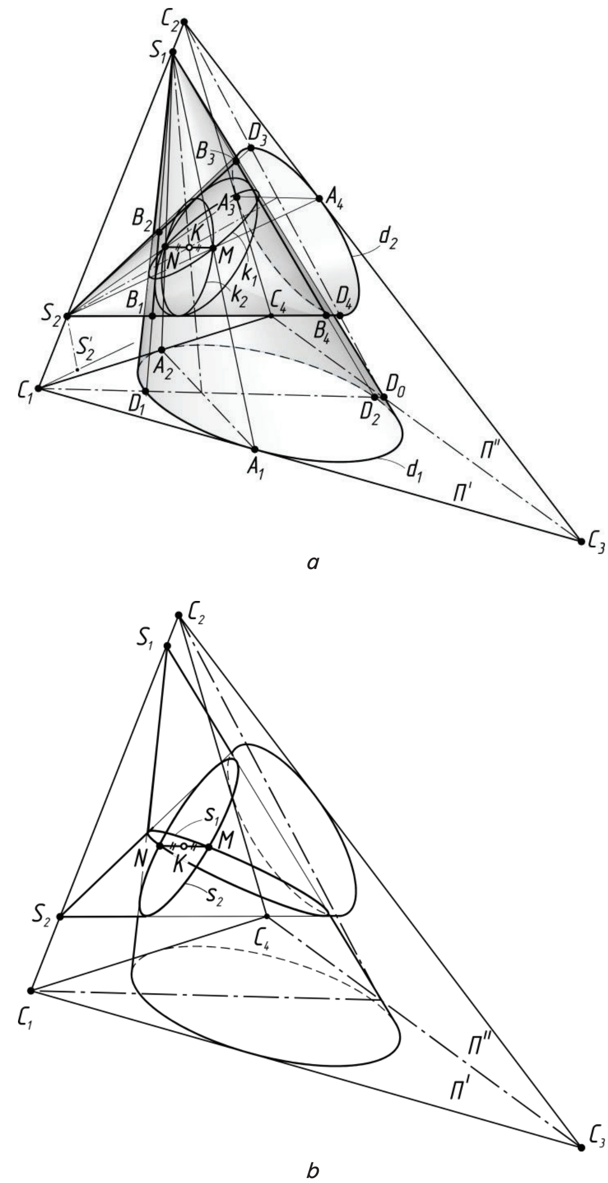


Fig. 2. The variant of construction of the geometric structure: *a* — geometric scheme; *b* — cross-section of quadric cones by flat curves

We will show that this structure has eight arbitrary parameters that can be implemented for modeling the quadric surface. For example, as follows: conic d_1 is constructed in an arbitrary way, it will take five parameters. The tangents to curve d_1, drawn from point C_1 (C_1A_1 and C_1A_2), are prolonged to the cross-section with plane Π″ in points C_3 and C_4. Planes $C_1C_2C_3$ and $C_1C_2C_4$ form the dihedral angle, to which both cones must be tangent. Curve d_2 is constructed on condition of tangency to straight lines C_2C_3 and C_2C_4, and three parameters are realized arbitrarily. Specifically, points A_3 and A_4 are assigned arbitrarily. Then there remains only one free parameter out of eight that is provided by the structure. That is why we will arbitrarily assign point D_3 or D_4, which will determine conic d_2. From that fact that this structure is eight-parameter one, it follows that there will be one-parameter set of inscribed surfaces.

The assigned geometric structure formalizes the problem of the construction of the quadric by two perspective images. Note that cones S_1 and S_2 are the quadric cones, but their axes are not only non-perpendicular to the image, but are unknown at all. It is neither supposed to establish the positions of the axes, nor make any equations.

The geriatrics of the cones, which are drawn through points of tangency A_1, A_2 and A_3, A_4. A_1 intersect with each other at points M and N. The line of intersection of cones will pass through these points. According to the theorem of G. Monge, this line disintegrates into two flat curves (conics s_1 and s_2 (Fig. 2, *b*)).

None of these curves can become the part of the inscribed surface. Indeed, even if there is an inscribed quadric that is incident to one of the lines of intersection of the cones, this line will determine two tangent cones to the quadric with one contact line, which is impossible. That is why one contact line k_1 must pass through straight line MN and cross cone S_1, and another k_2 – through the same straight line MN and cross cone S_2. Then only points M and N will be common points of the curves that determine the sought-for surface.

One tangent cone determines a four-parameter set of inscribed surfaces, if the contact line is not assigned, and a one-parameter set, if the contact line is assigned [21]. At two tangent cones, we have a one-parameter set of inscribed surfaces in the case of non-determined contact lines. When there is one tangent cone, the plane, which assigns the contact line, can be chosen arbitrarily. In the problem with two tangent cones, it will be a one-parameter set of planes of the bundle with axis MN. One contact line is sufficient to determine the surface, inscribed in two cones. However, it is useful to determine the contact line for the second cone in order to solve some structural problems (for example, finding the center of a quadric) and to set the standard algorithms for obtaining cross-sections of the surface by various bundles of planes.

If we assign them independently from each other, we will have a two-parameter set of pairs of curves, which contradicts to parametric analysis.

The next problem is to find out how by assigning one of the curves, for example k_1, which is the line of contact with cone S_1, to obtain curve k_2 – the line of contact with cone S_2. Plane S_1KS_2 will be called the plane of centers. All the cross-sections of cones S_1 and S_2 by the bundle of planes with axis MN will be crossed by it by diameters (property 2). By prolonging the plane of centers to planes Π′ and Π″, we obtain geriatrics of cones S_1D_1, S_2D_2, S_3D_3 and S_4D_4, the cross-section of which forms the rectangle with vertices B_i, $i=1...4$, which limits the zone of the plane of centers, which are inside both cones.

Each of curves k_1^i must be tangent to sides B_1B_2 and B_3B_4 of this rectangle, and each of curves k_2^i must touch sides B_2B_3 and B_1B_4.

Among all the curves, the planes of which pass through straight line MN and cross cones S_1 and S_2, there are lines s_1 and s_2 – the curves of intersection of two cones (according to the theorem by G. Monge). Such curves pass through the common points of the cones, and in the plane of centers, these are only points B_i, $i=1...4$. Curves s_1 and s_2 belong to the bundle of planes with axis MN and in the intersection with the plane of centers, their diameters B_1B_3 and B_2B_4 are identical to point K. That is why point K is the point of intersection of the diagonals of the rectangle with vertices B_i, $i=1...4$.

The plane of centers, like any other plane, should cross the inscribed quadric by the quadric curve, so the task of coordination of curves k_1 and k_2 can be replaced by the problem of coordination of their diameters when constructing the curve inscribed in the quadrangle of the plane of centers.

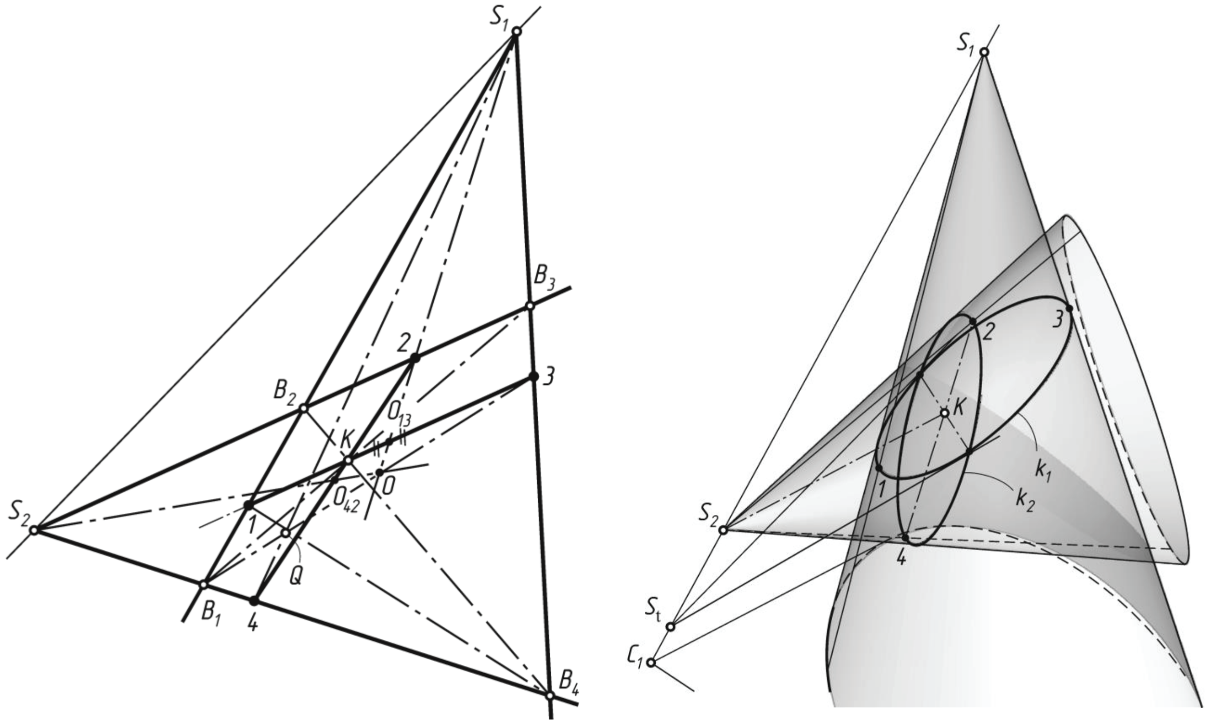
Fig. 3, *a* shows the life-size planes of centers. The sides of quadrangle B_1, B_2, B_3, B_4 are assigned by four parameters to construct the diametric cross-section of all surfaces that are modeled by assigned cones. The fifth parameter can be chosen arbitrarily. One of the most fruitful options is to assign the point of tangency of the curve to one of the sides. Let it be point 1 on tangent B_1B_2. By the Brianchon theorem «on conics inscribed in a quadrangle» [22], the point of contact with the opposite side be found on the straight line passing through point 1 and point K of intersection of the diagonals of the quadrangle. This assigns straight line 13, which being transferred to the space, along with straight line MK directly assigns the line of contact with cone S_1 (curve k_1 in Fig. 3, *b*).

In order to find the line of contact of cone S_2, consider triangle $B_1B_4S_1$ and accept points 1 and 3 as points of tangency to its two sides (Fig. 3, *a*). Then by the Brianchon theorem «on conics inscribed in a triangle» [22], the straight lines, connecting the points of tangency of the curve to the sides of the triangle with opposite vertices of the quadrangle, intersect at one point. This will be point Q, which is the point of intersection of straight lines $1B_4$ and $3B_1$. The straight line that connects the vertex of cone S_1 with the point of tangency of inscribed curve to straight line B_1B_4 must pass through this point. This is point 4, which determines point 2 (the point of tangency to side B_2B_3) in intersection with straight line k_4. Straight line 24 along with straight line MN in space determine the plane of curve k_2, which belongs to cone S_2 (Fig. 2, 3, *b*). Thus, straight line 13 completely determines the quadric that is tangent to both cones.

Come back to Fig. 3, *a*. Straight lines 13 and 24 are the diameters of the contact lines of the quadric surface with cones S_1 and S_2, respectively. It follows from this that their middles – points O_{42} and O_{13} – are the centers of contact lines.

It is known that the vertex of the tangent cone, the center of the contact line [21] and the center of the inscribed surface belong to one straight line – this is straight line S_1O_{13} for cone S_1, and straight line S_2O_{42} for cone S_2. These straight lines belong to the same plane of centers and in the cross-section determine one center of the surface – point O, modeled by two tangent cones. At a change of curve k_1 (Fig. 3, *b*), the position of the center will change, but in any case, the axes will belong to the plane of centers. That is why the following property is true:

Property 3. The centers of quadrics, inscribed in two cones, always belong to the plane that passes through the vertices of the cones and the point that is the middle of the segment, which connects double points of intersection of the cones.



a *b*

Fig. 3. Full-scale image of the plane of centers and modeling cross-sections of inscribed surface: *a* – image of the full-scale of the plane of centers; *b* – contact line of inscribed surface

6. Possible geometric schemes for the construction of quadric surfaces, the determinant of which includes two tangent cones

Consider possible schemes of construction of the quadric surface. Any bundle of planes with its own or not own axis should cross the surface by the quadric curves. Four bundles of planes, which have the following axes are natural for this structure:

– first bundle of planes with axis MN;

– second bundle of planes with axis S_1S_2;

– two following ones with the axis parallel to straight line MN, which passes through the axis of one or another cone.

The bundles of planes will not be considered separately, they will be generalized.

As an example, we explore bundle of planes Δ_i with axis S_1S_2 (Fig. 4). The limiting planes of this bundle are the planes of the dihedral angle with edge S_1S_2, to which points M and N of any modeled plane belong. The plane of centers also belongs to this bundle. The current plane of the bundle passes through the i-th point of straight line MN. The i-th plane intersects plane Π' along straight line C_1T_i, and plane Π'' along straight line C_2T_i. These straight lines, in turn, intersect curve d_1 and d_2 at points: T^i_{12}, T^i_{34}, T^i_{14}, T^i_{23}. Joining these points with vertices of the cones, to which these points belong. at the intersection of generatrix, we will obtain points B^i_1, B^i_2, B^i_3, B^i_4 – the quadrangle, the vertices of which will belong to curves s_1 and s_2, which follows from the construction of the points themselves. For subsequent solution of the problem, it is necessary to construct the curve, inscribed in this quadrangle.

To do this, it is necessary to have at least one point of tangency, which we will obtain at intersection of curves k_1 and k_2 by plane Δ_i. But indeed, all points of intersection will be the points of current cross-section of the quadrangle with vertices B^i_1, B^i_2, B^i_3, B^i_4. That is plane Δ_i with intercept curves k_1 and k_2 (Fig. 4) in the points that belong to the sides of quadrangles B^i_j, where $i=1...4$, $j=1...4$. These points were coordinated at the previous stage by joint plotting of curves k_1 and k_2.

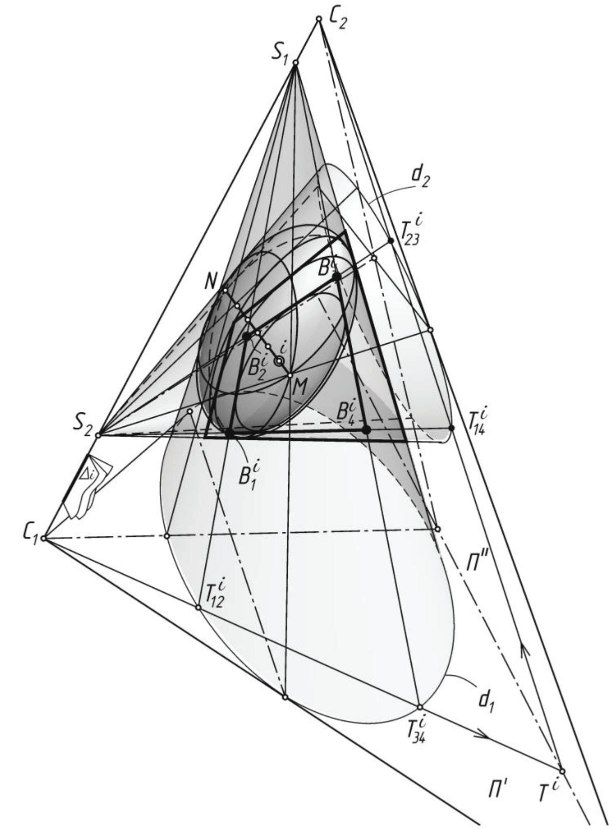


Fig. 4. Scheme of construction of a quadric with the help of the bundle of planes with axis S_1S_2

Knowledge about one of the points of tangency is sufficient for the computer plotting of the quadric curve. Modeling was carried out in the SolidWorks system, but one can use any other system that has similar means of operation with conics and quadrics.

Bundles of planes with axes, parallel to line *MN* and incident to vertices of cones, are shown in Fig. 5. To do this, the structural scheme in Fig. 2, *a* will be replaced with the other scheme, reduced to the hexagon (Fig. 5, *a*).

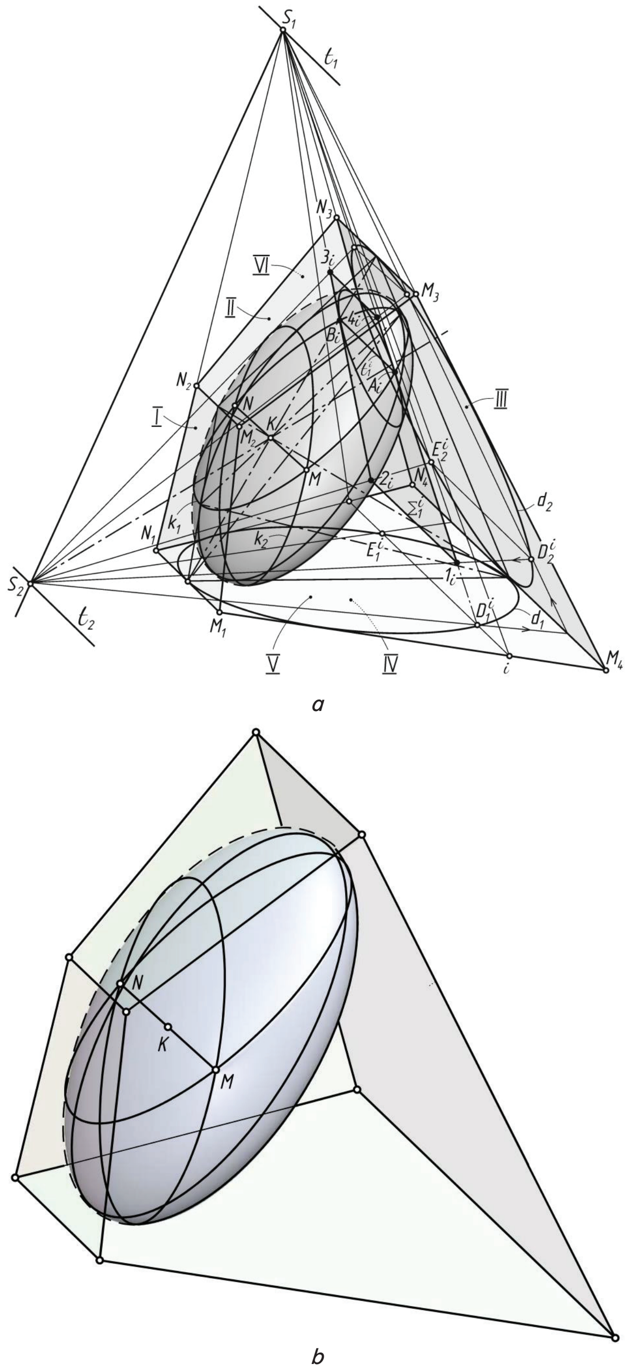


Fig. 5. A hexagon, which limits the quadric surface, assigned by two tangent cones: *a* — geometric scheme of construction of quadric; *b* — images of the quadric that is tangent to the hexagon faces

The hexagon faces will be two faces tangent to cone S_1 at points D_1 and D_2, two faces tangent to cone S_2 in points D_3 and D_4, and two faces of the dihedral angle, simultaneously tangent to both cones. It follows from property 1 that in this case, four side edges of the hexagon will be parallel to *MN*. In Fig. 5, such edges are designated as M_iN_i, where $i=1...4$. In Fig. 5, four faces that are incident to straight lines M_iN_i will be designated by Roman numerals: I, II, III, IV.

The faces, designated in Fig. 5 as V and VI, are actually common tangent planes to two cones and incident both to vertex S_1, and to vertex of cone S_2. Thus, the hexagon is formed as the intersection of two pyramids with vertices at points S_1 and S_2. As a result of this cross-section, we obtain a tetrahedral prism with the edges that are parallel to *MN*.

Curves k_1 and k_2, belonging to the eponymous cones, together give six points of tangency to the faces of assigned hexagonal of the inscribed quadric. This hexagon compared to the arbitrary one has additional properties, specifically: the straight lines, connecting the points of intersection of the diagonals on the opposite sides intersect at one point, and this point will be the middle of segment *MN*, i. e. point *K*.

Indeed, the point of intersection of diagonals is the object of projective geometry. That is why the points of intersection of the diagonals of all arbitrary cross-sections of this pyramid, which do not belong to the vertex, will be incident to the straight line that connects the vertex of the pyramid with the point of intersection of the diagonals of one of the cross-sections.

For pyramid S_1, among them there will be not only diagonals of planes II and IV, but also of diagonal planes $\{M_1N_2N_3M_4\}$, $\{N_3M_3M_2N_1\}$, $\{M_1M_2N_3N_4\}$, $\{M_4M_3N_2N_1\}$, $\{N_1N_3M_3M_1\}$, $\{M_4M_2N_2N_4\}$. But all these planes are the cross-section of not only pyramid S_1, but also of pyramid S_2 and the cut prism that is incident to edges M_iN_i, at $i=1...4$. That is why such planes intersect at one point, which belongs to straight line *MN*, and is the middle of this segment, according to the Gauss theorem [23].

This model can be also applied to construct a net-like frame of the quadric. To construct this frame, it is convenient to use two single-type bundles of planes, specifically, those that pass through straight lines t_1 and t_2, parallel to *MN*, which are incident to the vertices of cones S_1 and S_2. To separate a specific cross-section plane from bundle $\{t_1^i\}$, choose the current point on segment M_1M_4. To separate points from the bundle, choose the point on segment M_1N_4.

Fig. 5 shows as an example the rule for the construction of cross-section in transversal plane Σ_1^i, which passes through straight line t_1 and point *i* on straight line M_1M_4. As a result of performed constructions, we will obtain quadrangle with vertices $1_i2_i3_i4_i$, which are coordinated with the help of points D_1^i and D_2^i on curves d_1 and d_2, respectively, and line M_4N_4. Intersection of plane Σ_1^i with curve k_1 gives points A_i and B_i with S_1A_i та S_1B_i that are tangent to them. The fifth parameter for the construction of current cross-section t_1^i will be obtained as the intersection of plane Σ_1^i with the conic in the plane of centers. It is enough to construct the current cross-section of quadric. Based on this algorithm, we construct the one-parameter set of the conics of the modeled surface, specifically, the one-parameter set of the curves, which belong to the bundle of planes Σ_1^i. Similarly, we construct the one-parametric set of the curves that belong to bundle of planes Σ_2^i.

In the process of the presentation of material, we modeled the surfaces (Fig. 4, 5), which can be regarded as an example of modeling a quadric by two perspective images. Here are some examples of the construction of conjugation of quadric surfaces with the help of tangent cones.

7. Examples of the implementation of results of the study in the problems on conjugating quadrics

The problem of modeling the conjugation of quadric surfaces by flat curves can be split into two variants. The first one includes the situation when we have the assigned quadric and accept it as the basic surface. The second variant involves modeling both the basic surface and the one conjugate with it. The first variant is much easier because we have a contact line and a tangent cone. According to the second variant, we have the same algorithm that was introduced for modeling a quadric by two perspective images, that is, we have two tangent cones, which must be coordinated.

Here is an example when a basic surface is assigned, and it is possible to determine one or more contact lines on it.

In Fig. 6, *a*, we have a one-sheet rotation hyperboloid with the known surface center, point *G*. If we assign an arbitrary cross-section of this hyperboloid, specifically, conic k_1 with center *O*, the vertex of the cone, tangent to a one-sheet hyperboloid will belong to the line that connects center *O* and surface center *G*. To determine the vertices of cone *S*, we construct tangency plane Σ_i at arbitrary point A_i on contact line k_1. To determine the vertex of cone *S,* we construct at an arbitrary point A_i on contact line k_1 plane of tangency Σ_i to the one-sheet hyperboloid, which in the intersection with axis *GO* will determine the vertex of the tangency cone (point *S*). To construct the modeled surface, it is possible to select arbitrary point *B*, best of all, on axis *GO* (Fig. 6, *a*). A change of the position of point *B* makes it possible to obtain the surfaces that are different by shape. An example of one of them is presented in Fig. 6, *b*.

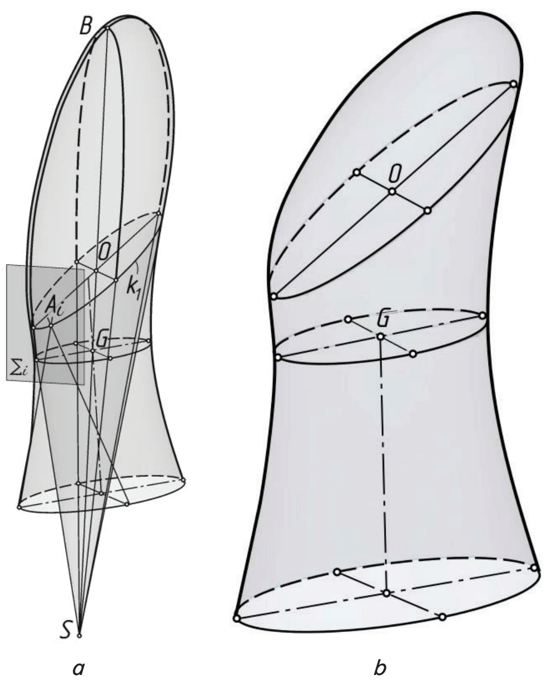


Fig. 6. Example of a change in the shape of conjugate quadrics: *a* — geometric scheme; *b* — example of the changed shape

Below, Fig. 7, 8 show the construction of two more conjugate quadrics. In the first example (Fig. 7, *a),* the basic surface is assigned, specifically, a one-sheet hyperboloid of the general form, which has two cross-sections k_1 and k_2, which are independent on each other, which will be accepted as contact lines. The axis of the figure is vertical. Point *G* is the center of the surface.

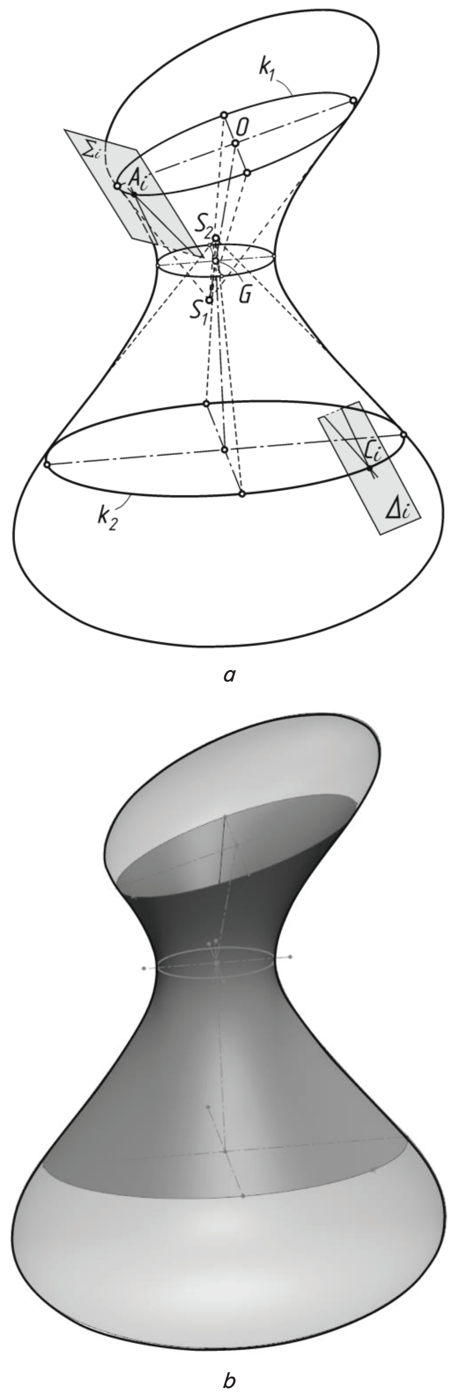
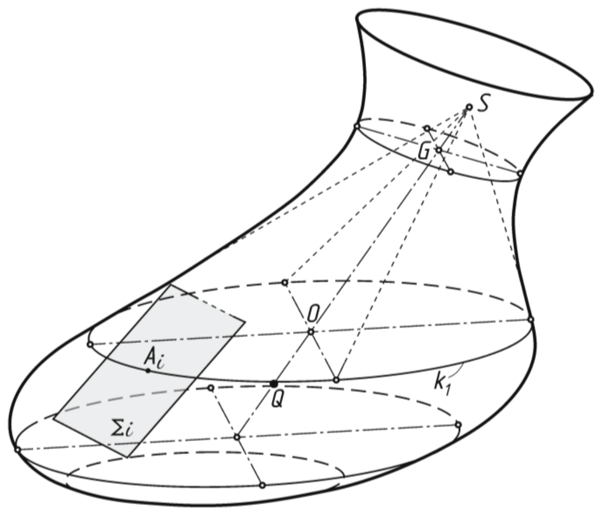


Fig. 7. Modeling the surfaces from three conjugate quadrics: *a* — geometric scheme; *b* — appearance

Like in the previous example, we assign tangent plane Σ_i at arbitrary point $A_i \in k_1$. This plane at the intersection with line *GO* (*O* is the center of conic k_1) determines vertex S_1 of the cone that is tangent to the one-sheet hyperboloid. To determine vertex S_2 of the cone that is tangent to contact line k_2, we assign plane Δ_i that is tangent to the one-sheet hyperboloid. The view of the shape is shown in Fig. 7, *b*. The positions of the points controlling the form are not shown.



a

b

Fig. 8. The surface of the vase from conjugate quadrics:
a — geometric scheme; *b* — appearance

The third example (Fig. 8) shows modeling of a vase. The axis of the figure is not vertical. Geometrical scheme of modeling the ellipsoid, conjugate with a one-sheet hyperboloid, and the appearance of the vase are shown in Fig. 8, *a*, *b*, respectively. The center of the ellipsoid, which must belong to straight line *OS* should be selected as the parameter of controlling the shape of the ellipsoid surface (the bottom part of the vase). It is point *Q* in Fig. 8, *a*.

All the examples, described in the article, are performed using the SolidWorks system. The obtained theoretical calculations and the examples of their application show the effectiveness of the algorithms and give grounds to argue that the use of the presented methods and design algorithms can enhance the capabilities of computer systems when using them in the work of designers in the process of creating actual objects.

8. Discussion of results of the conducted study on the possibility of constructing quadrics based on tangent cones and the prospects of their further development

The studies related to modeling of the quadrics, the determinant of which includes tangent cones, were performed by the paradigm of using the design methods for creating algorithms.

As a part of the described study, we established a series of geometric properties, useful for modeling of quadrics by contour lines on their perspective images. Specifically, it was established that by two quadric cones that have two common tangent planes, it is possible to construct the one-parameter set of quadrics that are tangent to these cones. The scheme (Fig. 4, 5), according to which a specific surface and the algorithms of the construction of the linear and net-like frames of the modeled surface are separated from this set.

In the generalized variant, the derived geometrical properties are as follows:

– if a quadric cone and a straight line of the arbitrary position are assigned, all the planes belonging to or parallel to this straight line determine conical cross-sections, the centers of which are incident to one plane. This plane can be considered the plane of centers;

– the centers of all the quadrics, inscribed in two cones, always belong to the same plane. This plane passes through the vertices of the cones and the point that is the middle of the segment, which connects the double points of intersection of the cones;

– the assigned contact line on one of the cones unambiguously determines the line of contact on the second cone and the sought-for surface, in general.

The studies and the algorithms relate to the case where the straight line connecting the vertices of the tangent cones is located in a subspace that is external to the cones. This restriction makes it possible to use the results of the study to solve problems in simpler cases of modeling the surfaces. The research and construction of the algorithms to solve the problem when a straight line that connects the vertices of the tangent cones passes in internal hollow of the cones is the subject of subsequent consideration.

The prospect of development is also the task of the construction of surfaces by three tangent cones. However, the most interesting thing, in terms of subsequent research prospects, is to reduce the algorithms to the variant of constructing a discrete conjugate network, which belongs to the quadric. This will be consistent with the modern approach in the new theoretical branch of development of discrete differential geometry. The first steps in this direction were made in the construction of the hexagon, to all the faces of which the modeled quadric should be tangent. The final development and implementation of this algorithm will allow constructing the surface by the conjugate network by the assigned tangent cones.

Both problems are essential within computer graphics.

8. Conclusions

1. Parametric analysis of the problem of modeling the quadric, the determinant of which included tangent cones, was carried out. In the framework of this problem, we carried out the theoretical research, the implementation of which makes it possible to construct the quadric surface according to their contour lines on the perspective images. These studies created the theoretical basis for constructing the objects of design from conjugate quadrics.

2. The method, by which, assigning the contact line on one cone, there is a possibility to find the contact line on the second cone and to find the center of the inscribed in these two cones of the quadric. An alternative method of modeling

the described surfaces was proposed. By this method, the cross-sections of all quadrics, tangent to two cones, are inscribed in quadrangles, the vertices of which belong to the lines of intersection of the assigned cones. Based on the geometric research, the fundamentally new algorithms were developed with the use of design methods of geometry for computer implementation of the problems of modeling objects by contour lines on their perspective images.

3. The capacity of the created and implemented algorithms for the construction of linear and net-like frames of quadrics is proved by the examples of the modeled design objects. The applied aspect of using the obtained scientific result is the expansion of possibilities of the existing computer systems in the process of their application for the construction of quadrics by the assigned tangency conditions.

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