

На основі методів симетрії і факторизації отримано загальне аналітичне рішення диференціального рівняння четвертого порядку для задачі про вільні вісесиметричні коливання кругової пластинки зі змінною товщиною. Законом зміни товщини є увігнута парабола $h=H_0(1-\mu r)^2$, де μ – постійний коефіцієнт, який визначає ступінь угнутої пластинки. Рішення представлено у функціях Бесселя нульового та першого порядку від дійсного і уявного аргументу. Розглянута кругова кільцева пластинка з жорстким закріпленням внутрішнього контуру і вільним зовнішнім краєм при трьох значеннях коефіцієнта μ . Визначено перші три власні значення задачі (частотні числа) і власні функції (форми коливань). Показано, що власні частоти перших трьох форм коливань зі зростанням угнутої (збільшенням μ) знижуються в різній мірі, яка визначається номером частотного числа λ_i ($i=1, 2, 3$). При $\mu=1,21417$ і $\mu=1,39127$ частоти в порівнянні з випадком $\mu=0,5985$ знижуються відповідно на (1; 1,3) %, (17,6; 24) %, (22,85; 30, 35) % для $\lambda_1, \lambda_2, \lambda_3$. Видно, що наявне істотне падіння частоти на вищих формах коливань (λ_2, λ_3) і незначне при основній формі (λ_1). Встановлено значення і координати екстремальних прогинів (пучностей коливань) і орієнтовні координати вузлових перерізів. Наведені числові параметри поряд з частотними показниками є засобом для ідентифікації коливальних властивостей пластинки при її вивченні на практиці. Побудовано графічні залежності для радіальних σ_r і тангенціальних σ_θ циклічних напружень при основній формі для кожного з трьох варіантів угнутої параболічної пластинки. Встановлено, що збільшення відношення крайових товщин, тобто увігнутості, призводить до підвищення σ_r в перерізах за межами закріплення. Ці напруження, що діють віддалено від вільного краю, наприклад, в закріпленні або в районі дії максимальних σ_θ в різній мірі перевершують σ_θ . Через це ці напруження представляють собою основну небезпеку з точки зору циклічної міцності пластинки при досягненні σ_r руйнівних значень. Відзначено можливість за рахунок підвищення увігнутості параболічної пластинки забезпечити оптимальне співвідношення між значеннями σ_r в закріпленні і σ_θ , які діють за межами від закріплення. Це співвідношення, яке дорівнює приблизно 1, забезпечується у випадку $\mu=1,39127$, що розглянуто в роботі

Ключові слова: кільцева пластинка, змінна товщина, метод симетрій, власне число, напружено-деформований стан

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ANALYSIS OF FREE OSCILLATIONS OF CIRCULAR PLATES WITH VARIABLE THICKNESS BASED ON THE SYMMETRY METHOD

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1. Introduction

This paper reports the implementation of the symmetry method in studying the free oscillations of the continuous ring or circular plates of variable thickness. Circular plates are widely used in many technical applications in engineering and construction [1, 2]. Plate-type elements of the membrane type have been used for medical purposes [3]. The plates are employed in high-speed rotational engineering systems as one of the main structural elements [4]. Typically, these elements are subject to destructive stresses caused by resonant oscillations during operation. Free oscillation analysis is necessary to enhance the technical resource of a structure as a whole by adjusting the resonance, which is possible if the natural frequencies are known. At the same time, there is an issue of finding a closed solution to the problems on the free oscillations of plates, addressed by a large body of research [5–7]. The above problems remain unresolved in many cases, despite their practical and theoretical significance. The relevance of the current work stems from the practice-dictated requests to provide for the desired opera-

tional resource of plate elements or nodes, which is impossible without a clear understanding of their cyclical stressed and strained state. This requirement could be primarily met on the basis of a theoretical analysis of oscillations.

2. Literature review and problem statement

The results of precise studies into the plates of variable thickness are very limited primarily due to the known mathematical difficulties arising from the analytical investigation of their oscillations.

Paper [8] shows that it is possible, under a free oscillation mode, based on the approximate Relay-Ritz method and Chebyshev's polynomial functions, to solve the problem on natural values. According to the proposed approach, the displacement coefficients are determined as a simplification tool for solving the problem. The paper reports the results of studying the oscillations of a plate with a linear-variable thickness on the elastic base. The specified methods, however, do not make it possible to derive an analytical solution to

the problem on the oscillations of a plate with the parabolic law of thickness change. It could be argued that the reason for this is the mathematical difficulties that arise when trying to use the ratios given in the work for this purpose.

Another approach to solving a problem on natural values is considered in paper [9]. In this case, an independent conjugal coordinate method is recommended to analyze the free oscillations of a plate. However, in fact, it is unclear how the specified approach could be extended to the problem involving a plate with a solid profile, different from linear.

The free oscillations of a plate of constant thickness are studied in [10, 11]. These works employed a differential method of transforming equations to derive the solutions to the problem of natural values. The authors do not consider the algorithm for finding the natural frequencies of a plate for the highest shapes of oscillations. In addition, work [10] graphically presents only the diagrams of deflections while the diagrams of cyclical stresses are lacking, which may raise doubts about the suitability of the proposed approach for practice. A circular plate was examined in paper [11] employing the Runge-Kutta method. The resulting ratios and the approach itself are difficult to apply to solve the problem on free oscillations of a variable-thickness plate.

Although article [12] provides a solution to the problem about a plate under a mode of axisymmetric free oscillations (the frequency numbers were defined), it does not give the frequency equation in an analytical form. To study the plates, the authors suggest using additional variable coefficients, which they believe are designed to improve the classical Kirchhoff plate theory. As a result, this leads to a significant complication of the problem.

The issue of fixing a plate under conditions of repeatedly-variable operational loads deserves special attention. This is explained by the fact that the classical statement of the problem on natural values requires mandatory accounting of boundary conditions. Investigating such a situation is addressed, for example, in work [13], where a finite element method was used to analyze free oscillations of a rigidly-clamped plate.

The issues relating to the construction of analytical solutions to the fourth-order oscillation equation with variable coefficients remain unresolved. Part of the solution to this problem is given in article [14], where a similar equation was studied on the basis of approximate approaches – the classical method by Galerkin and the energy one. The statement of the problem set out in the cited article does not make it possible to use the results to consider plates with a parabolic profile.

Prospective practical application of variable-thickness plates is addressed in works [15, 16] that show the importance of solving a problem on natural values when using plates under the conditions of cyclical air loads in aircraft. There remained, however, the issues related to the search for the natural frequencies of a plate under conditions of real loads; no regions of the location of nodes and antinodes of oscillations were specified. The absence of such information could lead to uncontrollable deformations of the plate.

All this suggests that it is appropriate to conduct a study on solving the problem on the oscillations of a plate of variable thickness with varying degrees of concaveness. The above literary sources employ a mathematical apparatus based only on the approximate numerical methods and approaches. The accuracy and reliability of the results, in this case, could be highly questionable. Analytical approaches, as opposed to approximate or numerical, make it possible to expand the existing estimated base of accurate solutions for

the plates subject to oscillations, thereby supplementing it with new results obtained in the final form.

3. The aim and objectives of the study

The aim of this study is to examine the oscillations and to analyze the stressed-strained state of a circular ring plate whose thickness changes in line with the law of concave parabola $h=H_0(1-\mu\rho)^2$ with varying degrees of concaveness, determined by the values of constant μ . In practice, this would make it possible to map out the optimal design of plate-type structural elements of variable thickness according to the criteria of mass-size, resonance frequencies, movements (shapes of oscillations) and cyclical stresses by choosing appropriate values for μ .

To accomplish the aim, the following tasks have been set:

- to build a general analytical solution to the fourth-order differential equation for the problem about a cyclical symmetrical bend of the circular plate of a parabolic profile with varying degrees of concaveness;
- to derive, based on a general solution, the formulae for calculating cyclical stresses in a plate;
- to determine, for a circular plate with a rigidly trimmed internal contour and a free one on the outside, by using the resulting solution, the first three natural values for the problem for each of the three cases of concaveness and to build, based on them, the natural functions (shapes of the plate's natural oscillations);
- to construct, as an example, the diagrams of radial and tangential stresses at the basic shape of oscillations.

4. The original differential equation and its general solution for a plate with a thickness of $h=H_0(1-\mu\rho)^2$

For a plate that abides the law of changing the thickness $h=H_0(1-\mu\rho)^2$, the equation of the shapes of natural oscillations takes the form [17]

$$LLW - (\lambda^4 + 4\mu^4)W = 0, \quad (1)$$

where

$$L = (1-\mu\rho)^2 \frac{d^2}{d\rho^2} + (P) \frac{d}{d\rho} - 2\mu^2;$$

$$P = \frac{[\rho(1-\mu\rho)^4]'}{\rho(1-\mu\rho)^2};$$

$W = W(\rho)$ – displacements (deflections) of a plate;

μ – arbitrary constant;

$\rho = r/R$ – relative radius;

r – variable radius;

R – constant radius;

$$\lambda^2 = 2\pi f \frac{R^2}{H_0} \sqrt{\frac{12(1-\nu^2)\gamma}{gE}}; \quad (2)$$

f is the frequency of natural oscillations; H_0 is the thickness in the center of a plate; ν, γ, E are the Poisson coefficient, the specific weight, the elasticity module of the plate materi-

al, respectively; g is the acceleration of gravity. Hereafter, the ν coefficient is accepted equal to $1/3$, which is true for most structural metallic materials.

The order IV equation in notation (1) makes it possible, based on a factorization method, to replace it with two order II equations

$$(1-\mu\rho)^2 W'' + \left(\frac{[\rho(1-\mu\rho)^4]'}{\rho(1-\mu\rho)^2} \right) W' - 2\mu^2 W \pm \sqrt{\lambda^4 + 4\mu^4} W = 0, \tag{3}$$

and then the solution to equation (1) could be defined as the sum of the general solutions to these two equations $W=W_1+W_2$, where W_1 is the solution to equation (3) at the plus sign before the root, and W_2 – at the minus sign.

Using a variable replacement

$$\rho = (1 - e^{-x}) / \mu,$$

the differential equation (3) could be rewritten as

$$W_{xx} + 2 \frac{D_x}{D} W_x + k^2 W = 0, \tag{4}$$

where

$$W_{xx} = W''; \quad D = D_0 (e^{-3x} - e^{-4x})^{1/2};$$

$$W_x = W'; \quad k^2 = -2 \pm \sqrt{\left(\frac{\lambda}{\mu}\right)^4 + 4}. \tag{5}$$

The equation (4) is similar in structure to the equation of the shapes of longitudinal oscillations of a rod with a variable cross-section, with a diameter of $D(x)$, so the solution to it could be derived by using the symmetry method. Since D_x/D is generally independent of the D_0 coefficient from ratios (5), this coefficient could be thought to be arbitrary, not affecting the outcome of the general solution. Note that the natural boundaries of the variable $\rho=0 \div 1$ for equations (3) are matched with the boundaries of the variable $x \geq 0$ for equations (4) at any $\rho \geq 0$ that do not exceed the marked limits of ρ . To derive a final solution to the problem, it is required that the function $W=W_1+W_2$ should meet the boundary conditions at x_1 and x_2 .

The equation (4) at D set by law (5) is not solvable in elementary or known special functions. However, its solution is easy to find by approximating $D(x)$ with some function $D_1(x)$, in which the solution could be defined in a closed form. As a result, the issue of the accuracy of solving the problem is transferred to the successful choice of the law $D_1(x)$ in the sense of its satisfactory approximation to $D(x)$ over the required interval ($x_1 \div x_2$). The symmetry method allows this choice to be implemented because, according to this method, the expression for $D_1(x)$ may contain a series of uncertain coefficients, the selection of which enables the required approximation.

The author of article [17] derived an approximation function $D_1(x)$ for a given problem, based on the symmetry method, in the following form

$$D_1 = 0.21 \frac{\sqrt{x}}{x^2 + 0.2483}. \tag{6}$$

It follows from the analysis of graphic dependences given in the work that solving a problem based on (6) yields acceptable results for technical applications. Given this, we shall for further analysis use the calculations that were earlier obtained by the author in article [17] to compute natural frequencies and to build the shapes of oscillations of a thin plate. In this case, to construct the first three shapes of oscillations, we shall employ the function of deflections, expressed through the variable x according to the dependence $x = -\ln(1-\mu\rho)$, recorded in a slightly modified form:

$$W_i = W_{1i} + W_{2i} = B_1 (x^2 + C_0) \left\{ \frac{A}{B_1} M + \frac{B}{B_1} N + \frac{A_1}{B_1} T - U \right\}. \tag{7}$$

where

$$M = \alpha_i J_0(\alpha_i x) - \frac{2x}{x^2 + C_0} J_1(\alpha_i x);$$

$$N = \alpha_i Y_0(\alpha_i x) - \frac{2x}{x^2 + C_0} Y_1(\alpha_i x);$$

$$T = \beta_i I_0(\beta_i x) - \frac{2x}{x^2 + C_0} I_1(\beta_i x);$$

$$U = \beta_i K_0(\beta_i x) - \frac{2x}{x^2 + C_0} K_1(\beta_i x).$$

Here, at known coefficients $A/B_1, B/B_1, A_1/B_1$, the B_1 multiplier is freely selected and is used in this case to normalize W_i , so that $W_i(\rho=\rho_2) = W_i(x=x_2) = 1$.

5. Formulae for calculating cyclical stresses in a plate

To analyze the stressed-strained state of a plate, we shall use the expressions, known from a theory of plates, to determine the maximally thick radial σ_r and tangential (circular) σ_θ normal stresses [18].

$$\sigma_r = -\frac{6D}{h^2} \left(W_{rr} + \frac{\nu}{r} W_r \right); \quad \sigma_\theta = -\frac{6D}{h^2} \left(\nu W_{rr} + \frac{1}{r} W_r \right), \tag{8}$$

where W_r, W_{rr} are the derivatives from radius r ;

$$D = Eh^3 / 12(1-\nu^2)$$

is the cylindrical rigidity.

Following the transition to the relative variable $\rho=r/R$ and onwards, to the variable $x = -\ln(1-\mu\rho)$, and after the introduction of $\nu=1/3$ to (8), we obtain

$$\sigma_r = \sigma_0 (W_{xx} + p W_x); \quad \sigma_\theta = \frac{\sigma_0}{3} (W_{xx} + q W_x), \tag{9}$$

where

$$\sigma_0 = \frac{9EH_0\mu^2}{16R^2};$$

$$p = 1 + \frac{1}{3(e^x - 1)}; \quad q = 1 + \frac{3}{e^x - 1}.$$

The minus sign is omitted at σ_0 due to the cyclicality of the stresses σ_r and σ_θ . Given that $W=W_1+W_2$; $k_1^2 = \alpha^2$; $k_2^2 = -\beta^2$, we find

$$\begin{cases} W_{xx} + pW_x = M(x)W' + \beta^2W_2 - \alpha^2W_1; \\ W_{xx} + qW_x = N(x)W' + \beta^2W_2 - \alpha^2W_1, \end{cases} \quad (10)$$

where

$$M(x) = p - \frac{F'}{F} = 1 + \frac{1}{3(e^x - 1)} + \frac{3x^2 - C_0}{x(x^2 + C_0)};$$

$$N(x) = q - \frac{F'}{F} = 1 + \frac{3}{e^x - 1} + \frac{3x^2 - C_0}{x(x^2 + C_0)}.$$

Substitution of the obtained expressions W_1' , W_2' from article [17], ratios (10) in formulae (9) leads to the desired dependences for cyclical stresses:

$$\begin{cases} \sigma_r = \sigma_0 B_1 (x^2 + C_0) \cdot (p - q + n + s); \\ \sigma_\theta = \frac{\sigma_0}{3} B_1 (x^2 + C_0) \cdot (t - u + v + z), \end{cases} \quad (11)$$

where

$$p = -\alpha^2 \frac{A}{B_1} \left[\alpha J_0 + \left(M - \frac{2x}{x^2 + C_0} \right) J_1 \right];$$

$$q = \alpha^2 \frac{B}{B_1} \left[\alpha Y_0 + \left(M - \frac{2x}{x^2 + C_0} \right) Y_1 \right];$$

$$n = \beta^2 \frac{A_1}{B_1} \left[\beta I_0 + \left(M - \frac{2x}{x^2 + C_0} \right) I_1 \right];$$

$$s = \beta^2 \left[-\beta K_0 + \left(M - \frac{2x}{x^2 + C_0} \right) K_1 \right];$$

$$t = -\alpha^2 \frac{A}{B_1} \left[\alpha J_0 + \left(N - \frac{2x}{x^2 + C_0} \right) J_1 \right];$$

$$u = \alpha^2 \frac{B}{B_1} \left[\alpha Y_0 + \left(N - \frac{2x}{x^2 + C_0} \right) Y_1 \right];$$

$$v = \beta^2 \frac{A_1}{B_1} \left[\beta I_0 + \left(N - \frac{2x}{x^2 + C_0} \right) I_1 \right];$$

$$z = \beta^2 \left[-\beta K_0 + \left(N - \frac{2x}{x^2 + C_0} \right) K_1 \right].$$

The Bessel's functions $[J(\alpha x), Y(\alpha x), I(\beta x), K(\beta x)]$ are denoted $[J, Y, I, K]$ for convenience.

6. Determining the natural frequencies, shapes of oscillations, and cyclical stresses

For the intended use of circular plates of variable thickness in technical devices, in accordance with the objectives of the current study, three options of thickness $h = H_0(1 - \mu\rho)^2$ were selected. The ratios of the limiting thicknesses $h(\rho_1)/h(\rho_2)$ at $\rho = \rho_1 = 0.1$ and $\rho = \rho_2 = 0.5$ are assigned as 1.8; 5; 8. The graphic illustrations (Fig. 1) show the plates with these parameters under numbers 1; 2; 3, which corresponds to the coefficient values $\mu_i = 0.5985$; 1.21417; 1.39127 ($i = 1, 2, 3$), respectively. Based on ratios (7) and boundary conditions for a circular plate with rigid fastening at $\rho = \rho_1 = 0.1$ and free at $\rho = \rho_2 = 0.5$, the following results were obtained.

We determined the frequency numbers λ_i as the solutions to the corresponding frequency equation (Table 1).

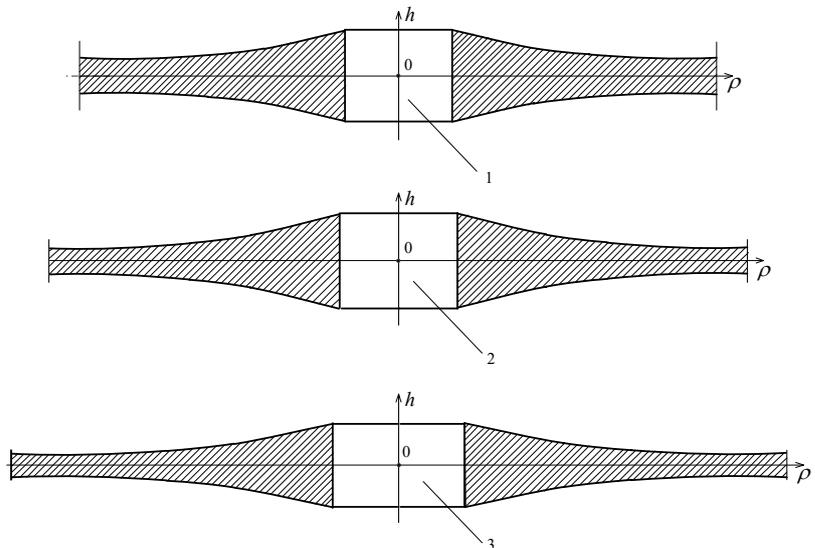


Fig. 1. Graphic representation of the profiles of circular plates depending on the values μ_i : 1 - $\mu_1 = 0.5985$; 2 - $\mu_2 = 1.21417$; 3 - $\mu_3 = 1.39127$

Table 1

The oscillations parameters of the plate of different concaveness

$h(\rho_1)/h(\rho_2)$	Frequency numbers λ_i			Coordinates of the antinodes of oscillations ρ_{mi}			Deflections at ρ_{mi}		
	λ_1	λ_2	λ_3	ρ_{m1}	ρ_{m2}	ρ_{m3}	$W_i(\rho_{mi})$		
1.8	4.29083	9.7437	16.073027	0.5	0.2904 0.5	0.22125 0.38559 0.5	1	0.75863 1	0.856244 0.655916 1
5	4.2500	8.02452	12.39974	0.5	0.3219 0.5	0.24617 0.4038 0.5	1	0.4551 1	0.410675 0.5043 1
8	4.23412	7.4097	11.19461	0.5	0.33736 0.5	0.25899 0.41261 0.5	1	0.4054 1	0.334157 0.479561 1

For all calculation based on variable x , taking into consideration the dependence

$$x = -\ln(1 - \mu\rho),$$

the following boundary values of variable x were established, which correspond to the values μ_i ($i=1, 2, 3$)

$$x_1 = 0,0617; 0,12944; 0,149808;$$

$$x_2 = 0,3556; 0,93416; 1,18952.$$

Based on (7), according to the obtained λ_i , we constructed the shapes (deflections) of natural oscillations (Fig. 2–4). With the help of (7), (11) and λ_i , the diagrams of the radial and tangential cyclical stresses were built for the main shape of oscillations (Fig. 5, 6). Table 1 gives the coordinates of the antinodes of oscillations ρ_{mi} and the corresponding values of maximum deflections $W_i(\rho_{mi})$.

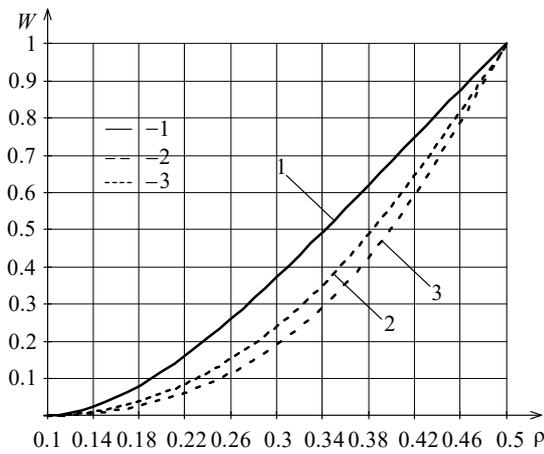


Fig. 2. Graphic representation of deflections at the first shape of natural oscillations of plates of different concaveness: 1 – μ_1 ; 2 – μ_2 ; 3 – μ_3

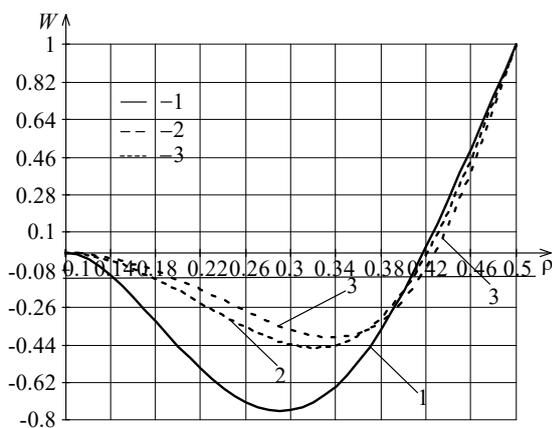


Fig. 3. Graphic representation of deflections on the second shape of natural oscillations of plates of different concaveness: 1 – μ_1 ; 2 – μ_2 ; 3 – μ_3

The estimated data, given in Table 1, as well as the graphic dependences, shown in Fig. 2–6, provide quite clear information about the oscillations of the plate considered.

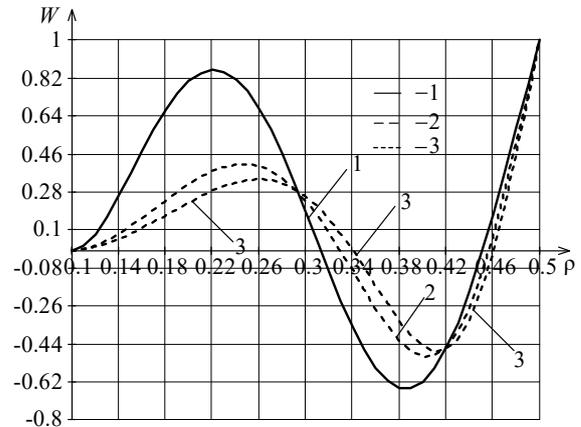


Fig. 4. Graphic representation of deflections at the third shape of natural oscillations of plates of different concaveness: 1 – μ_1 ; 2 – μ_2 ; 3 – μ_3

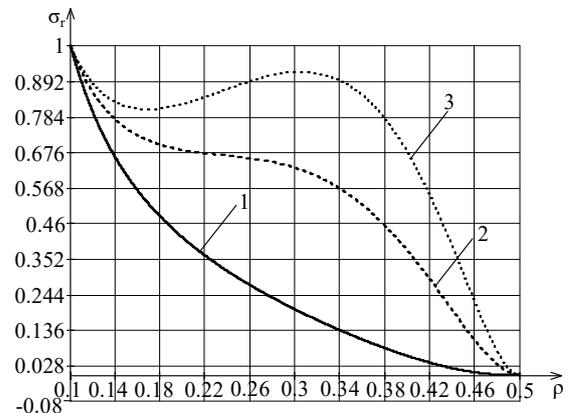


Fig. 5. Graphic representation of radial stresses σ_r at the first shape of oscillations for three values of μ_i : 1 – μ_1 ; 2 – μ_2 ; 3 – μ_3

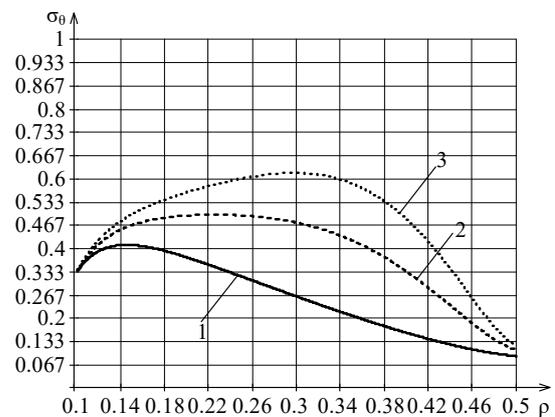


Fig. 6. Graphic representation of tangential stresses σ_θ at the first shape of oscillations for three values μ_i : 1 – μ_1 ; 2 – μ_2 ; 3 – μ_3

7. Discussing the results of solving the set problem

By using the function of deflections W_i , constructed on the basis of the symmetry method, and by applying the built algorithm, one could derive analytical expressions to calculate the radial σ_r and circular σ_θ stresses. That provides for an opportunity to study the distribution of stresses and to

generally analyze the stressed-strained state of a plate. The resulting graphic dependences for the shapes of oscillation of the plate make it possible to determine the location of nodes and antinodes, to estimate the magnitude of deflections.

According to the results of our calculations, it is possible to confirm the obvious that the degree of concaveness affects the weight of a plate. One could see that, based on the profiles (Fig. 1), the increase in the ratio $h(\rho_1)/h(\rho_2)$ makes it possible to reduce the mass of the plate element when designing. Based on the data from Table 1, the natural frequencies of the first three shapes of oscillations decrease to varying degrees with a rise in concaveness, determined by the number of the frequency number λ_i ($i=1, 2, 3$). Thus, it follows from the data in Table 1 that at $h(0.1)/h(0.5)=1.8; 5; 8$ ($\mu=0.5985; 1.21417; 1.39127$) the frequencies decrease, compared to μ_1 , respectively, for cases μ_2 and μ_3 , by (1.0; 1.3) % for λ_1 , by (17.6;24) % for λ_2 , by (22.85;30.35) % for λ_3 . These indicators are a benchmark for studying the oscillations of a given type of plates. One could see a significant drop in frequency on the higher shapes (λ_2, λ_3) and a slight drop at the basic shape(λ_1). To date, only one attempt to analytically determine a single frequency coefficient λ_1 is known, but based on a laborious and less accurate series method, and only for the plate of a linear-variable thickness [19].

Data from Table 1 could be applied to establish that the values of the normalized extreme deflections for cases μ_2, μ_3 ($h(0.1)/h(0.5)=5; 8$) are much lower compared to the plate at μ_1 ($h(0.1)/h(0.5)=1.8$). This fact is important for judging the effect of concaveness on the cylindrical rigidity of a plate under a mode of the axisymmetric cyclical bend. In this case, plate number 3 ($\mu=\mu_3$) has obviously the least rigidity.

Table 1 could be used to determine the coordinates of the extreme values of deflections (the antinodes of oscillations) and Fig. 3, 4 – to find the indicative coordinates of the nodes. These parameters are a means, along with frequency indicators, to identify the oscillatory properties of a plate when it is studied in practice.

The following should be noted as regards the pattern of distribution of stresses σ_r (Fig. 5) and σ_θ (Fig. 6) for the three variants of the circular plate. The stresses σ_r that operate far from the free edge, such as at the end constraint or in the region of maximal σ_θ , are greater to varying degrees than σ_θ . Therefore, these stresses are the main threat in terms of cyclical strength (resistance to fatigue) of the plate when σ_r reaches destructive values in the process of oscillations. The experiments set to determine cyclical strength by initiating intense axisymmetric resonance oscillations in such plates confirmed the validity of the stated assumption [20] because during such experiments the fatigue crack was always located perpendicular to the direction of the action of maximal σ_r . As the ratio of limiting boundary thicknesses increases, the maximum value of the radial stress σ_r is closer to the value of σ_r at the end constraint. For example, for $h(0.1)/h(0.5)=5$, the stress σ_r has an extremum only at the end constraint ($\rho=0.1$); so such a plate cannot be used for fatigue tests. At $h(0.1)/h(0.5)=8$, there is an extremum $\sigma_r \approx 0.93$ at $\rho \approx 0.3$. In this case, the ratio $\sigma_r(0.1)/\sigma_r(0.3)=1.075$ is little different from 1. It is obvious that with a further increase in concaveness (increase in μ) one could obtain the types of plates required for these purposes, with a ratio of $\sigma_r(0.1)/\sigma_r(0.3) < 1$.

Our study of the problem about the natural oscillations of a plate with the assigned profile has demonstrated the flexibility of the symmetry method. The main feature of the

current study is the technique whereby the equation of the shape of natural oscillations could be greatly simplified. The consequence of simplification is the possibility of obtaining an analytically closed solution to the formulated problem.

The algorithm for solving the problem in the case of the original parameters of the plate, which are different from those adopted in the estimation example, remains unchanged. It may be necessary, however, to modify the approximation function. However, according to the symmetry method, whose flexibility and multivariance has been confirmed in the current work, this circumstance does not present fundamental difficulties.

A feature of the obtained theoretical results is the real practical use of the symmetry method and the algorithm of its application to solve the problem about the oscillations of a plate with different-variant thickness $h=H_0(1-\mu\rho)^2$. The practical significance of our results is the possibility to directly use the estimated data obtained in the current work, in particular for the rational design of resonance sound and ultrasound systems based on plates as acoustically active elements.

8. Conclusions

1. A combination of the symmetry method and the factorization technique has helped to build a general analytical solution to the order IV differential equation for the problem about the cyclical axisymmetric bend of a circular plate, whose thickness changes in line with the law of concave parabola $h=H_0(1-\mu\rho)^2$ at varying degrees of its concaveness.

2. Based on the general solution, we have derived formulae to calculate the cyclical stresses in a plate. The feature of the formulae is their compactness achieved by transforming the known expressions containing derivatives from the displacement function of second-order for natural variable ρ to the form containing only the first derivatives for the auxiliary variable $x(\rho)$. The transformed formulae are convenient for directly using the deflection function $W=W(x)$ in them, obtained as a general solution.

3. We have determined the first three natural values for the problem (frequency numbers) and natural functions (oscillation shapes) for a circular plate with a rigid fastening of the inner contour ($\rho_1=0.1$) (1/0.1) and free on the external contour ($\rho_2=0.5$). It has been shown that the natural frequencies of the first three shapes of oscillations decrease to varying degrees with the increase in concaveness, determined by the number of the frequency number λ_i ($i=1, 2, 3$). At $\mu=1.21417$ and $\mu=1.39127$, the frequencies decrease, compared to the case of $\mu=0.598$, by (1; 1.3) %, (17.6; 24) %, (22.85; 30.35) %, respectively. One could see a significant drop in frequency on the higher shapes of oscillations (λ_2, λ_3) and a slight drop at the basic shape (λ_1). We have established the magnitudes and coordinates of extreme deflections (the antinodes of oscillations) and the indicative coordinates of the nodes. These numerical parameters, along with the frequency indicators, are a means of identifying the oscillatory properties of a plate when it is studied in practice.

4. We have constructed the diagrams (graphic dependences) of radial σ_r and tangential σ_θ cyclical stresses at the basic shape for each of the three variants of the parabolic plate concaveness. It has been established that the increase in the ratio of the edge thickness $h(0.1)/h(0.5)$, that is concaveness, leads to an increase in σ_r in the cross-sections

outside the end constraint. At $h(0.1)/h(0.5)=1.8;5$, the stresses σ_r have an extremum $\sigma_r=0.93$ at $\rho=0.3$. The ratio $\sigma_r(0.1)/\sigma_r(0.3)=1.075$ in this case is little different from 1. It has been concluded that a further increase in concave-

ness would lead to the values $\sigma_r(0.1)/\sigma_r(0.3)<1$, which makes it possible, in particular, to use such plates to experimentally determine the limits of materials endurance under a complex stressed state.

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