Удосконалено алгоритм методу базисних матриць для проведення аналізу властивостей лінійної системи (СЛАР) при різноманітних змінах в моделі. Зокрема, при включенні-виключенні групи рядків та стовпців (на основі "окаямлювання") без перерозв'язання задачі спочатку. Встановлено умови сумісності (несумісності) обмежень та побудовано вектори фундаментальної системи розв'язків у випадку сумісності. Досліджено вплив точності подання елементів моделі (довжина мантиси, величина порядку, порогу машинного нуля та переповнення) та варіантів організації проведення обчислень на властивості розв'язків. Зокрема, вплив на величину та повноту рангу на прикладі СЛАР з погано обумовленою матрицею обмежень. Програмна реалізація була розвинута на проведення обчислень за методами базисних матриць (МБМ) та Гауса, тобто використано довгу арифметику для моделей з раціональними елементами. Запропоновано алгоритми та комп'ютерну реалізацію методів типу Гауса та штучних базисних матриць (варіант методу базисних матриць) в середовищах Matlab та Visual C++ з використанням технології точних обчислень елементів методів, в першу чергу, для погано обумовлених систем різної розмірності.

На прикладі матриць Гільберта, які характеризуються як "незручні", проведено експеримент з метою аналізу властивостей лінійної системи при різних розмірностях, точності подання вхідних даних та сценаріях проведення обчислень. Розвинуто формати ("точний" та "неточний") подання елементів моделі (довжина мантиси, величина порядку, порогу машинного нуля та переповнення) та варіанти організації виконання основних операцій при проведенні обчислень та їх вплив на властивості розв'язків. Зокрема, простежено вплив на величину та повноту рангу на прикладі СЛАР з погано обумовленою матрицею обмежень

Ключові слова: метод базисних матриць, прямокутна матриця обмежень, погано обумовлена СЛАР

UDC 519.852:519.876
DOI: 10.15587/1729-4061.2019.184530

FORMING A METHODOLOGY OF BASIC MATRICES IN THE STUDY OF POORLY CONDITIONED LINEAR SYSTEMS

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Received date 20.09.2019
Accepted date 18.11.2019
Published date 28.12.2019

1. Introduction

It was historically predetermined that a considerable body of scientific research is largely focused on the study of properties of linear systems, in particular, systems of linear algebraic equations (SLAE) [1–3]. This is explained by the fact that linearity is an initial basic structure on which more complex structures are further "fit". Quite often, a nonlinear problem is replaced in some determined neighborhood by a linear (simplified) problem at each step of the algorithm under study. For example, Newton's classical method of solving systems of nonlinear equations contains in its structure (as simplification or linearization) solution of a system of linear algebraic equations. SLAE and linear programming problems (LPP) were used, for example, in first industrial implementations for military and economy needs as an auxiliary decision-making tool using information technology [3, 4]. It is not difficult to be convinced that typical SLAE (as LPP iteration) were implemented for a rectangular constraint matrix on an assumption of rank completeness [1–3]. Compatibility of SLAE with a rectangular constraint matrix is studied, for example, by application of the well-known Kronecker-Capelli theorem. In this case, solvability of the problem is reduced to comparison of rank magnitudes of matrices of main and extended constraint systems. When constructing a fundamental system of solutions (FSS), rank magnitude remains a fundamental problem of establishing the system properties [3]. The rank magnitude is defined as the highest order of a non-zero minor (analysis of sequence of the "framing" rectangular matrices). The rank magnitude correlates with the number of iterations at which leading element is non-zero in the Gaussian type methods.

On the other hand, the problem of determining rank magnitude of a linear system has turned out to be dependent on organization of computing technology. In particular, calculation practice revealed distinctions between mathematical (theoretical or exact) and machine (computer) representation of the model [1]. These distinctions (as irreversible errors) are an additional source of model and process inadequacy (for example, task inaccuracy). Moreover, in different computation scenarios, this inadequacy manifests itself in different ways both quantitatively and qualitatively (because of mantis length, order magnitude, rounding methods and computation organization). For example, in IEEE standard [1, 2], machine zero is a certain interval of belonging to a set of subnormal numbers in calculations. It can acquire specific (reference) boundaries at a given computation configuration. In general, mathematical (analytical) zero can be regarded as an exact assignment and machine (computer) one as an inaccurate assignment. On the other hand, fulfillment of the zero-non-zero condition is essential in organization of many algorithms, in particular, in the Gauss method (leading element) or in the simplex method as a condition of independence of the vector system (resistance), etc., and can acquire "own" quantitative content depending on computation process organization. In general, sequence of operations is also important when performing calculations because commutativity is not always met. Through "delineation" of the machine zero, influence on the rank magnitude and other model parameters are determined in calculations. As a mathematical category, the rank magnitude is a key property in analysis of structure properties of a problem (representation of a multifaceted set) which also depends on fulfillment of this condition. Quantitative inaccuracies in representation of model elements can cause qualitative distinctions in geometric structure (multifaceted set), in particular, dimension of the set, minimal face, etc.

These properties can be of significance in conditions of poor conditionality. Some poorly conditioned constraint matrices have a typical structure with their intrinsic peculiarities (embedded features of formation). In particular, the Hilbert matrix is mathematically fully ranked and non-degenerate. Conduction of calculations using this matrix can bring about contradictory (unpredictable) results, both accurate and with a great error. In this regard, approaches to construction of preconditioners aimed to "correct" such system property by directed transformation of the constraint matrix were proposed [5–8].

Some typical matrices (including those mentioned above) can serve as components when solving a series of problems of the same type in conditions of different dimensions. In this case, constraint matrices of larger dimension can be considered as sequences of column-row inclusions to matrices of smaller dimension. In particular, in the Galorkin scheme for the boundary-value problem, the basis function system can be expanded by inclusion of new functions to achieve greater solution precision which determines the process of increasing dimension of the SLAE constraint matrix by framing the Hilbert matrices.

It is getting important to establish both the rank magnitude and connections between solutions of the problems of different dimension in such changes.

It seems appropriate to develop new and improve existing methods and algorithms of analyzing the impact of changes in presentation of SLAE elements whose values are in the "zone" close to the machine zero, in particular, when solving systems of equations in different variants of model representation (mantis length, order value, etc.) taking into account peculiarities of the constraint matrix structure.

2. Literature review and problem statement

Today, there are dozens of versions of exact and probably hundreds of versions of iteration methods and algorithms of SLAE solution. The most common and applied of them are given in [1, 3]. In some cases, application of certain iterative methods in solving SLAE can lead to a loss of "continuity" (depending on initial solutions) which limits their use in analysis of system properties in a context of the given problem. This, in particular, can be traced in [1, 3]. However, important aspects concerning determination of the rank magnitude and its effect on properties have been neglected and not studied enough by the authors. In theoretical studies, the category of rank is key in establishing properties of the system, and in practical calculations, it depends on computation process organization and the problem properties. It seems appropriate to use the SLAE solution which thoroughly evaluates and considers magnitude of the rank and its completeness. Therefore, from the point of view of applicability in analysis of the problem properties, it is advisable to take the classic exact methods in which such a possibility is embedded as a basis. The Kramer and Gauss methods, including their adaptation to the structure of the constraint and generalization matrices are the most famous [1–3]. If we consider an SLAE with an arbitrary row to column ratio, then the Gaussian method can be considered as a universal one. The method allows one to solve the problem without limitations on structure and dimensions of the corresponding matrix. Such variants of this method are imbedded in some implementations of SLAE and LLP [3, 4]. The theoretical backgrounds of the SLAE solution methods proposed in both well-known [1, 3] and more recent [5–7] publications suggest the following. For their correct application, for example in calculations, it is important to determine magnitude of the rank and properties of its completeness, finding an appropriate rank matrix (basic), fulfillment of conditions of resistance, non-degeneracy of the basic matrix. In general, establishing these properties is a separate auxiliary problem of analysis.

The abovementioned classic methods of SLAE solution have some universal properties that make it possible to move in the course of iterations from the initial constraint matrix to one of a simple structure without limiting its dimensions. During these iterations (non-zero leading element in iterations is the condition of transition accuracy), all important parameters of the system (solvability, rank magnitude) can be established which makes it possible to find common solutions or establish incompatibility of the problem constraints. In the case of rank completeness, this approach was taken as a basis of the matrix rotation procedure and construction of variants of implementation of the simplex method for finding an optimal solution [3, 4]. However, specificity of the Gaussian method, as a tool for solving SLAEs [8], somewhat limits its use in analyzing properties of the system when solving other related problems. It should be noted that computational intensity of iterations of the method decreases in the course of calculations.

Along with the classical (Gaussian) scheme of studying SLAE and ZLP, there are variants using the basic matrix method (BMM) whose computational scheme works, tentatively speaking, on the contrary [9–11]. Transition is made to the initial model from the auxiliary SLAE with a trivial constraint matrix and known solution by means of equivalent transformations (sequential inclusion of the initial problem constraints). In the course of iterations, all important parameters are monitored such as condition of resistance (the leading element of transformation is non-zero), completeness of rank, non-degeneracy of the straight and inverted matrices, intermediate solutions, establishment of the problem solvability, control of the system conditionality. But what is the most valuable, ability to analyze impact of change of individual elements (rows, columns) in the model without having to solve the problem with making changes to the model. This reduces time spent in finding the best solution. Thus, the use of BMM not only enables solution of problems but also conduction of analysis.

The solution found by one of the methods can also determine statement of a number of new problems. For example, when establishing property of incompleteness of the constraint matrix rank or insolubility of SLAE or LPP, it may be necessary to restore the rank completeness by purposeful changes to the constraint matrix, identification of the constraints that cause incompatibly and their elimination from consideration or, if possible, restore solvability by purposeful changes in the model. This is characteristic in the case where simple rejection of constraints that create insolubility is unacceptable. If the SLAE as a process model is refined when it undergoes changes, then the problem of analyzing the SLAE properties as a result of changes in the model may be relevant but without re-solving. The problems in this formulation were investigated in [11], in particular, when expanding (or narrowing) the constraints matrix by inclusion (exclusion) of rows or columns. This may be relevant when using SLAE, for example, when solving a boundary value problem using the Galorkin method. Here, the system of basic functions can be expanded which causes a change in the constraint matrix ("framing" of the current one).

From a mathematical point of view, the category of rank (its magnitude, completeness) as the order of a maximum non-zero minor is determined by checking fulfillment of the condition of independence of the system of vectors (resistance) in iterations of a number of methods (the leading element is non-zero). It is significant in theoretical studies when analyzing properties of linear systems. On the other hand, practice of implementing methods and algorithms opens up significance of the rank category, however in the context of computer representation of the mathematical model [1, 2].

It is well known that accumulation of errors in rounding and performing basic operations using a computer is generally an irreversible source of errors and inadequacy of the process model in studies. Content of the category of machine zero was revealed in [1, 2] as a counterpart of mathematical zero which can be interpreted in some sense as an inaccurately set value (a certain membership interval). It is also known that rounding and computing operations can be organized differently in general. In particular, according to the IEEE format, when representing a model, the machine zero corresponds mathematically to a certain set (interval). For example, when calculating a mathematical master element and its machine counterpart (as an interval dependent on the chosen standard of representation of model numbers and organization of calculations), distinctions may arise when checking fulfillment of the "zero-non-zero" condition. In particular, inconsistencies appear between mathematical and machine representation of the model with different accuracy in the number of "fulfillments" of the conditions of independence of the vector system (the leading element of transformation is non-zero) at iterations in the zones close to the machine zero. For example, when calculating with the use of the Hilbert matrix, significant errors were detected: deviation of the found solution for different accuracy options [10]. Thus, it can be stated that the methods of investigating the model properties and relationship of the results at various parameters (especially close to 'zero-non-zero' transients) of experiments have not been developed sufficiently.

Accumulation of errors in calculations is promoted by poor conditionality of the system [1]. In conditions of poor conditionality, presence of the control of occurrence of calculations in a state of poor conditionality is the desirable property of methods and algorithms [1, 5–8]. It is known that conditionality estimation implies presence of norms (or their estimates) of straight and inverse matrices which is not always embedded in capabilities of a concrete computational scheme. Study [10] provides an overview of methods and algorithms for calculating variants of model accuracy, in particular, in exact numbers for poorly conditioned systems (with structure features).

Also, presence of a reference solution and a set of test tasks using specific methods to verify their properties is important in calculations. References to the respective libraries are given in [1, 8].

Development of so-called exact algorithms within the methods of calculating in rational numbers is another approach proposed in [10]. This makes it possible to directly check efficiency of calculations according to the algorithm, in particular, combined use of variants of implementation of algorithms and methods [1–3, 8–10] in different programming languages, for different levels of accuracy (mantis length, order value) in exact numbers for a typical problem. For example, using the Hilbert constraint matrix one can identify important regularities for further construction of the membership function [12, 13]. A well-constructed membership function can be used later in making decisions on how organize calculations to achieve a given level of calculation parameters.

Finding magnitude of the rank, establishing its completeness, the scheme of restoring its completeness and analysis of resolvability are imbedded in BMM [9–1]. Implementations have been developed in environment of various software products, different computation scenarios, in particular, in exact numbers, and for "growing-up" the accounting mechanism and fuzziness in model representation.

There is an obvious need to improve existing and develop new methods and algorithms that would provide an opportunity to analyze linear systems using a single methodology. In particular, this applies to the models further applicable in solution of SLAEs with a poorly conditioned constraint matrix, different scenarios of representation of the model elements (mantis lengths, order magnitude, etc.). Not only the fact of solving the problem but also availability of tools to analyze properties of the model in changes, different scenarios of calculation and establishing their relationships is very important. It is also important to study possibilities of influence ("from the inside") of such parameters as the leading element of iteration, conditions of the value resistance and the rank completeness taking into account thresholds of machine zero and overflow on key parameters of the methods and algorithms of analysis. This also applies to the ways of analyzing and processing transient situations ("near zero"). Pre-checked and confirmed consistency between all parameters of calculation should be considered as an important sign of computation correctness.

3. The aim and objectives of the study

The study objective is development of theoretical provisions for construction of algorithms of compatibility analysis and solution of linear systems (SLAE) which are provided with additional types of data ("accurate" and "inaccurate") to represent the model parameters, organize various options for performing basic operations during calculation, in particular, when values of the model elements in the "zone" are close to the machine zero when establishing properties of poorly conditioned linear systems.

To achieve this objective, the following tasks were set:

– improve SLAE solution algorithms as regards analysis of compatibility (incompatibility) of constraints, construction of structure of vectors of fundamental solution system in the case of compatibility and impact of changes in the model using the basic matrix method, in particular, when including and axcluding a group of rows and columns without re-solving the problem from beginning;

– study the effect of "exact" and "inexact" types of data entered to perform calculation of properties using the basic matrix method, in particular, effect on the magnitude and completeness of the rank on an example of a SLAE with a poorly conditioned constraint matrix;

– perform a computational experiment to analyze properties of a linear system with a poorly conditioned constraint matrix (the Hilbert matrix) for different parameters of data representation (mantis length, model dimensionalty) and variants of computation (including the case of computation in exact numbers).

4. Theoretical aspects of analysis of compatibility of constraints and construction of the linear system solution structure

It is known [4] that finding common solutions of a system of linear algebraic inhomogeneous equations involves a step of constructing a fundamental system of vectors, that is, a subspace of general solutions of the corresponding system of homogeneous equations shifted to the vector of inhomogeneous system solution. For overconditioned SLAEs, a problem of analyzing solvability in terms of compatibility appears as well.

It follows that the problem consists not only in development of a standard procedure for constructing common solutions (at compatibility of constraints) but also in establishment of the system solvability in general. When using SLAE models, a need arises to analyze effects from changes in elements, rows or columns of the constraint matrix on the structure of common solutions [9–11].

Let us there is a SLAE:

$$Ax=B, \tag{1}$$

where

$$A=\left\{a_{ij}\right\}_{i=\overline{1,m},\ j=\overline{1,n}}$$

are elements;

$$\left\{a_i\right\}_{i=\overline{1,m}}=\left\{a_{i1},a_{i2},\ldots,a_{in}\right\}_{i=\overline{1,m}}$$

are rows;

$$\left\{A_j\right\}_{j=\overline{1,n}}=\left\{a_{1j},a_{2j},\ldots,a_{mj}\right\}_{j=\overline{1,n}}$$

are columns of a matrix with dimensions (m, n);

$$x=(x_1, x_2,\ldots, x_n)^{\mathrm{T}}=(x_b, x_n)^{\mathrm{T}}$$

is vector of variables;

$$B=(b_1, b_2,\ldots, b_m)^{\mathrm{T}}=(B_b, B_n)^{\mathrm{T}}$$

is vector of the right-hand sides of constraints; T is transposition sign.

Model (1) is studied in E^n.

Introduce sets of indices

$$I=(1, 2,\ldots, m), J=(1, 2,\ldots, n), I_b=(i_1, i_2,\ldots, i_r),$$

$$J_b=(j_1, j_2,\ldots, j_r), J_n=J/J_b,\ I_n=I/I_b$$

are sets of indices of rows and columns (linearly independent) and the remaining rows (columns).

In the general case, for a rectangular constraint matrix, the rank $R=rank(A)<\min(m, n)$.

Assume that model (1) is studied in E^n space. Assuming the sets I_b, J_b are known, introduce the following into the matrix consideration:

$$A_b=\left\{a_{ij}\right\}_{\substack{i\in I_b,\\ j\in J_b}}=\left\{a_{(b)i}\right\}_{i\in I_b}=\left\{a_{ij_1},a_{ij_2},\ldots,a_{ij_r}\right\}_{i\in I_b};$$

$$A_{bj}=\left\{a_{i_1j},a_{i_2j},\ldots,a_{i_rj}\right\}^{\mathrm{T}}_{j\in J_b};$$

$$\overline{A}_b=\left\{a_{ij}\right\}_{\substack{i\notin I_b,\\ j\in J_b}}=\left\{\overline{a}_{bi}\right\}_{i\notin I_b}=$$

$$=\left\{a_{ij_1},a_{ij_2},\ldots,a_{ij_r}\right\}_{i\notin I_b};$$

$$\overline{A}_n=\left\{a_{ij}\right\}_{\substack{i\notin I_b,\\ j\notin J_b}}.$$

Introduce main definitions.

Definition 1. Call a rectangular matrix A_b, formed by intersection of r linearly independent rows $a_{i_1}, a_{i_2},\ldots,a_{i_r}$ $(i_1, i_2,\ldots, i_r)=I_b\subset I$ and columns $A_{j_1},A_{j_2},\ldots,A_{j_r}$ $(j_1, j_2,\ldots, j_r)=$ $=J_b\subset J$ (1) an r-basic matrix and call the solution

$$x_0=\left(x_{i_1},x_{i_2},\ldots,x_{i_r}\right)$$

of the corresponding system of equations $A_bx_b=B_b$, $B_b\subset B$ an r-basic solution.

Definition 2. Call two r-basic matrices with one different row or column adjacent matrices.

Basic matrices $(r=m)$ are successively changed in BMM iterations by substitution of rows. Let e_{ri} are elements of the matrix A_b^{-1}, inverted to A_b; $\alpha_r=(\alpha_{r1}, \alpha_{r2},\ldots, \alpha_{rm})$ is the vector of development of α_r by the rows of the basic matrix A_b; $x_j=(x_{1j}, x_{2j},\ldots, x_{mj})$ be the vector of development of the column A_j, $j\in J$ by columns of the basic matrix A_b. Elements of the BMM are coefficients of development of the normals to constraints. Denote coefficients of the inverted matrix by a bar at the top, i. e. A_b, α_r, e_{ri}, α_r.

According to Theorem 1 [9], corresponding relations were established between coefficients of development of constraint normals (1) by rows of the basic matrix and elements of the inverted matrices in two adjacent basic matrices. Conditions of nondegeneracy of the basic matrix when substituting normal α_l for the k-th row of the basic matrix A_b $(\alpha_{lk}\neq 0)$ were also established.

Let values of the matrix A rank and its r-basic matrix A_b be known.

Without limiting generality, according to the notation introduced, the system (1) can be rewritten in an equivalent form:

$$A_b x_b = B_b - A_n x_n; \quad (2)$$

$$\overline{A_n} x_b = B_n - \overline{A_n} x_n. \quad (3)$$

Let us study property of solvability of (1) for basic, non-trivial relations of dimension of the constraint matrix and the rank magnitude.

I. Let

$$r = rank(A) = \min(m,n) = m,\ m < n.$$

With such constraints, rows

$$\alpha_j = (\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jm})$$

and columns

$$A_j = (\alpha_{1j}, \alpha_{2j}, \ldots, \alpha_{mj}),\ j \in J_b$$

are linearly independent

$$I_b = I = (i_1, i_2, \ldots, i_r),\ J_n = J / J_b,\ I_n = I / I_b = \varnothing,$$

vector of variables is

$$x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} = (x_b, x_n)^{\mathrm{T}},\ x_b = (x_{i1}, x_{i2}, \ldots, x_{ir})^{\mathrm{T}} \subset x,$$

vector of the right-hand sides of the constraints is

$$B = (b_1, b_2, \ldots, b_m)^{\mathrm{T}} = B_b.$$

In this case, system (1) can be rewritten in an equivalent form and will be solvable by compatibility.

$$A_b x_b = B - A_n x_n. \quad (4)$$

The r-basic matrix A_b will be determined by intersection of rows and columns ($I_b = I, J_b = (i_1, i_2, \ldots, i_r) \subseteq J$).

Denote vectors $(x_{10}, x_{20}, \ldots, x_{m0})^{\mathrm{T}}$, $(x_{1j}, x_{2j}, \ldots, x_{mj})^{\mathrm{T}}$, $j \in J_n$ as solutions of the following SLAEs with a rectangular nondegenerate matrix A_b of the form:

$$A_b x_b = B_b, \quad (5)$$

$$A_b x_b = -A_j,\ j \in J_n \quad (6)$$

and form the following vector system on their basis

$$X_0 = \left(x_{10}, x_{20}, \ldots, x_{m0}, \underbrace{0, 0, \ldots, 0}_{n-r} \right),$$

$$X_i = \left(x_{1i}, x_{2i}, \ldots, x_{mi}, \underbrace{0, \ldots, 0, \overset{i}{\hat{1}}, 0, \ldots, 0}_{n-r} \right),$$

$$i = \overline{1, n-r},$$

where i is sequence number of column j in the set J_n. According to [4], such vector system defines a fundamental system of vectors, a variety of common solutions of (1) in the following form

$$X_0 + \sum_{s=1}^{n-r} \beta_s X_s,$$

where $\beta_1, \beta_2, \ldots, \beta_r$ are arbitrary real numbers.

Computational difficulty in finding a general solution of (1) lies not only in determining the rank magnitude but also in formation of a basic matrix and matrices inverted to it. This also applies to formation of a set of linearly independent rows and columns A_b, row and column development vectors, components of the vectors X_0, X_i, $i = \overline{1, n-r}$. Determination of the rank, construction of the basic matrix inverted to A_b (r-basic) can be made based on BMM by substitution of rows using relations of Theorem 1 [9].

Carrying out r simplex BMM iterations according to [9] for substituting rows (algorithm 1 in [9]) of the matrix A for rows of the matrix $E = E^{-1}$ (an auxiliary matrix of order n) determines basic matrix A_b. Inverted matrix A_b^{-1} of the system (1) is also determined by deleting corresponding rows and columns and the matrix nondegeneracy is controlled in steps.

Vectors

$$x_0 = \left(x_{10}, x_{20}, \ldots, x_{m0} \right) = A_b^{-1} B_b,$$

$$x_i = \left(x_{1j}, x_{2j}, \ldots, x_{mi} \right) = A_b^{-1} A_j,\ j \in J_n$$

are solutions of the SLAE of the form (5), (6) and, on the other hand, components of vectors X_0, X_i, $i = \overline{1, n-r}$ are determined.

II. Study of the system solvability when condition $m > n$, $r = rank(A) = \min(n, m) = n$ is met determines an equivalent system (1):

$$A_b x_b = B_b, \quad (7)$$

$$\overline{A_b} x_b = B_n. \quad (8)$$

The matrix A_b (r-basic) will be determined by intersection of rows and columns ($I_b = I$ and $J_b = (j_1, j_2, \ldots, j_r) \subseteq J$).

System (7) has a unique solution $x_b = A_b^{-1} B_b$. Collectively for (7), (8), overconditioning (of the constraint system (1)) $B_b \subset B$ is its characteristic feature. Additional complexity consists in analysis of the system (7), (8) for solvability in terms of compatibility, that is, admissibility of solution of (7) in the constraint system (8). To analyze this case, assume that

$$\alpha_i = (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{im}),\ i \in I_n,\ \alpha_i = \alpha_i A_b^{-1},$$

$$B_b = (b_{i1}, b_{i2}, \ldots, b_{ir})^{\mathrm{T}} \subset \mathrm{B},\ \bar{b}_i = \alpha_i B_b$$

are known and can be determined, for example, by BMM. It turns out that the following is true for this case.

Violation of at least one and the following relations is a necessary and sufficient condition for incompatibility of constraints (insolvability of the system (7), (8)) for which $m > n$, $r = rank(A) = \min(n, m) = n$:

$$\exists\, i,\ b_i = \bar{b}_i,\ i \in I_n. \quad (9)$$

Fulfillment of the following relation is a necessary and sufficient condition for compatibility of constraints (solvability of the system (1)) for which $m>n$, $r=rank(A)=\min(n,m)=n$:

$$b_i=\bar{b}_i,\quad i\in I_n. \tag{10}$$

III. Let

$r=rank(A)<\min(n, m)$.

The matrix A_b (r-basic) is determined by intersection of rows and columns ($J_b^r=(j_1, j_2, ..., j_r)\subset J$ and $I_b^r=(i_1, i_2, ..., i_r)\subset I$).

It is characteristic that a system equivalent to (1) takes the form of (2), (3), that is, analysis of its solvability is carried out in two steps. A common solution of (2) is constructed in the first step (case 1) and solvability of (2), (3) is analyzed in the second step. It should be noted that the solvability conditions (case II) can be extended to the general case of structural properties of the constraint matrix and the rank magnitude, that is, $r=rank(A)<\min(n, m)$.

Study of cases

$r=rank(A)<\min(n, m)$, $n=m$

and

$r=rank(A)=\min(n, m)$, $n=m$

reduces to the previous cases.

The above solvability conditions are constructive in analyzing the linear system properties when there are changes in individual elements, groups of rows and columns, or the constraint vector. In general, changes in the system determine magnitude of the constraint matrix rank, i. e. increase (decrease), re-formation of the new basic matrix (change in elements) and development of corresponding rows or columns i. e. structure of vectors of the fundamental solution system.

The theorem conditions do not contradict provisions of the well-known Kronecker-Capelli theorem on solvability of equation systems.

The use of BMM [9] provides analysis of impact of changes in the model (1) on overall solution. This can reduce total computation volume with slight changes in non-A_b elements (1). In this case, the problem is actually solved. In particular, changes in the vector B and components of the vectors A_j, $j\in J_n$ only determine recalculation of a perturbed subvector components and the compatibility check. Changes in the components of the basic matrix have a somewhat more complex mechanism of influence on the solution structure. In these circumstances, typical simplex iterations should be carried out to check rank and property of the matrix nondegeneracy resulting from the changes.

5. Algorithms for analyzing completeness of rank of the constraint matrices of linear systems by the method of basic matrices

In this section, the method of basic matrices was developed for analysis of rank completeness [9].

Let us consider a SLAE of the form:

$$Au=C, \tag{11}$$

where A is matrix with dimensions ($n\times m$);

$C=(c_1, c_2,..., c_n)^{\mathrm{T}}$

is a vector-column of dimension n;

$u=(u_1, u_2,..., u_n)^{\mathrm{T}}$; $u=(u_1, u_2,..., u_n)^{\mathrm{T}}$

is the sought dimension vector m; T is transposition sign;

$a_j=(a_{j1}, a_{j2},... , a_{jm})$, $j=1, 2,..., n$

are matrix A rows. Equation (1) is supplemented by an auxiliary SLAE of the form:

$$Iu=K, \tag{12}$$

where I is the unit-diagonal dimension matrix ($m\times m$) and

$$K=\underbrace{(1,1,....,1)}_{m}^{\mathrm{T}}$$

is the vector of dimension m. It is assumed that system (12) is usually trivial, with known properties. It plays only an auxiliary role, namely, construction of initial values of the BMM elements, in particular, of the inverted matrix and solution.

Method of basic matrices was taken as the basis for construction of the algorithm of analyzing the SLAE properties since, according to [9], it has the ability of:

– finding value of rank of the matrix of system constraints (11);

– finding SLAE solution (11);

– controlling conditionality of the system;

– analyzing the impact of changes in the model (11) as a result of refinements (without re-solving);

– building initial solutions of the problems based on trivial basic matrices (12) which excludes initial time-consuming computation.

Recall [9] where the idea of an ordinal basic matrix is the basis of the proposed method of artificial basic matrices (BMM variant). During iterations, the basic matrices are successively changed by inclusion/exclusion of row-normals of the problem constraints.

Submatrix A_b composed of m linearly independent row-normals ($i_1,i_2,...,i_m$) of constraints will be called artificial basic submatrix and solution of u_0 of the corresponding system of equations $A_bu=C^0$ where $C^0=(c_{i1}, c_{i2},..., c_{in})^{\mathrm{T}}$ will be called artificial basic solution.

Rank of the constraint matrix A will be considered complete if the condition $rank(A)=\min(n, m)$ is satisfied; m, n, are numbers of rows and columns of the constraint matrix, respectively.

Let e_{ri} be elements of the matrix A_b^{-1} inverted to A_b; $u_0=(u_{01}, u_{01},..., u_{0m})^{\mathrm{T}}$ is the basic solution; $\alpha_r=(\alpha_{r1}, \alpha_{r2}, ..., \alpha_{rm})$ is the vector of development of the vector-normal restriction $\alpha_ru\leq c_r$ by rows of the basic matrix A_b; $\Delta_r=\alpha_ru_0 - c_r$ is residual of the r-th constraint (1) at the vertex. All elements introduced in the new basic matrix $\overline{A_b}$, other than $\overline{A_b}$, in one row will be marked with a bar at the top.

According to Theorem 1 [9], corresponding relations were established between the coefficients of development of normals of constraints, elements of inverted matrices, basic solutions and residuals of constraints in two adjacent basic matrices.

On their basis, a scheme is constructed for determining the system (1) rank and solution of the system of equations by means of successive changes in basic matrices and corresponding artificial solutions.

According to Lemma 1 [9], fulfillment of the condition $\alpha_{lk}\neq 0$ is a necessary and sufficient condition of linear independence of the system of vectors $a_{i_1}, a_{i_2}, ..., a_{i_{k-1}}, a_l, a_{i_{k+1}}, ..., a_{i_m}$ formed by substitution of the vector α_1 for the vector a_{i_k} that occupies the k-th row in the basic matrix A_b. The lemma is fundamental in analysis of completeness of the constraint matrix rank and construction of new algorithms of the basic matrix method.

Corollary 1 (of Lemma 1 [9]). For existence of a unique solution (11), it is necessary and sufficient that $\alpha_{lk}^{(i)}\neq 0$, $i=\overline{1,r}$, $r=m=n$ where i is the iteration number; $\alpha_{lk}^{(i)}$ are leading elements of the BMM simplex iteration for substitution of normals of constraints (11) for rows of the basic matrix (12).

Corollary 2 (of Lemma 1 [9]). Matrix $\underline{A}$ of the main system (11) is not degenerate if $\alpha_{lk}^{(i)}\neq 0$, $i=\overline{1,m}$, $r=m=n$.

Corollary 3 (of Lemma 1 [9]). Rank of the system (11) is determined by the number of correct (as regards satisfaction of Lemma 1 conditions) substitutions of vector-normals (11) for rows of the constraint matrix (12) (according to Theorem 1 [9]).

The below are main stages of the algorithmic scheme for finding value of rank, corresponding basic matrix and solution of the system (11) based on known properties (12) of the system built according to the number of columns (variables) of the constraint matrix:

1. Perform simplex iterations for substitution of rows of constraints of the system (11) for rows of the basic matrix of the system (12) (algorithm 1 [11]) according to relations (5)–(9) [9].

2. Check that conditions of nondegeneracy are fulfilled ($\alpha_{l^{(r)}k^{(r)}}\neq 0$, r is the iteration number).

3. Find appropriate elements of the method: vectors of expansion by rows of the basic constraint matrices (12), inverted basic matrix, basic solutions $u_0^{(r)}$, where r is iteration number.

4. Control the number of iterations r in substitution of rows of the main system (11) for which conditions of non-degeneracy are satisfied for rows of the auxiliary system (12).

If the number of iterations of substitution of rows of the basic system (11) for which conditions of non-degeneracy $\left(\alpha_{lk}^{(i)}\neq 0\right)$ is equal to r and $r=m=n$, that is, for a rectangular matrix, for rows of the auxiliary system (12), a unique solution will be found according to the relation $A_b^{-1}\cdot c^0=u^0$.

If the number of iterations of substitution of rows of the basic system (1) for which conditions of non-degeneracy $\left(\alpha_{lk}^{(i)}\neq 0\right)$ are satisfied for the rows of the auxiliary system (2) is equal to r and $r=m<n$, that is, the constraint matrix (11) is "long" full-rank constraint matrix. The matrix A_b (r-basic) and the matrix inverted to it will be determined from the full-rank matrices (straight and inverted) (12) transformed in r iterations by deleting the substituted rows and columns, respectively, ($I_b=I$ and $J_b=(i_1, i_2,..., i_r)\subset J$) (Case I from the previous section). Next, find the FSS (a set of solutions) or one of the solutions can be separated.

If the number of iterations for which $\alpha_{lk}^{(i)}\neq 0$ is equal to r and $r=n<m$, it is a high constraint matrix. The matrix A_b (r-basic) and the matrix inverted to it will be determined from the full-rank matrices (straight and inverted) (12) transformed in r iterations by deleting substituted rows and columns, respectively ($J_b=J$ and $I_b=(i_1, i_2,..., i_r)\subset I$) (Case II).

The following is analysis of the problem solvability.

If the number of iterations for which $\alpha_{lk}^{(i)}\neq 0$ is equal to r and conditions $r<m$, $r<n$ are fulfilled, this means rank incompleteness for the SLAE (1), that is, the model needs further analysis for resolvability (Case III from the previous Section), namely, the matrix A_b (r-basic) and the matrix inverted to it will be determined from the full-rank matrices transformed in r iterations (straight and inverted) (12) by deleting substituted rows and columns, respectively, ($J_b^r=\left(j_1, j_2, ..., j_r\right)\subset J$ and $I_b^r=\left(i_1, i_2, ..., i_r\right)\subset I$) (Case III). Conditions of solvability shall be checked. In they are fulfilled, find the FSS (solution set).

5. 1. On completeness of the SLAE constraint matrix rank

Let the restriction $\alpha_l u=c_l$ from (11) is fulfilled, that is, $\alpha_{lk}=0$ (the k-th component of decomposition of the vector α_l in rows of A_b is zero). That is, "inclusion" of such a row into the basic matrix violates the condition of non-degeneracy of the "new" constraint matrix (indicating a possible incompleteness of the rank magnitude). Accordingly, for the perturbed restriction $\widetilde{a_l}\,u\leq \widetilde{c_l}$, in the form $\widetilde{a_l}=a_l+\dot{a_l}$, $\widetilde{c_l}=c_l+\dot{c_l}$, we can write that

$$\widetilde{\Delta}_l=\Delta_l+\dot{\Delta_l}=\widetilde{a_l}u_0-\widetilde{c_l} \quad\text{and}\quad \tilde{\alpha}_l=(\alpha_l+\dot{\alpha_l})=(a_l+\dot{a_l})A_b^{-1}.$$

Corollary 4. The condition of non-degeneracy of the matrix $\overline{A_b}$ formed by substitution of row $\widetilde{a_l}$ for row α_l which occupies the k-th row in the basic matrix is fulfillment of the condition $\exists i\ e_{ik}\neq 0$ where e_{ik} is an element of the inverse matrix A_b^{-1} such that α'_{li}. The new solution will be determined by the ratio

$$\bar{u}_{0j}=u_{0j}-\frac{e_{jk}}{\tilde{\alpha}_{lk}}\widetilde{\Delta}_l$$

if $\widetilde{\Delta}_l=0$, then $\bar{u}_{0j}=u_{0j}$, $j=\overline{1,m}$ (remains unchanged).

This is the condition of aimed recovery of nondegeneracy (rank completeness) of the basic matrix in the course of computation.

Corollary 5. If the leading element of simplex iteration $\alpha_{lk}\neq 0$, then the condition $\alpha'_{li}+\alpha_{lk}=0$ must be satisfied for the perturbation that preserves nondegeneracy of the "new" basic matrix (rank magnitude).

This is the condition of restoration of nondegeneracy (rank completeness).of the basic matrix with changes in the course of computation.

Corollary 6. Fulfillment of $\tilde{\alpha}_{lk}\neq 0$ is a necessary and sufficient condition for preserving nondegeneracy (rank completeness) of the new basic matrix $\overline{Ab}$ formed by substitution of the row $\tilde{a}_l$ for the row α_l occupying the k-th row in the basic matrix.

Validity of the corollaries follows directly from Theorem 1 [9] and organization of an algorithmic scheme for solving SLAE (11).

6. Analysis of the rank magnitude in "extension" and "narrowing" of the constraint matrix by "framing" the constraint matrix of the linear system

It is known that magnitude of the constraint matrix rank can be judged from properties of determinants of the major minors of the constraint matrix [1–3]. This is where the procedure of framing the minors applies. There are constraint matrices (for example Hilbert matrix) that are mathematically nondegenerate (full-rank matrices) but "can become" "non-rank matrices" in the case of "poor" organization of computation. It should be noted that the case of rank incompleteness (if the number of iterations for which $\alpha_{lk}^{(i)} \neq 0$ equals to r and conditions $r<m$, $r<n$ are fulfilled) can be considered as the most general one. This means that the property of rank completeness for SLAE (11) can be a starting property for its further study and, in particular, for restoring its completeness. The model may undergo further changes. Because of this, analysis of solvability when expanding (narrowing) dimension of the constraint matrix, i. e. including (excluding) new rows and columns or changing individual model elements [9, 11] can be considered as a variant of successive changes: the changes made for restoration of rank completeness and the like. Naturally, change in the model properties must be traced in such transformations.

The A_b (r-basic) matrix can be considered as a submatyrix of a matrix of larger dimension.

Elemental "portion" of changes in the model will be determined by framing of the basic (r-basic) matrix as well (by inclusion of a row and a column when expanding).

The next matrix will include the previous one, that is, we have a sequence of major minors of the matrix (order of such minors' changes). In a general case (Case III), the finite matrix A_b (r-basic) and the matrix inverse to it are determined by transformations in r iterations from the A matrix (12) by deleting substituted column-rows, respectively,

$$J_b=(i_1, i_2, ..., i_r)\subseteq J,\ I_b=(i_1, i_2, ..., i_r)\subset I.$$

Properties of a SLAE (11) with a rectangular matrix of constraints in perturbations taking place in the elements of a "group" of rows were studied in [11]. The algorithm provides for sequential replacement of rows of basic matrices respectively perturbed and calculation (replacement) of elements of the method in iterations. An important condition during iterations consists in preservation of non-degeneracy of the problem constraint matrix (Algorithm 1 [11]).

Similarly, the technology of impact of single column changes on solution of the SLAE [11] can be extended to analysis of impact of the changes in a group of columns by constructing a corresponding iterative procedure of accounting for the impact of changes in each column (Algorithm 2 [11]).

Based on Algorithms 1, 2, one can construct additions to the study of properties of the major minor expansion (changes in the group of row-columns). For example, by aimed framing with a group of rows and columns (when expanding the minor dimension: inclusion (exclusion) of new rows and columns (framing).

"Expansion" and "narrowing" of the matrix of SLAE constraints are studied in [11].

Let the initial r-basic matrix (with known properties) is a part of a "larger" constraint matrix ("framing"). Without loss of generality, we can assume that we have a major minor: the r-basic constraint matrix A_b.

Let us study the effect of framing a given minor with rows and columns of the "larger" constraint matrix.

Introduce auxiliary block-and-cell matrices structurally close to A_b and A_b^{-1} (r-basic matrix and inverted matrix) of the form:

$$A_0=\begin{pmatrix} a_{11} & a_{11} & \dots & a_{1r} & 0 \\ a_{21} & a_{22} & \dots & a_{2r} & 0 \\ . & . & \dots & . & 0 \\ a_{r1} & a_{r1} & \dots & a_{rr} & 0 \\ 0 & 0 & \dots & 0 & I_{r+1r+1} \end{pmatrix},$$

$$E_0=\begin{pmatrix} e_{11} & e_{11} & \dots & e_{1r} & 0 \\ e_{21} & e_{22} & \dots & e_{2r} & 0 \\ . & . & \dots & . & 0 \\ e_{r1} & e_{r1} & \dots & e_{rr} & 0 \\ 0 & 0 & \dots & 0 & I_{r+1\,r+1} \end{pmatrix},$$

where $I_{r+p\ r+p}$ is the unit-diagonal matrix of dimension p, and A_0 and E_0 contain A_b and A_b^{-1}, respectively, (introduced earlier).

By the known r-basic matrix and the A_b (minor) matrix inverted to it and A_b^{-1}, it is possible to determine properties of the framing matrix $\overline{A}$, $A_b \subset \overline{A}$ of the following form:

$$A_b=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r} \\ a_{21} & a_{22} & \dots & a_{2r} \\ . & . & \dots & . \\ a_{r1} & a_{r2} & \dots & a_{rr} \end{pmatrix},$$

$$\overline{A}=\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1r+p} \\ a_{21} & a_{22} & \dots & a_{2r+p} \\ . & . & \dots & . \\ a_{r+p1} & a_{r2} & \dots & a_{r+pr+p} \end{pmatrix},$$

which contain $r\times r$ and $(r+p)\times(r+p)$ elements, respectively, i. e. p columns and rows of framing are added to the minor (r-basic matrix A_b).

It is easy to make sure that:

– matrices A_0 and E_0 are straight and inverted matrices;

– establishment of properties of $\overline{A}$ matrix ("extended" matrix) in the assumption of known A_b and A_b^{-1} is based on sequential application of algorithms 2 (by columns) and 1 (by rows), respectively.

That is, based on A_b та A_b^{-1}:

– form auxiliary matrices A_0 and E_0;

– realize iterative transition from A_0 with inverted E_0 to $\overline{A}_b$, $\overline{A}_b^{-1}$ by successive replacement ("inclusion") of columns and rows ("framing");

– check conditions of resistance (increasing the minor rank) and recalculate elements of the method, in particular, changes in solution, and the inverse matrix;

– failure to comply with the condition of resistance indicates that the corresponding row and column are "unsuitable" for framing.

A new basic matrix (of full or incomplete rank, i. e. a maximum minor with a non-zero determinant) is formed in the course of iterations in accordance with the performed "extension" of the constraint matrix.

On the contrary, the matrix "narrowing", that is, establishment of A_b and A_b^{-1} properties by known $\overline{A}$, $\overline{A}^{-1}$ is based on "inverted" sequential application of algorithms 2 (by columns) and 1 (by rows), respectively. Transition to auxiliary A_0 and inverted E_0 matrices that include A_b and A_b^{-1} matrices is performed by successive replacement ("exclusion") of columns and rows with transition to the A_0 and

E_0 structure. A_b and A_b^{-1} are "separated" from them at the next stage.

7. On computational aspects of linear systems analysis by the method of basic matrices

It is known [1, 2] that IEEE, the standard of binary arithmetic, is the universally accepted standard for representing numbers. It was implemented at major computation centers and in all PC types. The IEEE standard provides 2 main types of floating-point numbers (32-bit and 64-bit). Accordingly, a concept of machine zero threshold "approximately" at a level of 2^{-126} and overflow threshold at a level of 2^{+128} was introduced for the first level. Zero threshold of 2^{-1022} and overflow threshold of 2^{+1024} was introduced for the second type. That is, the threshold of machine zero defines the zone of subnormal numbers as a set (in a certain neighborhood to zero) and the threshold of overflow limits the zone of normalized numbers (beyond the zone of subnormal numbers). Categories of mantis length, order value (and the method of rounding, performing operations) are fundamental in the introduced IEEE standard. They form basis of number representation, establish levels of overflow and machine zero thresholds. This determines basic quantitative estimates of errors in representation of numbers and in computation. In general, the standard may be organized in different ways, and the input parameters mentioned may differ. That is, loss of accuracy in representation of numbers caused by rounding is usually small.

7. 1. Leading element as an inaccurately determined value

When performing exact mathematical calculations in classical methods of Gaussian type, condition of inequality of the leading element of transformation to zero attracts attention. From a mathematical point of view, according to the fundamental formulas, division by the leading element in transformations is carried out (usually, division by zero is inadmissible). At the same time, this condition indicates linear dependence of rows, that is, non-full rank of the constraint matrix as well. In conditions of machine implementation of the method, "zero-non-zero" condition (inaccurate set) as pertaining to the zone of subnormal numbers gives the calculations a new content, i. e. influence on the rank magnitude.

7. 2. The condition of resistance as an inaccurate value in BMM

In the basic matrix method (BMM), the condition of resistance (as independence of the vector system) of Lemma [9] also indicates significance of the "zero-non-zero" condition (belonging to an inaccurate set) in the method medium.

It was found that the rank magnitude correlates with the number of fulfillments of conditions of resistance in the BMM scheme. Completeness of the rank as a condition of coincidence with the maximum number of fulfillments of resistance conditions.

In various scenarios of computation organization, the rank magnitude (its completeness) is clear and unambiguously defined in terms of mathematical representation, "acts" as inaccurate (belonging to the set) since it depends directly on the real threshold of mathematical zero (of the set, the zone of subnormal numbers). This occurs in the BMM condition of resistance and in inequality of the leading element to zero in the Gaussian method and generally in methods of the "simplex" type.

It is known that the Hilbert matrix is mathematically nondegenerate, and in a "successful" machine representation, it can give large errors in obtained solutions in the course of calculations. Study [10] has developed an appropriate way of representing numbers, performing basic operations and calculations in exact numbers. The Gaussian and BMM methods were implemented in exact numbers and appropriate experiments were performed for the CLAE, in particular, with the Hilbert constraint matrix. Subsequently, based on the exact format, a new type of number representation ("inaccurate" or with fixed decimal point) was developed, rounding, and basic operations were performed. Due to this format of representation, a possibility appeared to programmatically control the calculation process with various lengths of fractional part (to the right of the decimal point) and unlimited integers (within overflow). In this case, depending on the fractional part of the number, the method of rounding and performing basic operations, "own" machine zero is correspondingly set. This opened the possibility of calculating with typical models of different dimensions, different lengths of the fractional part (16, 32, 64, 128, 256), both in standard-exact and "inaccurate" algorithms of formats of number representation. Algorithms of the Gauss and BMM methods were used as basic. The Hilbert matrix was chosen as a typical structure. Against the background of poor conditionality of the Hilbert matrix, manifestations of discrepancies between mathematical (analytical) and machine representation of zero were found during the experiment, especially in the zones close to zero (Eps). The reference [14] provides relevant results and tools that were used in the experiment organization.

Some general features of the experiment should be emphasized:

– preservation of rank completeness and high accuracy of solution were observed in high-precision calculations with a great mantis length;

– a certain boundary exists at which rank completeness is still preserved but the error of the found solution becomes significant;

– a phase of rank completeness drop and a significant error in solution is observed beginning from a certain moment of time, that is, the moment of machine insolvability or inconsistency of SLAE solutions.

In the first case, we have a congruency between the standard mathematical solution and the machine solution. The second case indicates the effect of calculation errors which initially causes perturbations (in preserving linear independence of the vector-normals of constraints) and deviation of the solution (growth of the absolute error).

Further accumulation of errors leads to "convergence" (bonding) of the vectors of normals (loss of rank completeness) and formation of diversity in solvability or contradictory constraints (in incompatibility).

For example, if mantis of the fractional part is 32 binary digits, then 2^{-11} is the minimum machine zero neighborhood (value of 2^{-32}) at which the rank becomes incomplete for the dimension 10 of the constraint matrix. The following approximate dependence of the Eps neighborhood on the mantis length (Mant) is observed in the examples: $\mathrm{Eps}=2^{-\mathrm{Mant}+21}$.

In general, a non-monotonic dependence of the rank drop on the dimension is established if the rank "falls" for some dimension, it does not necessarily mean that it will “fall” for a higher dimension as well. Probably, the property of non-commutativity of computations (Table 1) also has its affect in a general case.

Table 1

Relationship between computation parameters and the rank completeness according to the experimental results

Mant	Eps	Dim	Rank	Completeness
32	2^{-11}	10	7	«fell»
32	2^{-11}	11	9	«fell»
32	2^{-11}	12	10	«fell»
32	2^{-11}	13	13	«not fell»
32	2^{-11}	14	12	«fell»
64	2^{-43}	10	10	«not fell»

In the Table 1, Mant is length of the mantis of binary digits of the fractional part (the mantis of the integer part is not limited), Eps is machine zero, Dim is dimension of the constraint matrix, Completeness is measure of the rank completeness (“fell”/”not fell”). Since mathematically rank of the Hilbert matrix must be complete, solution is unique but according to the computation results, different situations are possible in conditions of poor conditionality, so the term (“fell”/”not fell”) characterizing rank completeness was introduced.

In the case of the experiment with solution of SLAE by Gaussian method with a large-dimensional Hilbert matrix (Dim=50, Mant=32, Eps=2^{-11}), a situation was observed when all elements of the matrix under the leading element fell into the Eps-zero neighborhood. It is interesting that corresponding elements were not zeros in exact numbers. This influenced change of order of approximate calculations relative to the order of calculations in exact numbers. In doing so, the rank remained complete and a solution was obtained that had nothing in common with exact solution.

In practice, errors in the input data (mantis “trimming”) and poor conditionality of the Hilbert matrix leads to “instability” of calculations: these errors changed for the worse.

This indicates the need to combine calculations in exact and approximate (i. e. with fixed decimal point) numbers.

8. Discussion of results obtained in analysis of poorly conditioned matrix structures by means of algorithms of the basic matrix method

Theoretically, the following was established:

– significant role of the rank category in formation of solution structure, i. e. subspaces;

– significance of mathematical zero (of the leading element or condition of resistance) in formation of subspace dimensions (in the BMM structure).

A non-monotonic dependence of the rank drop on dimension of the linear system and other calculation parameters was experimentally established.

In general, the assumption of non-equivalence of theoretical mathematical zero (of the number) and real machine zero (inaccurately specified set) in “working” conditions with poorly conditioned systems was confirmed.

Presence of various options of computation organization (including exact number calculations) makes it possible to obtain versatile information about the model and mathematical (ideal or reference) model in exact numbers and machine (real) model with different accuracy of representation and calculation. It was established that the rank value is a key category in mathematical analytical (theoretical). analysis of properties of linear systems. The rank magnitude for SLAE in calculations is fundamental qualitative category for solution representation. Poor conditioning (for example the Hilbert matrix) leads in practice to “instability” of calculations in the zones close to the machine zero. The rank magnitude (obtained in calculations) outlines starting possibilities (and limitations) of methods and algorithms, creates preconditions for forming solutions that may be unique, not unique or non-existing.

On the other hand, quantitative value of the rank magnitude, its property of completeness substantially depends on the value of the leading element (or fulfillment of the condition of resistance in the method of basic matrices). It should be noted that definition of its value as “zero-non-zero” corresponds to the definition of belonging (or nonbelonging) to a subset of subnormality or close to it. This can be represented as a ratio of mathematically exact zero to inexact machine zero. Of course, boundaries of subnormal numbers are determined by the threshold of machine zero (for example, in the IEEE format) in PC representation of numbers with floating point. That is, for theses boundaries, mantis length, magnitude of order are extremely significant components. This can also be attributed to incorrect fixed point format entered.

Non-monotonic dependence of the rank drop on dimension was experimentally established. This indicates that availability of reference solution (finding it in exact numbers) is important when performing calculations. Development of “so-called” “exact” algorithms within the methods of calculating rational numbers makes it possible to check efficiency of algorithmic computations in a direct way. In particular, combined use of different variants of implementation of the method’s algorithm with different accuracy types (mantis length, order value) in exact numbers for a typical problem can reveal important regularities, for example for constructing membership functions and also for further use in deciding the order of computation organization with achievement of a given level of parameters (for example the rank completeness degree), i. e. mathematical apparatus of fuzzy sets, construction of membership functions.

Of course, calculation in different scenarios must be interrelated and explainable. In exact number calculation as equivalent of a perfect mathematical calculation and at different approximations (values of mantis and order), calculation can be recognized as an equivalent of a real one which determines solution to some extent.

9. Conclusions

1. Algorithm of the method of basic matrices of analysis of changes in inclusion-exclusion and changes in a group of rows and columns of SLAE without re-solving the task from beginning was improved. Conditions of compatibility (incompatibility) of system restrictions, uniqueness of solution, etc. were established. Structure of vectors of the fundamental solution system in a case of compatibility was elaborated.

2. Formats (“exact” and “inexact”) of representation of the model elements (mantis length, order value, thresholds of machine zero and overflow) as well as variants of orga-

nization of performing basic operations during calculations and their influence on solution properties were developed. In particular, influence on rank magnitude and completeness was traced on an example of an SLAE with a poorly conditioned constraint matrix.

3. It is known that when constructing the Lagrangian interpolation polynomial, application of the least-squares method or solving a boundary-value problem, etc., improvement of approximate solution is achieved by extending the corresponding system of basic functions. With this extension, the SLAE constraint matrix is changed by means of framing and the system may be poorly conditioned. Probably, values of the method elements can fall into the zone close to the machine zero during calculation. This necessitates refinement of organization and execution of calculations which can be achieved by the use of "exact" and "inexact" data types in representation of the method elements. That is why an experiment was conducted to analyze properties of the linear system at different dimensions, accuracy of input data and computation scenarios including those in exact numbers using the Hilbert matrix as an example. It was established that non-monotonous dependence of rank fall on dimensions, errors in input data and poor conditionality leads to "instability" of calculation.

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