Розглянуто задачу імовірнісного аналізу складної динамічної системи, яка в процесі функціонування в випадкові моменти часу переходить з одного стану в інший. Запропоновано методику розрахунку умовних ймовірностей попадання системи в заданий момент часу t в заданий стан за умови, що в початковий момент часу система перебувала в будь-якому з можливих станів. Вихідні дані для аналізу представляють собою безліч експериментально отриманих значень тривалості перебування системи в кожному з станів до відходу в інший стан. Апроксимація одержуваних при цьому гістограм з використанням розподілу Ерланга дає набір щільності розподілу тривалості перебування системи в можливих станах до відходу в інші стани. При цьому вибір належного порядку розподілу Ерланга забезпечує отримання адекватного опису напівмарковських процесів, що протікають в системі. Запропоновано математичну модель, що зв'язує отримані щільності розподілу з функціями, що визначають вірогідну динаміку системи. Модель описує випадковий процес переходів системи з будь-якого можливого початкового стану в будь-який інший стан протягом заданого тимчасового інтервалу. З використанням моделі отримана система інтегральних рівнянь щодо шуканих функцій, що описують імовірнісний процес переходів. Для вирішення цих рівнянь використано перетворення Лапласа. В результаті рішення системи інтегральних рівнянь отримані функції, що задають розподіл ймовірностей станів системи в будь-який момент часу t. Ці ж функції описують також і асимптотичний розподіл ймовірностей станів. Наведено наочний приклад вирішення задачі для випадку, коли щільності розподілу тривалостей перебування системи в можливих станах описані розподілами Ерланга другого порядку. Процедура вирішення задачі описана детально для найбільш природного окремого випадку, коли початковим є стан H_0

Ключові слова: динамічна система з безліччю можливих станів, випадковий процес переходів, інтегральні рівняння динаміки, перетворення Лапласа

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METHODOLOGY OF PROBABILISTIC ANALYSIS OF STATE DYNAMICS OF MULTI-DIMENSIONAL SEMI-MARKOV DYNAMIC SYSTEMS

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1. Introduction

A significant part of mathematical models of dynamic systems functioning processes are constructed and described in terms of general graph theory [1, 2]. It is usually assumed that the system can be in one of the possible states at any time and passes from that state to another under the influence of some random process. The distribution laws of the duration of stay in each of the possible states before transition to another state are taken to be defined (or can be obtained by statistical processing of the original data). In many practical cases, the problem of finding the stationary probability distribution of the system states can be formulated and solved. However, the theoretical and practical interest is to solve a more complex problem – finding the probability distribution of the system stay in its possible states at an arbitrary point of time after the start of functioning from a given initial state.

The solution to this problem under the most general assumptions regarding the nature of the influencing random process is practically impossible. However, this solution can be obtained for an important special case when the process is Markov [3, 4]. Comprehensive results are obtained for the case when the random process determining transitions from one state to another is discrete in the phase state space, and the distribution law of intervals between transitions is exponential [5, 6]. Relations for calculating the final state probability distributions are also obtained for semi-Markov systems, when the distribution densities of the length of stay before transition to another state are integrable functions [7, 8]. However, the problem of obtaining simple relations for calculating the probabilities of stay in each of the states at an arbitrary point in time has not been studied enough. The problem is as follows. There is no methodology relating two mathematical objects. The first is the distribution density of the duration of the system stay in each of the states until the transition to another state. The second is the desired functions describing the probability dynamics of the system stay in its possible states. The solution to this problem is an urgent task.

2. Literature review and problem statement

The problem of analysis of semi-Markov systems is discussed in numerous publications. [9] considers the problem of evaluating the efficiency of the system, the model of which is a queuing system with non-uniform arrivals. In this case, the final state probability distribution for the nested Markov chain is sought. In [10], the production system is investigated using the semi-Markov model. The analysis ends with the calculation of the final probability distribution of the system states [11] describes a queuing system with arbitrary arrivals. The result of the study is a stationary state probability distribution. In [12], the possibility of using semi-Markov models for the analysis of computer networks, transport networks, and Internet of things objects is investigated. The decision regarding the efficiency of the system is made on the basis of the final state probability distribution obtained. [13] explores a queuing system with non-Poisson arrivals and non-exponential service in order to obtain stationary performance characteristics. In [14], a queuing system with semi-Markov arrivals is investigated. The analysis of the system ends with the calculation of the final state probability distribution. Finally, in [15], an analysis of a queuing system with an arbitrary distribution of random service duration is made. To evaluate the efficiency of the system, the obtained stationary state probability distribution is used.

As a result of the review of the known publications on the problem of analysis of semi-Markov systems, the following conclusion can be drawn. The known theoretical results of the study of semi-Markov systems are limited to the calculation of the final probability distribution of the system states. This is sufficient for solving some practical problems. However, in many cases, for example, when solving problems of evaluating the efficiency of restored systems, it is essential to know the probability dynamics of the system stay in a set of functional states. The same problem is important for critical multi-channel service systems. The availability of such systems is determined by the probability that the number of normally functioning channels is not lower than the given one. Thus, in the theoretically and practically important direction of studying an extensive class of complex systems, the functioning model of which is described in terms of the theory of semi-Markov processes, there is a significant gap associated with studying the state probability dynamics of such systems.

3. The aim and objectives of the study

The aim of the study is to develop a methodology for determining the probability dynamics of the system stay in its possible states. When solving many practical problems, it is important to know not only the stationary distribution of the probabilities of the system states, but also the values of these probabilities at any time. This information makes it possible to solve the problems of system state management.

To achieve this aim, the following objectives are set.

– to develop a mathematical model establishing a relationship between a given set of distribution densities of random durations of the system stay in its possible states and functions describing the state probability dynamics;

– to develop a method for obtaining analytical relationships for the direct calculation of the probabilities of the system stay in possible states at an arbitrary point in time;

– to consider the implementation of the developed methodology for calculating the relationships describing the probabilistic dynamics of the states of the semi-Markov system using a specific example.

4. Development of a mathematical model of the probability dynamics of system states

We introduce a mathematical model of the probabilistic dynamics of the system states as follows. Let the semi-Markov system be in one of n possible states $(H_1,...,H_n)$. The system functions in an external environment, under which it passes from one state to another. A formal description of the mechanism of the environment and system interaction is given by the following set of distribution densities of random variables:

$f_{ij}(t)$ – distribution density of the duration of the system stay in the state H_i before transition to the state H_j; $i=1,2,...,n,\ \ j=1,2,...,n.$

The random dynamics of the system states is described by the set of functions:

$G_{ij}(t)$ – conditional probability of being in the state H_j at the time t, if at the initial moment the system was in the state H_i, $i=1,2,...,n,\ \ j=1,2,...,n.$

To find the unknown functions $G_{ij}(t)$, we introduce a system of integral equations

$$G_{ij}(t)=\int_0^t f_{ik}(\tau)G_{kj}(t-\tau)\mathrm{d}\tau,\quad i=1,2,...,n,\quad j=1,2,...,n.$$

Consider the implementation of the method using a simple example of a system with two possible states H_0 and H_1.

For this system, we introduce:

$f_{01}(t)$ – distribution density of the duration of the system stay in the state H_0 before transition to the state H_1;

$f_{10}(t)$ – distribution density of the duration of the system stay in the state H_1 before transition to the state H_0;

$G_{00}(t)$ – conditional probability of being in the state H_0 at the time t if the object is in the state H_0 at the initial time;

$G_{01}(t)$ – conditional probability of being in the state H_1 at the time t if the object is in the state H_0 at the initial time;

$G_{10}(t)$ – conditional probability of being in the state H_0 at the time t if the object is in the state H_1 at the initial time;

$G_{11}(t)$ – conditional probability of being in the state H_1 at the time t if the object is in the state H_1 at the initial time.

We record the system of equations for the unknown functions $G_{00}(t),\ G_{01}(t),\ G_{10}(t),\ G_{11}(t)$.

The object that is in the state H_0 at the initial time may be in the state H_0 when one of two possible independent options is implemented. Firstly, the object can stay in H_0 without leaving this state for the entire interval $[0,t]$. Secondly, the object can leave the state H_0 at some time $\tau\in[0,t]$, then returning to the state H_0 by the time t. Hence:

$$G_{00}(t)=\left(1-\int_0^t f_{01}(\tau)\mathrm{d}\tau\right)+\int_0^t f_{01}(\tau)\cdot G_{10}(t-\tau)\mathrm{d}\tau. \tag{1}$$

The object that is in the state H_0 at the initial time can be in the state H_1 passing into that state at the time $\tau\in[0,t)$, then in the interval $(\tau,t]$ remaining until the time t, making a number of transitions from the state H_1 returning to it by the time t. Wherein:

$$G_{01}(t)=\int_0^t f_{01}(\tau)\cdot G_{11}((-\tau)\mathrm{d}\tau. \tag{2}$$

The object that is in the state H_1 at the initial time can be in the state H_0 passing into that state at the time $\tau\in[0,t)$, then in the interval $(\tau,t]$ remaining until the time t, making a number of transitions from the state H_0 returning to it by the time t. Wherein:

$$G_{10}(t)=\int_0^t f_{10}(\tau)\cdot G_{00}(t-\tau)\mathrm{d}\tau. \tag{3}$$

Finally, the object that is in the state H_1 at the initial time can stay in this state until the time t, or, leaving that state at the time $\tau\in[0,t)$, return to it at the time t. Wherein:

$$G_{11}(t)=\left(1-\int_0^t f_{10}(\tau)\mathrm{d}\tau\right)+\int_0^t f_{10}(\tau)\cdot G_{01}(t-\tau)\mathrm{d}\tau. \tag{4}$$

The system of integral equations (1)–(4) forms a mathematical model that relates the known distribution densities of the lengths of the system stay in possible states and the desired functions that describe the probabilistic dynamics of the system. We use this model.

Note that when constructing the model, no restrictions were imposed on the nature of the densities. Thus, this model can be used for probabilistic analysis of any semi-Markov system. The resulting system of equations (1)–(4) is solved using the Laplace transform [16–18].

The Laplace transform of the function $u(t)$ is the function:

$$L(u(t))=\int_0^\infty u(t)e^{-st}\mathrm{d}t. \tag{5}$$

To simplify recording, it is convenient to introduce $L(u(t))=u^*(s)$.

Taking into account the properties of the Laplace transform, we record equations (1)–(4) in operator form.

If we integrate (5) in parts, then:

$$L\left(\int_0^t u(\tau)\mathrm{d}\tau\right)=\int_0^\infty e^{-st}\left(\int_0^t u(\tau)\mathrm{d}\tau\right)=$$

$$=-\frac{1}{s}\left[\left(e^{-st}\int_0^t u(\tau)\mathrm{d}\tau\right)\Bigg|_0^\infty-\int_0^\infty e^{-st}u(\tau)\mathrm{d}\tau\right]=$$

$$=\frac{1}{s}L(u(\tau))=\frac{1}{s}u^*(s).$$

In this case, the Laplace images of the relations (1)–(4) will have the form:

$$G_{00}^*(s)=\frac{1}{s}\left(1-f_{01}^*(s)\right)+f_{01}^*(s)\cdot G_{10}^*(s), \tag{6}$$

$$G_{01}^*(s)=f_{01}^*(s)\cdot G_{11}^*(s), \tag{7}$$

$$G_{10}^*(s)=f_{10}^*(s)\cdot G_{00}^*(s), \tag{8}$$

$$G_{11}^*(s)=\frac{1}{s}\left(1-f_{10}^*(s)\right)+f_{10}^*(s)\cdot G_{01}^*(s). \tag{9}$$

The resulting system of equations breaks down into two pairs {(6), (8)} and {(7), (9)}, each containing two unknown functions. We have:

$$G_{00}^*(s)=\frac{1}{s}\left(1-f_{01}^*(s)\right)+f_{01}^*(s)\cdot G_{10}^*(s),$$

$$G_{10}^*(s)=f_{10}^*(s)\cdot G_{00}^*(s).$$

Substituting the second of these equations into the first, we get:

$$G_{00}^*(s)=\frac{1}{s}\left(1-f_{01}^*(s)\right)+f_{01}^*(s)\cdot f_{10}^*(s)\cdot G_{00}^*(s).$$

Hence:

$$G_{00}^*(s)\cdot\left(1-f_{01}^*(s)\cdot f_{10}^*(s)\right)=\frac{1}{s}\left(1-f_{01}^*(s)\right),$$

$$G_{00}^*(s)=\frac{1}{s}\cdot\frac{1-f_{01}^*(s)}{1-f_{01}^*(s)\cdot f_{10}^*(s)}. \tag{10}$$

Substituting (10) in (8), we get:

$$G_{10}^*(s)=\frac{1}{s}\cdot\frac{\left(1-f_{01}^*(s)\right)\cdot f_{10}^*(s)}{1-f_{01}^*(s)\cdot f_{10}^*(s)}. \tag{11}$$

Similarly:

$$G_{01}^*(s)=f_{01}^*(s)\cdot G_{11}^*(s),$$

$$G_{11}^*(s)=\frac{1}{s}\left(1-f_{10}^*(s)\right)+f_{10}^*(s)\cdot f_{01}^*(s)\cdot G_{11}^*(s).$$

Hence:

$$G_{11}^*(s)\cdot\left(1-f_{10}^*(s)\cdot f_{01}^*(s)\right)=\frac{1}{s}\left(1-f_{10}^*(s)\right),$$

$$G_{11}^*(s)=\frac{1}{s}\cdot\frac{1-f_{10}^*(s)}{1-f_{10}^*(s)\cdot f_{01}^*(s)}. \tag{12}$$

Now substitute (12) in (7):

$$G_{01}^*(s)=\frac{1}{s}\cdot\frac{\left(1-f_{10}^*(s)\right)\cdot f_{01}^*(s)}{1-f_{10}^*(s)\cdot f_{01}^*(s)}. \tag{13}$$

We use the obtained general relations describing the Laplace images of the desired functions to solve a specific problem. Let the restored system be in one of two states:

– H_0 – the system is functioning normally;
– H_1 – the system is restored after failure.

We first perform calculations for the simplest case when the system is Markov. We set the distribution density of the duration of stay in each of the states before transition to another state as follows:

$$f_{01}(t)=\lambda e^{-\lambda t}, \quad f_{10}(t)=\mu e^{-\mu t}.$$

Wherein:

$$f_{01}^*(s)=\frac{\lambda}{s+\lambda}, \quad f_{10}^*(s)=\frac{\mu}{s+\mu}. \tag{14}$$

Substituting (14) in (10)–(13), we obtain analytical descriptions of the images $G_{00}^*(s)$, $G_{01}^*(s)$, $G_{10}^*(s)$, $G_{11}^*(s)$ corresponding to the given initial data.

Wherein:

$$G_{00}^*(s)=\frac{1}{s}\cdot\frac{1-\dfrac{\lambda}{s+\lambda}}{1-\dfrac{\lambda}{s+\lambda}\cdot\dfrac{\mu}{s+\mu}}=$$

$$=\frac{1}{s}\cdot\frac{s(s+\mu)}{s^2+s(\lambda+\mu)}=\frac{s+\mu}{s\cdot(s+\lambda+\mu)}. \tag{15}$$

We perform the inverse Laplace transform by decomposing (15) into elementary fractions. Find the roots of the polynomial in the denominator, solving the equation, whence:

$$s_1=0, \quad s_2=-(\lambda+\mu).$$

Now we rewrite (15) as follows:

$$\frac{s+\mu}{s(s+\lambda+\mu)}=\frac{\alpha}{s-s_1}+\frac{\beta}{s-s_2}=$$

$$=\frac{\alpha}{s}+\frac{\beta}{s+\lambda+\mu}. \tag{16}$$

After bringing to a common denominator, we get:

$$\frac{s+\mu}{s(s+\lambda+\mu)}=\frac{(s+\lambda+\mu)\alpha+s\beta}{s(s+\lambda+\mu)}=$$

$$=\frac{s(\alpha+\beta)+\alpha(\lambda+\mu)}{s(s+\lambda+\mu)}. \tag{17}$$

We find the unknown coefficients α and β in (16) by solving the system of equations obtained after equating the coefficients of like powers of s in the numerators of fractions on the left and right in (17). We have:

$$\begin{cases}\alpha+\beta=1,\\ \alpha\lambda+\alpha\mu=\mu.\end{cases}$$

Hence:

$$\alpha=\frac{\mu}{\lambda+\mu}, \quad \beta=\frac{\lambda}{\lambda+\mu}. \tag{18}$$

Substituting (18) in (16), we get:

$$G_{00}^*(s)=\frac{s+\mu}{s(s+\lambda+\mu)}=$$

$$=\frac{\mu}{\lambda+\mu}\cdot\frac{1}{s}+\frac{\lambda}{\lambda+\mu}\cdot\frac{1}{s+\lambda+\mu}.$$

Using the correspondence table of the functions and their Laplace transforms, we record:

$$G_{00}(t)=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t}. \tag{19}$$

The inverse transformations for the images of other functions describing the laws of probability distribution are given without explanation.

$$G_{10}^*(s)=\frac{1}{s}\cdot\frac{\left(1-f_{01}^*(s)\right)\cdot f_{10}^*(s)}{1-f_{01}^*(s)\cdot f_{10}^*(s)}=$$

$$=\frac{1}{s}\cdot\frac{\left(1-\dfrac{\lambda}{s+\lambda}\right)\cdot\dfrac{\mu}{s+\mu}}{1-\dfrac{\lambda}{s+\lambda}\cdot\dfrac{\mu}{s+\mu}}=\frac{1}{s}\cdot\frac{s\mu}{s^2+s(\lambda+\mu)}=$$

$$=\frac{\mu}{s(s+\lambda+\mu)}=\frac{\alpha}{s}+\frac{\beta}{s+\lambda+\mu}=$$

$$=\frac{\alpha s+\alpha(\lambda+\mu)+\beta s}{s(s+\lambda+\mu)}=\frac{s(\alpha+\beta)+\alpha(\lambda+\mu)}{s(s+\lambda+\mu)},$$

$$\begin{cases}\alpha+\beta=0,\\ \alpha(\lambda+\mu)=\mu.\end{cases}$$

Hence:

$$\alpha=\frac{\mu}{\lambda+\mu}, \quad \beta=\frac{\mu}{\lambda+\mu}.$$

$$G_{00}^*(s)=\frac{\mu}{\lambda+\mu}\cdot\frac{1}{s}-\frac{\mu}{\lambda+\mu}\cdot\frac{1}{s+\lambda+\mu}. \tag{20}$$

$$G_{00}(t)=\frac{\mu}{\lambda+\mu}-\frac{\mu}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t}. \tag{21}$$

Then:

$$G_{11}^*(s)=\frac{1}{s}\cdot\frac{1-f_{10}^*(s)}{1-f_{10}^*(s)\cdot f_{01}^*(s)}=$$

$$=\frac{1}{s}\cdot\frac{1-\dfrac{\mu}{s+\mu}}{1-\dfrac{\mu}{s+\mu}\cdot\dfrac{\lambda}{s+\lambda}}=\frac{s+\lambda}{s(s+\lambda+\mu)}=$$

$$=\frac{\alpha}{s}+\frac{\beta}{s+\lambda+\mu}=\frac{\alpha(s+\lambda+\mu)+\beta s}{s(s+\lambda+\mu)}=$$

$$=\frac{s(\alpha+\beta)+\alpha(\lambda+\mu)}{s(s+\lambda+\mu)}\infty.$$

$$\begin{cases}\alpha+\beta=1,\\ \alpha(\lambda+\mu)=\lambda.\end{cases}$$

Hence:

$$\alpha=\frac{\lambda}{\lambda+\mu}, \quad \beta=\frac{\mu}{\lambda+\mu}.$$

$$G_{11}^{*}(s)=\frac{\lambda}{\lambda+\mu}\cdot\frac{1}{s}+\frac{\mu}{\lambda+\mu}\cdot\frac{1}{s+\lambda+\mu}.$$

$$G_{11}(s)=\frac{\lambda}{\lambda+\mu}+\frac{\mu}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t}. \quad (22)$$

Finally,

$$G_{01}^{*}(s)=\frac{1}{s}\cdot\frac{\left(1-f_{10}^{*}(s)\right)\cdot f_{01}^{*}(s)}{1-f_{10}^{*}(s)\cdot f_{01}^{*}(s)}=$$

$$=\frac{1}{s}\cdot\frac{\left(1-\frac{\mu}{s+\mu}\right)\cdot\frac{\lambda}{s+\lambda}}{1-\frac{\mu}{s+\mu}\cdot\frac{\lambda}{s+\mu}}=\frac{1}{s}\cdot\frac{s\lambda}{s^{2}+s(\lambda+\mu)}=$$

$$=\frac{\lambda}{s(s+\lambda+\mu)}=\frac{\alpha}{s}+\frac{\beta}{s+\lambda+\mu}=$$

$$=\frac{\alpha(s+\lambda+\mu)+\beta s}{s(s+\lambda+\mu)}=\frac{s(\alpha+\beta)+\alpha(\lambda+\mu)}{s(s+\lambda+\mu)},$$

$$\begin{cases}\alpha+\beta=0,\\ \alpha(\lambda+\mu)=\lambda.\end{cases}$$

Hence:

$$\alpha=\frac{\lambda}{\lambda+\mu},\quad \beta=-\frac{\lambda}{\lambda+\mu}.$$

$$G_{01}^{*}(s)=\frac{\lambda}{\lambda+\mu}\cdot\frac{1}{s}-\frac{\lambda}{\lambda+\mu}\cdot\frac{1}{s+\lambda+\mu}.$$

$$G_{01}(t)=\frac{\lambda}{\lambda+\mu}-\frac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t}. \quad (23)$$

If the initial state of the system is normal functioning, the result of solving the problem is the following functions describing the dynamics of states:

$$G_{00}(t)=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t},$$

$$G_{01}(t)=\frac{\lambda}{\lambda+\mu}-\frac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t}.$$

In this case, of course, $G_{00}(t)+G_{01}(t)=1$.

The obtained relations determine the values of the probabilities of the system stay in the states H_0 and H_1 at an arbitrary time t. In particular, it follows that these values asymptotically approach their stationary values:

$$P(H_0)=\frac{\mu}{\lambda+\mu},\quad P(H_1)=\frac{\lambda}{\lambda+\mu}. \quad (24)$$

Let us now consider a more complex situation when the system functioning processes are semi-Markov. Let us describe the distribution densities of the length of the system stay in each of the states before transition to another state by second-order Erlang distributions:

$$f_{01}(t)=t\lambda^{2}e^{-\lambda t},\quad f_{10}(t)=t\mu^{2}e^{-\mu t}.$$

The Laplace images of these functions have the form:

$$f_{01}^{*}(s)=\frac{\lambda^{2}}{(s+\lambda)^{2}},\quad f_{10}^{*}(s)=\frac{\mu^{2}}{(s+\mu)^{2}}. \quad (25)$$

It is clear that out of the four functions $G_{00}(t)$, $G_{01}(t)$, $G_{10}(t)$, $G_{11}(t)$, only the first of them $G_{00}(t)$ is of practical interest. Accordingly, we substitute (25) in (10). In this case, we get:

$$G_{00}^{*}(s)=\frac{1}{s}\cdot\frac{1-f_{01}^{*}(s)}{1-f_{01}^{*}(s)\cdot f_{10}^{*}(s)}=\frac{1}{s}\cdot\frac{1-\frac{\lambda^{2}}{(s+\lambda)^{2}}}{1-\frac{\lambda^{2}}{(s+\lambda)^{2}}\cdot\frac{\mu^{2}}{(s+\lambda)^{2}}}=$$

$$=\frac{1}{s}\cdot\frac{\left[(s+\lambda)^{2}-\lambda^{2}\right]\cdot(s+\lambda)^{2}\cdot(s+\mu)^{2}}{(s+\lambda)^{2}\cdot\left[(s+\lambda)^{2}\cdot(s+\mu)^{2}-\lambda^{2}\cdot\mu^{2}\right]}=$$

$$=\frac{1}{s}\cdot\frac{\left(s^{2}+2s\lambda\right)\cdot(s+\mu)^{2}}{(s+\lambda)^{2}\cdot(s+\mu)^{2}-\lambda^{2}\cdot\mu^{2}}=$$

$$=\frac{(s+2\lambda)\cdot(s+\mu)^{2}}{\left[(s+\lambda)\cdot(s+\mu)-\lambda\mu\right]\cdot\left[(s+\lambda)\cdot(s+\mu)+\lambda\mu\right]}=$$

$$=\frac{(s+2\lambda)\cdot\left(s^{2}+2s\mu+\mu^{2}\right)}{s\cdot(s+\lambda+\mu)\cdot\left(s^{2}+s\cdot(\lambda+\mu)+2\lambda\mu\right)}=$$

$$=\frac{s^{3}+2s^{2}\cdot(\lambda+\mu)+s\cdot\left(4\lambda\mu+\mu^{2}\right)+2\lambda\mu^{2}}{s\cdot(s+\lambda+\mu)\cdot\left(s^{2}+(\lambda+\mu)+2\lambda\mu\right)}. \quad (26)$$

The structure and analytical representation of the problem solution depend on the nature of the denominator roots in (26). The first two roots are determined directly:

$$s_0=0,\quad s_1=-(\lambda+\mu).$$

The last two roots are obtained by solving the equation:

$$s^{2}+s(\lambda+\mu)+2\lambda\mu=0. \quad (27)$$

Wherein:

$$s_{2,3}=-\frac{\lambda+\mu}{2}\pm\sqrt{\frac{(\lambda+\mu)^{2}}{4}-2\lambda\mu}=-\frac{\lambda+\mu}{2}\pm\sqrt{D}. \quad (28)$$

If is the discriminant $D>0$, then the equation (27) has two different real roots:

$$s_2=-\frac{\lambda+\mu}{2}+\sqrt{D},$$

$$s_3=-\frac{\lambda+\mu}{2}-\sqrt{D}. \quad (29)$$

If the discriminant $D=0$, then the roots

$$s_2=s_3=-\frac{\lambda+\mu}{2}.$$

If, finally, $D<0$, then the roots s_2 and s_3 are complex.

In all these cases, the expression for the inverse Laplace transform is found using the decomposition of (26) into elementary fractions.

If all the roots are real and different, this decomposition is as follows:

$$G_{00}^*(s)=\frac{\alpha_0}{s_0}+\frac{\alpha_1}{s-s_1}+\frac{\alpha_2}{s-s_2}+\frac{\alpha_3}{s-s_3}=$$
$$=\frac{\alpha_0(s-s_1)(s-s_2)(s-s_3)+\alpha_1 s(s-s_2)(s-s_3)}{s(s-s_1)(s-s_2)(s-s_3)}+$$
$$+\frac{\alpha_2 s(s-s_1)(s-s_3)+\alpha_3 s(s-s_1)(s-s_2)}{s(s-s_1)(s-s_2)(s-s_3)}=$$
$$=\frac{A(s)}{s(s-s_1)(s-s_2)(s-s_3)}. \tag{30}$$

After multiplication and cancellation of like terms, this decomposition takes the form:

$$A(s)=s^3(\alpha_0+\alpha_1+\alpha_2+\alpha_3))+$$
$$+s^2\begin{bmatrix}-\alpha_0(s_1+s_2+s_3)+\alpha_1(s_2+s_3)+\\+\alpha_2(s_1+s_3)+\alpha_3(s_1+s_2)\end{bmatrix}+$$
$$+s\begin{bmatrix}\alpha_0(s_1s_2+s_1s_3+s_2s_3)+\\+\alpha_1s_2s_3+\alpha_2s_1s_3+\alpha_3s_1s_2\end{bmatrix}-\alpha_0s_1s_2s_3. \tag{31}$$

Now, equating the coefficients of like powers of s in (26) and (31), we obtain a system of equations with respect to α_0, α_1, α_2, α_3:

$$\begin{cases}\alpha_0+\alpha_1+\alpha_2+\alpha_3=1,\\-\alpha_0(s_1+s_2+s_3)+\alpha_1(s_2+s_3)+\\+\alpha_2(s_1+s_3)+\alpha_3(s_1+s_2)=2(\lambda+\mu),\\\alpha_0(s_1s_2+s_1s_3+s_2s_3)+\alpha_1s_2s_3+\\+\alpha_2s_1s_3+\alpha_3s_1s_2=4\lambda\mu+\mu^2,\\-\alpha_0s_1s_2s_3=2\lambda\mu^2.\end{cases} \tag{32}$$

Since

$$D=\frac{(\lambda+\mu)^2}{4}-2\lambda\mu,$$

then:

$$s_2+s_3=-\frac{\lambda+\mu}{2}+\sqrt{D}-\frac{\lambda+\mu}{2}-\sqrt{D}=-(\lambda+\mu),$$

$$s_1+s_3=-(\lambda+\mu)-\frac{\lambda+\mu}{2}+\sqrt{D}=-\frac{3}{2}(\lambda+\mu)+\sqrt{D},$$

$$s_1+s_2=-(\lambda+\mu)-\frac{\lambda+\mu}{2}-\sqrt{D}=-\frac{3}{2}(\lambda+\mu)-\sqrt{D},$$

$$s_1+s_2+s_3=-2(\lambda+\mu), \tag{33}$$

$$s_2s_3=\left(\frac{\lambda+\mu}{2}\right)^2-\left[\left(\frac{\lambda+\mu}{2}\right)^2-2\lambda\mu\right]=2\lambda\mu,$$

$$s_1s_2=-\frac{\lambda+\mu}{2}\left(-\frac{\lambda+\mu}{2}-\sqrt{D}\right)=\frac{(\lambda+\mu)^2}{4}+\frac{\lambda+\mu}{2}\sqrt{D},$$

$$s_1s_3=-\frac{\lambda+\mu}{2}\left(-\frac{\lambda+\mu}{2}+\sqrt{D}\right)=\frac{(\lambda+\mu)^2}{4}-\frac{\lambda+\mu}{2}\sqrt{D},$$

$$s_1s_2s_3=-(\lambda+\mu)\cdot 2\lambda\mu.$$

In view of (33), the solution of the system of equations (32) is as follows:

$$\alpha_0=\frac{\mu}{\lambda+\mu},\quad \alpha_1=-\frac{\lambda}{\lambda+\mu},$$

$$\alpha_2=\alpha_3=\frac{\lambda}{\lambda+\mu}. \tag{34}$$

Now, using (30) and (34), we record the result of the inverse transformation $G_{00}^*(s)$:

$$G_{00}(t)=\frac{\mu}{\lambda+\mu}-\frac{\lambda}{\mu}e^{-(\lambda+\mu)t}+$$
$$+\frac{\lambda}{\lambda+\mu}e^{-\left(\frac{\lambda+\mu}{2}+D\right)t}+\frac{\lambda}{\lambda+\mu}e^{-\left(\frac{\lambda+\mu}{2}-D\right)t}. \tag{35}$$

Since in the present case

$$\left(\frac{\lambda+\mu}{4}\right)^2>2\lambda\mu,$$

the probability (35) of the system stay in the state H_0 at the time t has a stationary value equal to, as in the Markov case,

$$P(H_0)=\frac{\mu}{\lambda+\mu}.$$

Let now $D<0$. The roots of the equation (27) are equal to:

$$s_1=-(\lambda+\mu),$$

$$s_2=-\frac{(\lambda+\mu)}{2}+i\sqrt{|D|},$$

$$s_3=-\frac{(\lambda+\mu)}{2}-i\sqrt{|D|},$$

and the decomposition of (26) into elementary factors has the form:

$$G_{00}^*(s)=\frac{\alpha_0}{s_0}+\frac{\alpha_1}{s-s_1}+\frac{\alpha_2 s+\alpha_3}{s^2-s(s_2+s_3)+s_2s_3}=$$
$$=\frac{\alpha_0(s-s_1)(s^2-s(s_2+s_3)+s_2s_3)}{s(s-s_1)(s^2-s(s_2+s_3)+s_2s_3)}+$$
$$+\frac{\alpha_1 s(s^2-s(s_2+s_3)+s_2s_3)+\alpha_1 s^2(s-s_1)+\alpha_3 s(s-s_1)}{s(s-s_1)(s^2-s(s_2+s_3)+s_2s_3)}=$$
$$=\frac{1}{B}\begin{bmatrix}s^3(\alpha_0+\alpha_1+\alpha_2)-s^2\begin{pmatrix}\alpha_0(s_1+s_2+s_3)+\alpha_1s_2+\\+\alpha_1s_3+\alpha_2s_1-\alpha_3\end{pmatrix}+\\+s(\alpha_0(s_1s_2+s_1s_3+s_2s_3)+\alpha_1s_2s_3-\alpha_3s_1)-\alpha_0s_1s_2s_3\end{bmatrix}, \tag{36}$$

$$B=s(s-s_1)(s^2-s(s_2+s_3)+s_2s_3).$$

After equating the coefficients of like powers of s in (26) and (36), we obtain the system of equations:

$$\begin{cases} \alpha_1+\alpha_2+\alpha_3=1, \\ -\alpha_0(s_1+s_2+s_3)+\alpha_1 s_2+\alpha_1 s_3+\alpha_2 s_1-\alpha_3=2(\lambda+\mu), \\ \alpha_0(s_1 s_2+s_1 s_3+s_2 s_3)+\alpha_1 s_2 s_3-\alpha_3 s_1=4\lambda\mu+\mu^2, \\ -\alpha_0 s_1 s_2 s_3=2\lambda\mu^2. \end{cases}$$

The solution to this system:

$$\alpha_0=\frac{\mu}{\lambda+\mu},$$

$$\alpha_1=\alpha_2=\frac{\lambda}{2(\lambda+\mu)},$$

$$\alpha_3=-\frac{\mu}{2(\lambda+\mu)^2}\left[(\mu+2\lambda)^2+\lambda^2\right].$$

Thus:

$$G_{00}^*(s)=\frac{\mu}{\lambda+\mu}\cdot\frac{1}{s}+\frac{\lambda}{2(\lambda+\mu)}\frac{1}{s-s_1}+ \\ +\frac{\frac{\lambda}{2(\lambda+\mu)}\cdot s+\frac{\mu}{2(\lambda+\mu)^2}\left[(\mu+2\lambda)^2+\lambda^2\right]}{s^2-s(s_2+s_3)+s_2 s_3}. \tag{37}$$

We bring the third term in (37) to the form convenient for performing the inverse Laplace transform. Wherein:

$$s^2-s(s_2+s_3)+s_2 s_3= \\ =s^2-2s\frac{s_2+s_3}{2}+\frac{(s_2+s_3)^2}{4}+s_2 s_3-\frac{(s_2+s_3)^2}{4}= \\ =\left(s-\frac{s_2+s_3}{2}\right)^2-\left(\frac{s_2+s_3}{2}\right)^2+2\lambda\mu= \\ =\left(s+\frac{\lambda+\mu}{2}\right)^2-\frac{(\lambda+\mu)^2}{4}+2\lambda\mu= \\ =(s+b)^2+a^2,\ a>0. \tag{38}$$

Then:

$$\alpha_1 s+\alpha_3=\frac{\lambda s}{2(\lambda+\mu)}+\frac{\mu}{2(\lambda+\mu)^2}\left[\mu^2+4\lambda\mu+5\lambda^2\right]= \\ =\frac{\lambda}{2(\lambda+\mu)}\left[s+\frac{\mu}{\lambda(\lambda+\mu)}\left(\mu^2+4\lambda\mu+5\lambda^2\right)\right]= \\ =\frac{\lambda}{2(\lambda+\mu)}\left[s+\frac{\mu}{\lambda(\lambda+\mu)}\left((\lambda+\mu)^2+2\lambda\mu+4\lambda^2\right)\right]= \\ =\frac{\lambda}{2(\lambda+\mu)}\left[s+\frac{\mu}{\lambda}(\lambda+\mu)+\frac{\mu(2\lambda\mu+4\lambda^2)}{\lambda(\lambda+\mu)}\right]= \\ =\frac{\lambda}{2(\lambda+\mu)}\left[s+\frac{\lambda+\mu}{2}+\frac{\lambda\mu+\mu^2-\lambda^2}{2\lambda}+\frac{\mu(2\lambda\mu+4\lambda^2)}{\lambda(\lambda+\mu)}\right]= \\ =\frac{\lambda}{2(\lambda+\mu)}\left[s+\frac{\lambda+\mu}{2}+\frac{\mu^3-\lambda^3+6\lambda\mu^2+8\lambda^2\mu}{2\lambda(\lambda+\mu)}\right]= \\ =c(s+b+d). \tag{39}$$

In view of (38) and (39), the relation (37) takes the form:

$$G_{00}^*(s)=\frac{\mu}{\lambda+\mu}\cdot\frac{1}{s}+\frac{\lambda}{2(\lambda+\mu)}\frac{1}{s-s_1}+ \\ +c\cdot\frac{s+\frac{\lambda+\mu}{2}}{(s+b)^2+a^2}+cd\cdot\frac{1}{(s+b)^2+a^2}. \tag{40}$$

We get the result of the inverse transformation $G_{00}^*(s)$:

$$G_{00}(t)=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{2(\lambda+\mu)}e^{-\frac{(\lambda+\mu)}{2}t}+ \\ +c\cdot\cos at\cdot e^{-\frac{(\lambda+\mu)}{2}t}+\frac{cd}{a}\sin at\cdot e^{-\frac{(\lambda+\mu)}{2}t}. \tag{41}$$

The solution to the problem is completed. The resulting relation determines the probability of the system stay in the state H_0 at any time t. If, as in the considered special case, reliability analysis of the restored system is carried out, this relation allows to reasonably formulate and solve the optimization problem of increasing the system efficiency using standard parameter management technologies.

5. Discussion of the results of developing the methodology of probabilistic analysis of non-Markov systems

Thus, the methodology of probabilistic analysis of semi-Markov systems is proposed. The methodology is based on the model of the probability dynamics of the system states. The model contains a set of integral equations for the unknown functions describing the probabilistic dynamics of the system (1) –(4). The solution of these integral equations is obtained using the Laplace transform (5)–(13). As a result of solving the integral equations, the desired relation (41) is obtained for calculating the conditional probability of the object stay in the state H_0 at an arbitrary time t if the object was in the state H_0 at the initial time. Thus, the proposed methodology, in contrast to the known ones, allows not only to calculate the final probability distribution of the system, but also the probability value of any state at an arbitrary time t. The obtained relations, firstly, make it possible to solve the problems of evaluating the efficiency of a system depending on the values of a given set of its parameters. Secondly, they can be used to optimize the limited resource allocation management in order to increase system efficiency.

Note that the proposed methodology for analyzing the functioning of dynamic systems is generalized in the following directions.

Firstly, this methodology can be applied if the analyzed system has $m>2$ states. In this case, it is necessary to introduce and analytically describe m^2 conditional probabilities of the system stay in each of the states at the time t, provided that at the initial time the system was in another state. It is important that the complexity of solving this problem does not depend on the number of system possible states.

Secondly, when solving the problem of statistical processing of initial data on the duration of system transformation in each of the possible states, it is advisable to use a more adequate model than the above – Erlang distribution of an arbitrary order n, that is

$$f(t)=\frac{t^{n-1}\lambda^n}{(n-1)!}e^{-\lambda t}.$$

The accuracy of the histogram approximation naturally increases, but the complexity of the problem solution remains practically the same due to the fact that the Laplace transform of the n order Erlang distribution has the form:

$$L(f(t))=\frac{\lambda^n}{(s-\lambda)^n}.$$

The resulting feature consists in the need to find the roots of an algebraic equation of $2n,\ n>2$ degree. Of course, an analytical solution to this problem is impossible, but numerical is always feasible, which significantly increases the applicability of the proposed method.

6. Conclusions

1. The methodology for analyzing the system, which in the process of functioning at random times passes from one state to another is proposed. The methodology is based on the proposed mathematical model of the relationship between the distribution densities of the duration of the system stay in possible states and the functions describing the system dynamics. To describe the probabilistic mechanism of transitions, the Erlang distribution is used.

2. The methodology allows obtaining relations for calculating the conditional probabilities of the system stay in any of its possible states at the time t, provided that the system was in another state at the initial time.

3. The implementation of the proposed methodology for solving a specific problem of probabilistic analysis of reliability of the restored system is considered.

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