Наведено алгоритм аналітичного розв'язку однієї з задач механіки пружних тіл, що пов'язана з вивченням власних коливань складеної двохступеневої пластинки, в якій увігнута частина плавно сполучається з частиною постійної товщини. Окреслено особливості формулювання граничних і перехідних умов, які необхідно дотримуватись при розгляді власних коливань двохступеневої пластинки.

Отримано співвідношення, які дозволяють вивчити розподіл прогинів і визначити значення амплітуд згінних коливань пластинки. Зазначено, що форми коливань побудовано на основі положень розроблених та розвинутих раніше авторами методів симетрії та факторизації. Зокрема знайдено, що прогини можна дослідити через вирази, які визначаються через суму відповідних розв'язків двох лінійних диференціальних рівнянь другого порядку зі змінними коефіцієнтами.

На основі запропонованого підходу визначено систему з вісьмох однорідних алгебраїчних рівнянь, яка дозволила побудувати частотне рівняння для пластинки, що жорстко закріплена за внутрішнім контуром і є вільною на зовнішньому контурі. Знайдено значення власних частот пластинки для перших трьох форм власних коливань. Причому, задля апробації та для розширення набору пластинок різної конфігурації розглянуто пластинки з двома видами ввігнутості у їх змінній частині.

Нові підходи та отримані на їх основі співвідношення можуть бути корисними для подальшого розвитку методів розв'язку подібних задач математичної фізики на власні значення. Практичним уособленням цього є задачі про коливання пластинок змінної товщини різної форми

Ключові слова: власні частоти, форми коливань, аналітичний розв'язок, кільцева пластинка, вільні коливання, метод симетрій

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CONSTRUCTION OF AN ALGORITHM TO ANALYTICALLY SOLVE A PROBLEM ON THE FREE VIBRATIONS OF A COMPOSITE PLATE OF VARIABLE THICKNESS

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1. Introduction

Multifaceted research into mechanical vibrations remains a pressing challenge for many fields of technology and equipment. Particular attention has traditionally been drawn to the vibrations of elements in the structures used for various purposes. These elements include turbine rotors, turbine blades and disks, assemblies for vehicles and aircraft [1]. Plates of different configurations are widely used in mechanical engineering as an important type of design elements. However, analyzing their vibrations is one of the most important and difficult particular tasks in mechanics [2]. It should be noted that the theory of elastic bodies vibrations is an integral part of such a scientific discipline as applied acoustics, so plate vibrations could be considered, specifically, as the acoustic vibrations [3]. Such vibrations, especially in the high-frequency range (ultrasonic and near-ultrasonic), form an important part of modern technologies created through their targeted use. The most well-known examples of the use of vibrations relate to the methods for determining strength under conditions of variable load on rods or plates at elevated frequencies, cleaning parts, making or machining components made from super-solid materials with increased precision. Such technologies and their varieties are becoming increasingly widely used for medical purposes [4]. For example, active acoustic elements are used in the equipment for the treatment of hearing diseases, urological problems (contactless destruction of kidney stones), in dentistry when filling teeth [5]. Thus, it is still a relevant task to study the vibrations of elastic bodies.

2. Literature review and problem statement

When searching for and studying the literary sources related to the stated problem about vibrations of a composite plate, one must focus only on single plates of variable thickness. Of additional interest are the works that address plates of different outlines, other than the annular ones, as well as the plate elements in the composition of structures. What is common to such plates and the specified one is the need to consider the fourth-order equations with variable coefficients.

Paper [6] employed a series method to derive a solution about the vibrations of an annular plate of linear-variable thickness, the results of which relate only to the principal frequency and mode of vibrations. The reason for the limited single result is probably the cumbersomeness and complexity of the analytical expressions used.

Study [7] considers a similar problem, but the solution was obtained by the Ritz method using Chebyshev's polynomials, which is an obvious drawback.

In [8, 9], the vibrations of a plate of variable thickness were studied by a numerical method, but the scheme for finding the natural frequencies for higher vibration modes was not considered.

Paper [10] applied a finite difference method to investigate the vibrations of a composite plate containing a variable thickness area in the form of a radius transition. The solution is approximate, neither frequencies nor modes of vibrations were given in an explicit (analytical) form.

Article [11] examines a problem about the vibrations of a fully clamped elastic elliptical panel. The solution was derived in Mathieu's functions by a method of separating variables in elliptical coordinates. Five natural frequencies were found. The solution is accurate and well-studied.

In work [12], a problem on the vibrations of an annular plate of variable thickness is solved approximately, using the approximation of movements based on spline functions.

Study [13] examined the movement of an annular composite plate at free vibrations by the method of colloquialisms. The solution is approximate.

In article [14], the object of the research is the thin plates of various outlines (elliptical and rectangular; the authors use an energy approach in combination with the method of transformations by Kolosov-Muskhelishvili. The solution and the results are approximate.

Numerical methods are used for rectangular plates in studies [15, 16]. The results are approximate.

The free vibrations of an annular plate on elastic support according to the Winkler-Pasternak hypothesis were considered in [17]. Natural frequencies were determined by the Galerkin method.

Work [18] addresses the vibrations of an annular plate with a ring edge. Displacements are sought through the Fourier series. The problem on finding the natural frequencies was noted and its solution is given through the use of numerical methods.

Our review has revealed that, except for known approaches and methods, no new ones have been proposed, presumably because of the lack of them. This suggests that it is appropriate to conduct a study into the problem-solving about plate vibrations, whose mathematical model are the fourth-order equations with variable coefficients. A part of this issue is the problems about the vibrations of plates with variable thickness, including composite plates, in which the part with variable thickness is smoothly aligned to the part of a permanent thickness. Even though there is no scientific problem about composite plates in principle, resolving the stated technically laborious problem about the vibrations of composite plates has an important practical significance.

3. The aim and objectives of the study

The aim of this study is to build an algorithm to analytically solve a problem about the natural vibrations of a composite two-stage annular plate with steps of the variable (concave) and constant thickness.

To accomplish the aim, the following tasks have been set:

– based on the symmetry method, build a common solution to the problem about the natural axisymmetric vibrations of a composite annular plate of the predefined configuration;

– to establish the ratios for the boundary and transitional conditions for a composite annular plate, which is rigidly fixed along the inner contour and is free along the outer one;

– to derive a frequency equation for the specified plate;

– to calculate the natural frequencies and build the vibration modes for two types of a plate in a given configuration with varying degrees of concaveness in their variable part.

4. Building a common solution for a composite plate

An annular plate (Fig. 1) is considered, whose variable thickness changes according to the law $h=h_0H(\rho)$, where $H=(1-\mu\rho)^2$, and the conjugated region has a constant thickness $h^*=\text{const}$. Here, h_0, μ are the constants, $\rho=r/R$ is the dimensionless radius, r is the variable radius; R is the radius of the plate. The variable section of the plate is limited to the radii $\rho=\rho_1\div\rho_2$; constant – radii $\rho=\rho_2\div 1$. If $\rho=0$, then $h=h_0$. When the sections of the plate are aligned, at $\rho=\rho_2$, we obtain $h^*=h_0(1-\mu\rho_2)^2$. This thickness remains unchanged at $\rho=\rho_2\div 1$.

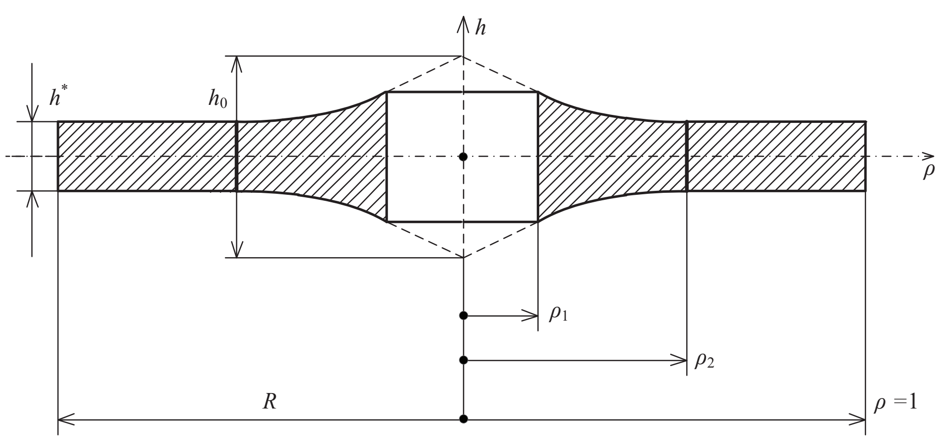

Fig. 1. Graphic representation of a composite plate

Deflections W along a section of variable thickness are determined by the sum of the solutions $W=W_1+W_2$ to the following differential equations [19]:

$$\left.\begin{aligned} W_1''+\frac{S'}{S}W_1'+\frac{\mu^2\lambda_1^2}{H}W_1=0,\\ W_2''+\frac{S'}{S}W_2'-\frac{\mu^2\lambda_2^2}{H}W_2=0,\end{aligned}\right\} \tag{1}$$

where

$$S=\rho H^2;$$

$$\lambda_1^2=\sqrt{k^4/\mu^4+4}-2;\;\; \lambda_2^2=\sqrt{k^4/\mu^4+4}+2;$$

$$k^2=\frac{\omega R^2}{h_0}\sqrt{\frac{12(1-\nu^2)\gamma}{gE}}=\mu^2\lambda_1\lambda_2=\mu^2\lambda_1\sqrt{\lambda_1^2+4}, \tag{2}$$

$\omega=2\pi f$ is the annular frequency, f is the cyclical frequency; v is the Poisson coefficient; g is the acceleration of gravity; γ is the specific weight; E is an elasticity module.

Equations (1) are converted to the following form after the variables are replaced:

$$\left.\begin{aligned} y_1''+2\frac{D'}{D}y_1'+\lambda_1^2y_1=0;\\ y_2''+2\frac{D'}{D}y_2'-\lambda_2^2y_2=0,\end{aligned}\right\} \tag{3}$$

where

$$W_{1,2}(\rho)=y_{1,2}(x);\;\; x=-\ln(1-\mu\rho);$$

$$D=\sqrt{e^{-3x}-e^{-4x}}=e^{-2x}\sqrt{e^x-1}=\frac{\sqrt{e^x-1}}{e^{2x}}.$$

Based on the approximation, we derived:

$$D=\sqrt{e^{-3x}-e^{-4x}}\approx D_0\frac{\sqrt{x}}{x^2+C_0},$$

$$D_0=0.21,\;\; C_0=0.2483. \tag{4}$$

In this case, according to the symmetry method, the exact solutions to equations (3) will be determined from expressions:

$$\left.\begin{aligned} y_1=2xy_{01}'+\lambda_1^2(x^2+C_0)y_{01};\\ y_2=2xy_{02}'-\lambda_2^2(x^2+C_0)y_{02},\end{aligned}\right\} \tag{5}$$

where

$$y_{01}=AJ_0(\lambda_1x)+BY_0(\lambda_1x);$$

$$y_{01}'=-\lambda_1\left[AJ_1(\lambda_1x)+BY_1(\lambda_1x)\right];$$

$$y_{02}=A_1I_0(\lambda_2x)+B_1K_0(\lambda_2x);$$

$$y_{02}'=-\lambda_2\left[-A_1I_1(\lambda_2x)+B_1K_1(\lambda_2x)\right];$$

D_0, C_0 are the constants set according to (4); A, B, A_1, B_1 are the arbitrary constants.

Solutions (5) in the expanded form are recorded as:

$$\left.\begin{aligned} &y_1=A\left[\lambda_1(x^2+C_0)J_0-2xJ_1\right]+\\ &+B\left[\lambda_1(x^2+C_0)Y_0-2xY_1\right];\\ &y_2=-A_1\left[\lambda_2(x^2+C_0)I_0-2xI_1\right]-\\ &-B_1\left[\lambda_2(x^2+C_0)K_0+2xK_1\right],\end{aligned}\right\} \tag{6}$$

with designations:

$$(J_0.J_1,Y_0,Y_1)=\left[J_0(\lambda_1x),J_1(\lambda_1x),Y_0(\lambda_1x),Y_1(\lambda_1x)\right];$$

$$(I_0,I_1,K_0,K_1)=\left[I_0(\lambda_2x),I_1(\lambda_2x),K_0(\lambda_2x),K_1(\lambda_2x)\right].$$

Deflections W_0 for the region of constant thickness (Fig. 1) are determined by the sum of the solutions $W_0=W_{01}+W_{02}$, where W_{01}, W_{02} are the solutions to the following differential equations (here, derivatives for variable ρ) derived from (1) at H=const:

$$\left.\begin{aligned} W_{01}''+\frac{1}{\rho}W_{01}'+\lambda^2W_{01}=0;\\ W_{02}''+\frac{1}{\rho}W_{02}'-\lambda^2W_{02}=0,\end{aligned}\right\} \tag{7}$$

where

$$\lambda^2=\frac{\omega R^2}{h_0^*}\sqrt{\frac{12(1-\nu^2)\gamma}{gE}}. \tag{8}$$

Here, $h_0^*=h_0(1-\mu\rho_2)^2$, hence, considering (2) and (8), we obtain:

$$\lambda=\frac{k}{(1-\mu\rho_2)}=\frac{\mu}{1-\mu\rho_2}\lambda_1\lambda_2=\frac{\mu}{1-\mu\rho_2}\lambda_1\sqrt{\lambda_1{}^2+4}. \tag{9}$$

Known solutions follow from (7):

$$\left.\begin{aligned} W_{01}=aJ_0(\lambda\rho)+bY_0(\lambda\rho);\\ W_{02}=a_1I_0(\lambda\rho)+b_1K_0(\lambda\rho),\end{aligned}\right\} \tag{10}$$

that is,

$$W_0=aJ_0(\lambda\rho)+bY_0(\lambda\rho)+a_1I_0(\lambda\rho)+b_1K_0(\lambda\rho). \tag{11}$$

Thus, solutions (6) and (11) provide an opportunity, based on the boundary conditions, to derive a solution to the problem for solid or annular composite plates with different way of fixing them.

5. Boundary and transitional conditions for a composite annular plate, rigidly fixed along the inner contour

If the plate is rigidly fixed along the inner contour ($\rho=\rho_1$) and is free along the outer one ($\rho=1$), then it is necessary to satisfy the following boundary conditions:

– in the first case (rigidly fixed):

$$(W)_{\rho=\rho_1}=0;\;\left(\frac{dW}{d\rho}\right)_{\rho=\rho_1}=0; \tag{12}$$

– in the second case (a free edge):

$$\begin{cases}\left(\dfrac{d^2W_0}{d\rho^2}+\dfrac{\nu}{\rho}\dfrac{dW_0}{d\rho}\right)_{\rho=1}=0;\\ \left(\dfrac{d^3W_0}{d\rho^3}+\dfrac{1}{\rho}\dfrac{d^2W_0}{d\rho^2}-\dfrac{1}{\rho^2}\dfrac{dW_0}{d\rho}\right)_{\rho=1}=0.\end{cases}\quad(13)$$

In addition, it is also necessary to meet the conditions for the conjugation (transitional conditions) at $\rho=\rho_2$, which ultimately take the form:

$$W=W_0;\ \ \frac{dW}{d\rho}=\frac{dW_0}{d\rho};$$

$$\frac{d^2W}{d\rho^2}=\frac{d^2W_0}{d\rho^2};\ \ \frac{d^3W}{d\rho^3}=\frac{d^3W_0}{d\rho^3}.\quad(14)$$

Because $W=W_1+W_2$; $W_0=W_{01}+W_{02}$, the conditions (13), (14) containing higher derivatives from functions (W, W_0) could be transformed so that they are expressed only through ($W_{1,2}$, $W'_{1,2}$) and ($W_{01,02}$, $W'_{01,02}$), which simplifies subsequent analytical considerations.

We find from (1), given $W=W_1+W_2$:

$$W''+\frac{S'}{S}W'+\frac{\mu^2}{H}\left(\lambda_1^2W_1-\lambda_2^2W_2\right)=0;$$

$$\left[W''+\frac{S'}{S}W'+\frac{\mu^2}{H}\left(\lambda_1^2W_1-\lambda_2^2W_2\right)\right]'=0,$$

hence:

$$W''=-\frac{S'}{S}W_1'-\frac{S'}{S}W_2'-\frac{\mu^2\lambda_1^2}{H}W_1+\frac{\mu^2\lambda_2^2}{H}W_2;\quad(15)$$

$$W'''=\left[\frac{S'^2}{S^2}-\left(\frac{S'}{S}\right)'-\frac{\mu^2\lambda_1^2}{H}\right]W_1'+$$

$$+\left[\frac{S'^2}{S^2}-\left(\frac{S'}{S}\right)'+\frac{\mu^2\lambda_2^2}{H}\right]W_2'+K,\quad(16)$$

where

$$K=\frac{\mu^2\lambda_1^2}{H}\cdot\frac{(SH)'}{SH}W_1-\frac{\mu^2\lambda_2^2}{H}\cdot\frac{(SH)'}{SH}W_2.$$

From (7), by assuming $W_0=W_{01}+W_{02}$, we similarly obtain:

$$W_0''=-\frac{1}{\rho}W_{01}'-\frac{1}{\rho}W_{02}'-\lambda^2W_{01}+\lambda^2W_{02};\quad(17)$$

$$\left(W_0''\right)'=\frac{2-\lambda^2\rho^2}{\rho^2}W_{01}'+\frac{2+\lambda^2\rho^2}{\rho^2}W_{02}'+$$

$$+\frac{\lambda^2}{\rho}W_{01}-\frac{\lambda^2}{\rho}W_{02}.\quad(18)$$

Considering (17), (18), the boundary conditions (13) at $\nu=1/3$ take the form, respectively:

$$\left(\frac{4}{3\lambda^2}W_{01}'+W_{01}-W_{02}\right)_{\rho=1}=0;\ \left(W_{01}'-W_{02}'\right)_{\rho=1}=0.\quad(19)$$

The third conditions from (14) considering:

$$\left(W=W_0\right)_{\rho=\rho_2};\ \left(W'=W_0'\right)_{\rho=\rho_2}\quad(20)$$

and given (15) to (18) at $S=\rho H^2$ takes the form:

$$\left[\begin{array}{l}2\dfrac{H'}{H}W_0'-\dfrac{\mu^2\lambda_2^2}{H}W_0+\\+\lambda^2\left(W_{02}-W_{01}\right)+\dfrac{\mu^2\left(\lambda_1^2+\lambda_2^2\right)}{H}W_1\end{array}\right]_{\substack{\rho=\rho_2\\x=x_2}}=0.\quad(21)$$

The last condition from (14) under the same preconditions, and when taking into consideration the expression $\left(W_1\right)_{\substack{\rho=\rho_2\\x=x_2}}$ in it, derived from (21), is recorded as:

$$\left[\begin{array}{l}(L)W_0'+\lambda^2\left(W_{02}'-W_{01}'\right)+\\+3\lambda^2\dfrac{H'}{H}\left(W_{02}-W_{01}\right)+\dfrac{\mu^2\left(\lambda_1^2+\lambda_2^2\right)}{H}W'\end{array}\right]_{\substack{\rho=\rho_2\\x=x_2}}=0,\quad(22)$$

where

$$L=\frac{4\mu}{\rho H}-\frac{\mu^2\lambda_2^2}{H}.$$

The transformed boundary and transitional conditions, obtained for direct use, make it possible, by applying solutions (6) and (10), to obtain a frequency equation.

6. Derivation of a frequency equation

After fitting expressions W, W', W_0, W_0' into (12), (19), (20)÷(22), at the corresponding boundary values ρ or x, we obtain eight homogeneous algebraic equations relative to the desired constants A_i, B_i, a_i, b_i. By equating the determinator of this system to zero, we obtain a frequency equation. To solve the stated problem, we record the expressions for the required functions. Because $W(\rho)=W_1(\rho)+W_2(\rho)=$ $=y_1(x)+y_2(x)$, then from (6):

$$W=\begin{Bmatrix}A\left[\lambda_1\left(x^2+C_0\right)J_0\left(\lambda_1x\right)-2xJ_1\left(\lambda_1x\right)\right]+\\+B\left[\lambda_1\left(x^2+C_0\right)Y_0\left(\lambda_1x\right)-2xY_1\left(\lambda_1x\right)\right]\end{Bmatrix}-M,\quad(23)$$

where

$$M=\begin{Bmatrix}A_1\left[\lambda_2\left(x^2+C_0\right)I_0\left(\lambda_2x\right)-2xI_1\left(\lambda_2x\right)\right]+\\+B_1\left[\lambda_2\left(x^2+C_0\right)K_0\left(\lambda_2x\right)+2xK_1\left(\lambda_2x\right)\right]\end{Bmatrix}$$

and hence

$$W'=\frac{dW}{d\rho}=x'\frac{dW}{dx}=x'\left(\frac{dW_1}{dx}+\frac{dW_2}{dx}\right)=x'\left(\frac{dy_1}{dx}+\frac{dy_2}{dx}\right).$$

By using known formulae for differentiating the Bessel functions [20], that is:

$$\left.\begin{aligned} &J_0'(\lambda_1 x) = -\lambda_1 J_1(\lambda_1 x);\\ &Y_0'(\lambda_1 x) = -\lambda_1 Y_1(\lambda_1 x);\\ &I_0'(\lambda_2 x) = \lambda_2 I_1(\lambda_2 x);\\ &K_0(\lambda_2 x) = -\lambda_2 K_1(\lambda_2 x);\\ &J_1'(\lambda_1 x) = \lambda_1\left[J_0(\lambda_1 x) - \frac{J_1(\lambda_1 x)}{\lambda_1 x}\right];\\ &Y_1'(\lambda_1 x) = \lambda_1\left[Y_0(\lambda_1 x) - \frac{Y_1(\lambda_1 x)}{\lambda_1 x}\right];\\ &I_1'(\lambda_2 x) = \lambda_2\left[I_0(\lambda_2 x) - \frac{I_1(\lambda_2 x)}{\lambda_2 x}\right];\\ &K_1'(\lambda_2 x) = -\lambda_2\left[K_0(\lambda_2 x) + \frac{K_1(\lambda_2 x)}{\lambda_2 x}\right], \end{aligned}\right\} \tag{24}$$

we derive

$$\left.\begin{aligned} &W_1' = -\lambda_1^2(x^2 + C_0)x'\left[AJ_1(\lambda_1 x) + BY_1(\lambda_1 x)\right];\\ &W_2' = -\lambda_2^2(x^2 + C_0)x'\left[AI_1(\lambda_2 x) - B_1 K_1(\lambda_2 x)\right]; \end{aligned}\right\} \tag{25}$$

$$W' = -(x^2 + C_0)x'\left\{\begin{aligned} &\lambda_1^2\left[AJ_1(\lambda_1 x) + BY_1(\lambda_1 x)\right] +\\ &+\lambda_2^2\left[AI_1(\lambda_2 x) - B_1 K_1(\lambda_2 x)\right] \end{aligned}\right\}. \tag{26}$$

We obtain from (10) and (11), in line with (24),

$$\left.\begin{aligned} &W_{01}' = -\lambda\left[aJ_1(\lambda\rho) + bY_1(\lambda\rho)\right];\\ &W_{02}' = -\lambda\left[-a_1 I_1(\lambda\rho) + b_1 K_1(\lambda\rho)\right], \end{aligned}\right\} \tag{27}$$

$$W_0' = -\lambda\begin{bmatrix} aJ_1(\lambda\rho) + bY_1(\lambda\rho) -\\ -\, a_1 I_1(\lambda\rho) + b_1 K_1(\lambda\rho) \end{bmatrix}. \tag{28}$$

Following the introduction of expressions (23) and (26) to (12), at $\rho=\rho_1$, $x=x_1$, we derive equations:

$$\begin{cases} Aq + Bp + A_1 a_1 + B_1 p_1 = 0;\\ Aq_2 + Bp_2 + A_1 q_3 + B_1 p_3 = 0, \end{cases} \tag{29}$$

where at $x=x_1$

$$q = \left[\lambda_1(x^2 + C_0)J_0(\lambda_1 x) - 2xJ_1(\lambda_1 x)\right];$$

$$p = \left[\lambda_1(x^2 + C_0)Y_0(\lambda_1 x) - 2xY_1(\lambda_1 x)\right];$$

$$q_1 = -\left[\lambda_2(x^2 + C_0)I_0(\lambda_2 x) - 2xI_1(\lambda_2 x)\right];$$

$$p_1 = -\left[\lambda_2(x^2 + C_0)K_0(\lambda_2 x) + 2xK_1(\lambda_2 x)\right];$$

$$q_2 = \lambda_1^2 J_1(\lambda_1 x);\quad p_2 = \lambda_1^2 Y_1(\lambda_1 x);$$

$$q_3 = \lambda_2^2 I_1(\lambda_2 x);\quad p_3 = -\lambda_2^2 K_1(\lambda_2 x).$$

The two equations follow from (19) considering (10) and (27) at $\rho=1$

$$\begin{cases} a\left[J_0(\lambda) - \dfrac{2}{3\lambda}J_1(\lambda)\right] + b\left[Y_0(\lambda) - \dfrac{2}{3\lambda}Y_1(\lambda)\right] +\\ + a_1\left(-I_0(\lambda) + \dfrac{2}{3\lambda}I_1(\lambda)\right) +\\ + b_1\left(-K_0(\lambda) - \dfrac{2}{3\lambda}K_1(\lambda)\right) = 0;\\ aJ_1(\lambda) + bY_1(\lambda) + a_1 I_1(\lambda) - b_1 K_1(\lambda) = 0. \end{cases} \tag{30}$$

Similarly, from (20), considering (11), (23), (26), (28) at $\rho=\rho_2$, $x=x_2$, we derive two more equations:

$$\begin{cases} AQ + BG + A_1 Q_1 + B_1 G_1 -\\ -\begin{bmatrix} aJ_0(\lambda\rho_2) + bY_0(\lambda\rho_2) +\\ +a_1 I_0(\lambda\rho_2) + b_1 K_0(\lambda\rho_2) \end{bmatrix} = 0;\\ AQ_2 + BG_2 + A_1 Q_3 + B_1 G_3 -\\ -\begin{bmatrix} aJ_1(\lambda\rho_2) + bY_1(\lambda\rho_2) -\\ -a_1 I_1(\lambda\rho_2) + b_1 K_1(\lambda\rho_2) \end{bmatrix} = 0, \end{cases} \tag{31}$$

where at $x=x_2$

$$Q = \left[\lambda_1(x^2 + C_0)J_0(\lambda_1 x) - 2xJ_1(\lambda_1 x)\right];$$

$$G = \left[\lambda_1(x^2 + C_0)Y_0(\lambda_1 x) - 2xY_1(\lambda_1 x)\right];$$

$$Q_1 = -\left[\lambda_2(x^2 + C_0)I_0(\lambda_2 x) - 2xI_1(\lambda_2 x)\right];$$

$$G_1 = -\left[\lambda_2(x^2 + C_0)K_0(\lambda_2 x) + 2xK_1(\lambda_2 x)\right];$$

$$Q_2 = \frac{\lambda_1^2}{\lambda}(x^2 + C_0)x'J_1(\lambda_1 x);$$

$$G_2 = \frac{\lambda_1^2}{\lambda}(x^2 + C_0)x'Y_1(\lambda_1 x);$$

$$Q_3 = \frac{\lambda_2^2}{\lambda}(x^2 + C_0)x'I_1(\lambda_2 x);$$

$$G_3 = -\frac{\lambda_2^2}{\lambda}(x^2 + C_0)x'K_1(\lambda_2 x).$$

From (21) and (22), by using (6), (10), (11), (25), (27), (28), we write down, believing $\rho=\rho_2$, $x=x_2$, the following two equations, respectively:

$$\begin{cases} AL + BR + aM + bN + a_1 M_1 + b_1 N_1 = 0;\\ AL_1 + BR_1 + aM_2 + bN_2 + a_1 M_3 + b_1 N_3 = 0, \end{cases} \tag{32}$$

where at $\rho=\rho_2$, $x=x_2$

$$L = \frac{\mu^2(\lambda_1^2 + \lambda_2^2)}{H}\left[\lambda_1(x^2 + C_0)J_0(\lambda_1 x) - 2xJ_1(\lambda_1 x)\right];$$

$$R = \frac{\mu^2(\lambda_1^2 + \lambda_2^2)}{H}\left[\lambda_1(x^2 + C_0)Y_0(\lambda_1 x) - 2xY_1(\lambda_1 x)\right];$$

$$M = -2\lambda\frac{H'}{H}J_1(\lambda\rho) - \left(\frac{\mu^2\lambda_2^2}{H} + \lambda^2\right)J_0(\lambda\rho);$$

$$N = -2\lambda\frac{H'}{H}Y_1(\lambda\rho) - \left(\frac{\mu^2\lambda_2^2}{H} + \lambda^2\right)Y_0(\lambda\rho);$$

$$M_1 = 2\lambda\frac{H'}{H}I_1(\lambda\rho) - \left(\frac{\mu^2\lambda_2^2}{H} - \lambda^2\right)I_0(\lambda\rho);$$

$$N_1 = -2\lambda\frac{H'}{H}K_1(\lambda\rho) - \left(\frac{\mu^2\lambda_2^2}{H} - \lambda^2\right)K_0(\lambda\rho);$$

$$L_1 = \mu^2\left(\lambda_1^2 + \lambda_2^2\right)\lambda_1^{\,2}\frac{\left(x^2 + C_0\right)x'}{H}J_1(\lambda_1 x);$$

$$R_1 = \mu^2\left(\lambda_1^2 + \lambda_2^2\right)\lambda_1^{\,2}\frac{\left(x^2 + C_0\right)x'}{H}Y_1(\lambda_1 x);$$

$$M_2 = \left[\lambda\left(\frac{4\mu}{\rho H} - \frac{\mu^2\lambda_2^2}{H}\right) - \lambda^3\right]J_1(\lambda\rho) + 3\lambda^2\frac{H'}{H}J_0(\lambda\rho);$$

$$N_2 = \left[\lambda\left(\frac{4\mu}{\rho H} - \frac{\mu^2\lambda_2^2}{H}\right) - \lambda^3\right]Y_1(\lambda\rho) + 3\lambda^2\frac{H'}{H}Y_0(\lambda\rho);$$

$$M_3 = -\left[\lambda\left(\frac{4\mu}{\rho H} - \frac{\mu^2\lambda_2^2}{H}\right) + \lambda^3\right]I_1(\lambda\rho) - 3\lambda^2\frac{H'}{H}I_0(\lambda\rho);$$

$$N_3 = \left[\lambda\left(\frac{4\mu}{\rho H} - \frac{\mu^2\lambda_2^2}{H}\right) + \lambda^3\right]K_1(\lambda\rho) - 3\lambda^2\frac{H'}{H}K_0(\lambda\rho).$$

The frequency equation follows from the system of equations (29) to (32):

$$\begin{vmatrix}
q & p & q_1 & p_1 & 0 & 0 & 0 & 0 \\
q_2 & p_2 & q_3 & p_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & J_0(\lambda) - \frac{2}{3\lambda}J_1(\lambda) & J_0(\lambda) - \frac{2}{3\lambda}J_1(\lambda) & -I_0(\lambda) + \frac{2}{3\lambda}I_1(\lambda) & -K_0(\lambda) - \frac{2}{3\lambda}K_1(\lambda) \\
0 & 0 & 0 & 0 & J_1(\lambda) & Y_1(\lambda) & I_1(\lambda) & -K_1(\lambda) \\
Q & G & Q_1 & G_1 & -J_0(\lambda\rho_2) & -Y_0(\lambda\rho_2) & -I_0(\lambda\rho_2) & -K_0(\lambda\rho_2) \\
Q_2 & G_2 & Q_3 & G_3 & -J_1(\lambda\rho_2) & -Y_1(\lambda\rho_2) & I_1(\lambda\rho_2) & -K_1(\lambda\rho_2) \\
L & R & 0 & 0 & M & N & M_1 & N_1 \\
L_1 & R_1 & 0 & 0 & M_2 & N_2 & M_3 & N_3
\end{vmatrix} = 0. \quad (33)$$

The resulting frequency equation makes it possible to derive the natural frequencies of the composite plate of the predefined profile.

7. Calculating natural frequencies, building vibration modes

To illustrate the effectiveness of the devised procedure for solving the stated problem about natural values, we selected two types of plates with varying degrees of concaveness in their variable part.

Plate 1 has the following parameters: $\mu=0.5985$; $\rho_1=0.1$; $\rho_2=0.5$; $x_1=0.0617$; $x_2=0.3556$; $k=\lambda\cdot 0.70075$.

Plate 2 is characterized by the following parameters: $\mu=0.6725$; $\rho_1=0.25$; $\rho_2=0.5$; $x_1=0.184064$; $x_2=0.409827$; $k=\lambda\cdot 0.66375$.

The results of calculating natural (frequency) numbers from equation (33) for the first three natural vibration modes of plates 1 and 2 are given in Table 1.

Table 1

Frequency numbers of plates

Number of a vibration mode	Plate 1	Plate 2
I	$k=1.62759$	$k=1.71890$
II	$k=4.15673$	$k=4.47488$
III	$k=6.61722$	$k=7.48927$

To construct the vibration modes (deflections) from the system of equations (29) to (32), after substituting the derived numbers k_j ($j=1, 2, 3$), we determine coefficients A_i, B_i, a_i, b_i included in (11) and (23). The graphic representation of deflections under the first mode of vibrations of plates 1 and 2 is shown in Fig. 2, under the second mode – in Fig. 3, and under the third mode – in Fig. 4. Fig. 2–4 can be used to judge the differences in the distribution of nodes and antinodes of vibrations depending on the degree of concaveness (parameter μ) and the values of radius ρ_1 of the fixing contour. Note that based on the well-known features of plate deformations, the maximum radial stresses, which are the most dangerous, operate in the zone of maximum deflections. From Fig. 2–4 one could establish those sections of the plate where such stresses occur. It is obvious that from this point of view the most favorable situation is the one in which the maximum of stresses is shifted towards the thickened part of the plate. In that sense, as shown in Fig. 2–4, plate 1 is preferable.

A quantitative assessment of the stressed-deformed state of the entire plate could be given by using, to calculate the stresses, the formulae, known from theory, to which one should introduce the natural frequencies and functions of deflections derived in this paper, following the example of work [21].

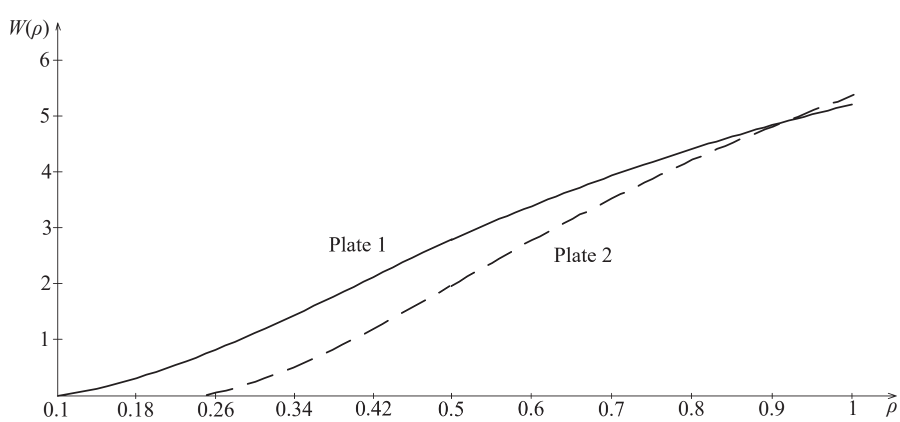

Fig. 2. Graphic representation of deflections under the first mode of vibrations

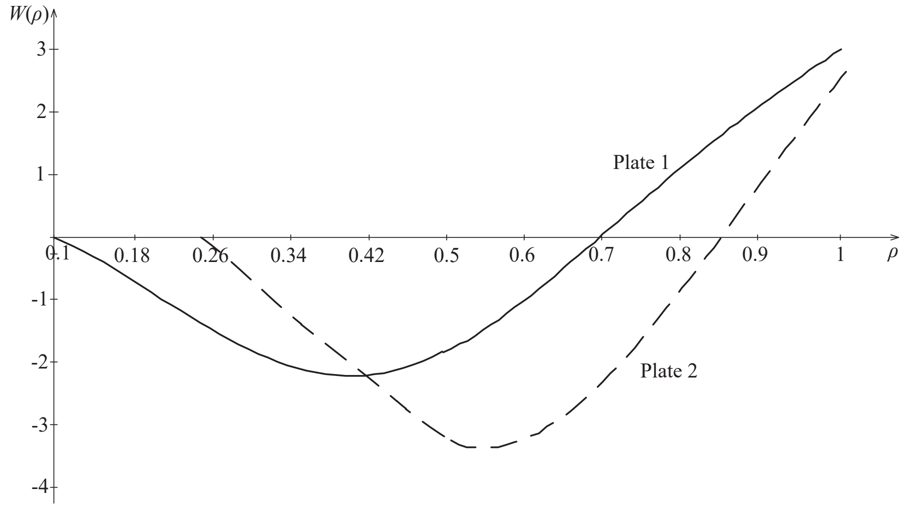

Fig. 3. Graphic representation of deflections under the second mode of vibrations

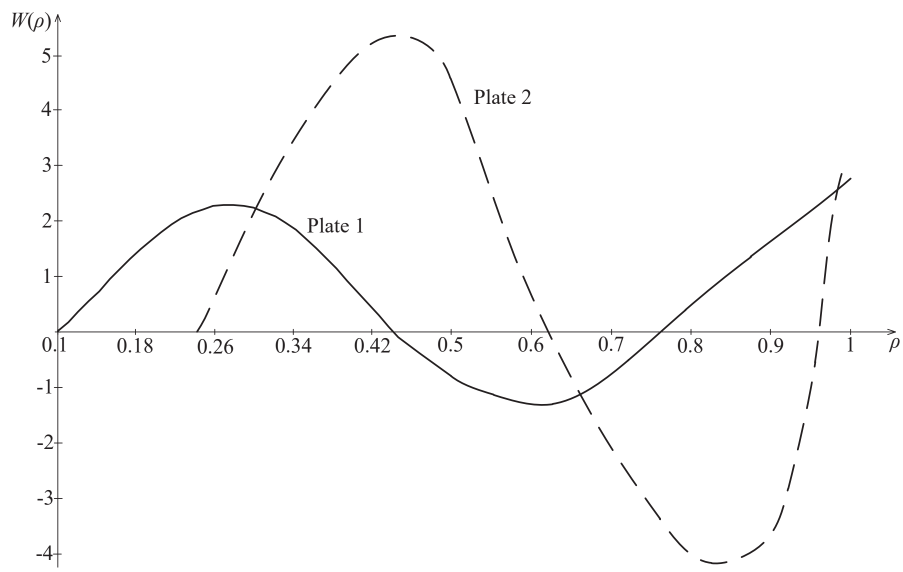

Fig. 4. Graphic representation of deflections under the third mode of vibrations

8. Discussion of results from solving a problem on the vibrations of a composite plate

A common solution, suitable for studying the vibrations of solid or annular composite plates of the predefined geometry under any boundary conditions, is based on common solutions for the two smoothly conjugated sections. For a section of variable thickness, the sum of solutions (6) determines its deflections. To build the deflections along a section of permanent thickness, solution (11) is applicable.

The principal issue in solving such problems for composite plates is the construction of solutions for a section of variable thickness, as at present there is no general method for solving differential equations of the fourth-order with variable coefficients. In a given case, for the adopted parabolic law of change in thickness, the analytical solution was derived through the application of the symmetry method, a new method for problems on natural values, which we constructed for this class of problems. Special features in the symmetry method for vibration problems imply the possibility to obtain precise solutions to the second-order equations with variable coefficients, built so that the number of such solutions is fundamentally unlimited. Solutions (6) were derived for two second-order equations (3) after they had replaced the original fourth-order equation of vibrations. The limitations of the method stem from a solution scheme, according to which it is necessary to pre-reduce the order of a differential equation to the order not higher than the second order. This requires the application of a factorization method, which, while not universal, has limited capabilities.

Overcoming the above limitations and difficulties may only be possible in the search for new approaches or the advancement of the symmetry method towards its more effective application for problems involving higher-order equations.

The transformed boundary and transitional conditions (12) to (14), expressed only through deflections, as well as their first derivatives, make it easier to calculate since the Bessel functions used here would be represented only by zero and first-order functions.

The equation system (29) to (32), obtained after the implementation of boundary and transitional conditions (12) to (14), leads to a frequency equation (33) in the form of a determinant of the eighth order equal to zero. The elements of this determinant are expressed in a complete, more compact form due to the above lower-order Bessel functions. This makes it easier to calculate frequencies because it makes it much easier to program the frequency determinant.

The frequency numbers, calculated from equation (33) for a plate with two types of concaveness of a parabolic section and a different diameter of the fixing contour, have allowed us to build the modes of their natural vibrations that correspond to these numbers. The vibration modes in the form of a graphic representation of deflections (Fig. 2–4) provide a clear picture of the nature of their change along the current radius of the plates. It is obvious that by following the above procedure it is possible to similarly study the vibrations of an annular plate of a given type with different geometric parameters and with other boundary conditions, including vibrations of a solid composite plate.

9. Conclusions

1. Based on the symmetry method, a general analytical solution has been derived to the problem on the natural axisymmetric vibrations of an annular plate, composed of a part, concave in line with a parabolic law $h=h_0(1-\mu\rho)^2$, conjugated to the part of permanent thickness. The solution makes it possible to study vibrations of both solid and annular plates of a given type at different ways of fixing them.

2. We have established the transformed ratios for the boundary and transitional conditions for a composite annular plate of the predefined configuration, rigidly fixed along the inner contour and free along the outer contour. A special feature of the proposed ratios is their dependence only on the functions of deflections and their first derivatives, which significantly simplifies analytical calculations.

3. The frequency equation has been given in the form of a determinant of the eighth order equal to zero and the scheme of its derivation for a composite annular plate, rigidly fixed along the inner contour.

4. As an example of the feasibility of the algorithm created, we have computed the first three natural frequencies (numbers k_j) for two types of a plate of the predefined configuration, but with varying degrees of concaveness, determined by the μ parameter, and with different values for the radius of the fixing contour ρ_1. For the case $\mu=0.5985$ and $\rho_1=0.1$ (plate 1), the frequency numbers (Table 1) k_j ($j=1, 2, 3$) are lower, respectively, by (5.3; 7.1; 11.6) %, than those for plate 2 ($\mu=0.6725$, $\rho_1=0.25$). The difference in frequencies is due to the difference in curved rigidity, determined by the thickness and the radial size of a plate $(1-\rho_1)$. If there is little difference in thickness throughout the entire length, the rigidity would be determined by the values $1-\rho_1=0.9$; 0.75 for plates 1 and 2, respectively. The smaller $(1-\rho_1)$, the more rigid the rigidity, and the higher its frequency.

Based on the derived frequency numbers, the vibration modes have been constructed and their brief analysis has been carried out. In particular, it has been shown that plate 1 is more preferable from the point of view of a hypothetical ultimate resource since the maximum of dangerous radial stresses acting in the zone of antinodes of displacements ($\rho_I\approx0.4$; $\rho_{II}\approx0.25$) is shifted towards the thickened part more than that of plate 2, where the antinodes ($\rho_I\approx0.55$; $\rho_{II}\approx0.45$) are close to a thinner transition section. The above analysis is indicative and more of a qualitative nature since the task of quantifying the expected results was not set. If one needs a complete analysis of the advantages or advantages of this type of plates from an operational standpoint, it is necessary to conduct a targeted study using the algorithm described in this paper.

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