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ANALYSIS OF FREE OSCILLATIONS OF ROUND THIN PLATES OF VARIABLE THICKNESS WITH A POINT SUPPORT

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Для задачі про вільні вісесиметричні коливання кругової пластинки зі змінною товщиною отримано загальний аналітичний розв'язок диференціального рівняння IV зі змінними коефіцієнтами. Товщина пластинки змінюється вздовж радіуса ρ за параболічним законом $h=H_0(1-\mu\rho)^2$. При побудові розв'язку використовувався синтез методу факторизації з методом симетрій. За допомогою методу факторизації рішення початкового рівняння IV порядку представлено як сума рішень двох відповідним чином побудованих рівнянь II порядку. Методом симетрії знайдені точні рішення цих двох рівнянь.

Задача з точковим закріпленням пластинки розглянута як граничний випадок задачі про жорстке закріплення внутрішнього контуру кільцевої пластинки, у якій $\rho \rightarrow 0$. З цієї метою виконано трансформацію загального розв'язку до виду, який задалегідь задовольняє умовам на жорсткій точковій опорі. Результатом такого перетворення є більш просте рішення, яке містить замість чотирьох шуканих постійних інтегрування лише дві. Внаслідок цього, частотне рівняння для пластинки при будь-яких умовах на зовнішньому контурі істотно спрощується, через те що воно отримано з визначника другого порядку. З частотного рівняння для пластинки з точковою опорою і з вільним краєм при $\mu=1,39127$, що відповідає відношенню граничних товщин, рівному 10,8, визначено перші п'ять власних чисел λ_i ($i=1 \div 5$). Для λ_i ($i=1 \div 3$) в якості графічної ілюстрації побудовано форми коливань. Наведено числові значення амплітудних коефіцієнтів, координати (відносні радіуси) пучностей коливань і вузлових кіл для кожної з п'яти форм коливань ($i=1 \div 5$). Знайдені числові значення параметрів коливань, які на практиці можуть бути використані для первинної ідентифікації виду коливальної системи та її можливих характеристик в разі закріплення пластинки за внутрішнім контуром малого діаметра. Цій же меті може слугувати критеріальне відношення діаметра контуру закріплення до діаметру пластинки. Якщо це відношення дорівнює або менше 0,2, то закріплення допустимо вважати точковим. У цьому випадку розрахунок коливань кільцевої пластинки із закріпленням за внутрішнім контуром можна вести за алгоритмом, який викладено для пластинки з точковою опорою

Ключові слова: вільні коливання, метод симетрій, тонка пластинка, точкова опора, аналітичний розв'язок

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1. Introduction

A series of problems on the natural oscillations of circular plates of various thicknesses have been considered in studies [1–3]. They also outline issues that reveal the importance and essence of the problem, review the ways to solve it with appropriate literary references.

In practice, there are cases when a circular plate in engineering structures is fixed inside the inner contour whose diameter is much smaller than that of the outer one. This situation necessitates considering a problem on the oscillations of a solid plate with a point support. The transition from natural conditions inside the internal contour to the conditions of point fastening is predetermined by the desire to simplify the plate estimation model without a significant loss of accuracy. For a plate of constant thickness with a support in the center, the results of solving the problem are reported in [4–7]. There is a basic issue related to a variable-thickness plate – the search for a general analytical solution based on

the original differential equation of free oscillations. If it is solved, it is possible to consider a series of applied problems on the oscillations of solid or circular, free, or fixed-in-different ways plates, including those with a point support. The relevance of the issue of calculating the oscillations of a plate of variable thickness with a point support is due to the need to improve the efficiency of estimation models, which is inextricably linked to their simplification, as is noted for the case of point fastening.

2. Literature review and problem statement

Our analysis of publications [8–15] has revealed that the solution to the problem of free oscillations of the plate with a point support is very general in the scientific papers related to the field of applied plate theory. At the same time, the focus of research is on ways and finding analytical solutions to the problem of free oscillations. In this aspect, one should

note the use of both a variety of classic approaches to solving the specified problem and modern algorithms. However, the numerous solution methods reviewed and the particular results obtained are difficult to use to analyze the oscillations of a round plate of variable thickness with a point support. The reason for this seems to be the difficulty of finding an accurate analytical solution to the IV-order differential equation with variable coefficients that will describe the oscillations of a variable-thickness plate.

5 natural frequencies for a plate with two point supports were obtained in work [8] based on the theory by Uffland-Mindlin. However, no variant for a variable thickness plate has been found.

The “cloud” method and the plate theory by Midlin were considered in paper [9] to solve a problem on the free oscillations of a plate with a point support. In this case, the authors considered plates of different shapes and with different fixation, except for a thin plate of variable thickness. As a result, the reported results of the study are unsuitable for use in solving a problem on the oscillations of a plate with a point support.

Study [10] considers a numerical approach to solving a problem on the oscillations of a complicated plate of rectangular configuration. Given the problem statement and the algorithm to solve it, the resulting expressions cannot be used for a plate with a point support.

Works [11–13] give analytical solutions to a problem on the free oscillations of rectangular plates. To find a solution, numerical approaches are used – a simplex superposition method, a method of finite elements, and an integrated method of transformation. It is obvious that the proposed approaches are difficult to use for analytical research into the plates of variable thickness with a point support.

A paired coordinate method is used in [14] to calculate the natural frequencies and build the plate oscillation shapes. However, the considered case of a thin plate with cutouts is difficult to adapt to the problem involving a solid plate with a point support. The same refers to the analysis of the ratios from article [15]. There, the Relay-Ritz method was proposed to construct the first oscillation shape of a thick plate. The cited article does not include any variants for a plate with a point support.

Thus, the above review of the scientific literature testifies to the absence of publications on the analytical solution to the problem on the oscillations of a round plate of variable thickness with a point support. The reason, of course, is not the lack of demand, as evidenced by the noted interest in the plates, including those with point supports, but, apparently, the difficulties associated with finding a solution to mathematical problems. Building an analytical solution to the problem on the free oscillations of a round plate with a point support would complement the theory of thin plates with new results, which, in itself, has scientific and applied value.

3. The aim and objectives of the study

The aim of this study is to derive an analytical solution to the problem on the free oscillations of a round plate, which changes in line with the law of concave parabola if its center includes a rigid point support.

To accomplish the aim, the following tasks have been set:

- to derive, based on the synthesis of symmetry and factorization methods, a general analytical solution to the

differential equation for a problem on the free axisymmetric oscillations of a plate whose thickness $h=H_0(1-\mu\rho)^2$;

- to transform the general solution to the form that meets the conditions on the support;

- to derive a frequency equation, calculate natural frequencies, and build the first three oscillation shapes for a plate with a free contour.

4. Differential equation and its overall solution for a variable thickness plate

For a plate whose thickness $h=H_0(1-\mu\rho)^2$, work [3] gives a differential equation for the shapes of natural axisymmetric bending oscillations, which can be represented in the following form

$$\left[(1-\mu\rho)^2 \frac{d^2}{d\rho^2} + \frac{[\rho(1-\mu\rho)^4]'}{\rho(1-\mu\rho)^2} \frac{d}{d\rho} - 2\mu^2 \right]^2 \times \\ \times W - (4\mu^4 + \lambda^4)W = 0. \tag{1}$$

Here, $W=W(\rho)$ are the plate displacements (deflections); μ is an arbitrary constant. $\rho=r/R$ is the relative variable; r is the variable radius; R is the plate radius.

$$\lambda^2 = 2\pi f \frac{R^2}{H_0} \sqrt{\frac{12(1-\nu^2)\gamma}{gE}}; \tag{2}$$

f is the natural oscillation frequency; H_0 is the thickness in the plate center; ν , γ , E are, respectively, the Poisson coefficient, specific weight, a material’s elasticity module; g is the acceleration of gravity. Hereafter, the Poisson coefficient is accepted to equal 1/3, based on which equation (1) is written for this particular case.

The notation of IV-order equation in form (1) makes it possible, applying the factorization method, to derive its solution in the form of the sum of solutions $W=W_1+W_2$ to the following two II-order equations

$$(1-\mu\rho)^2 W_{1,2}'' + \frac{[\rho(1-\mu\rho)^4]'}{\rho(1-\mu\rho)^2} W_{1,2}' - \\ - 2\mu^2 W_{1,2} \pm \sqrt{\lambda^4 + 4\mu^4} W_{1,2} = 0. \tag{3}$$

Equations (3) can be written in a generalized form

$$\left. \begin{aligned} W_1'' + \frac{S'}{S} W_1' + \frac{\mu^2 \alpha^2}{H} W_1 &= 0; \\ W_2'' + \frac{S'}{S} W_2' - \frac{\mu^2 \beta^2}{H} W_2 &= 0; \end{aligned} \right\} \tag{4}$$

where $S=\rho H^2=\rho(1-\mu\rho)^4$ and

$$\alpha^2 = \sqrt{\frac{\lambda^4}{\mu^4} + 4} - 2; \\ \beta^2 = \sqrt{\frac{\lambda^4}{\mu^4} + 4} + 2. \tag{5}$$

By replacing the variable $\rho=(1/\mu)(1-e^{-x})$, equation (4) can be written in the following form

$$\left. \begin{aligned} W_1'' + 2\frac{D'}{D}W_1' + \alpha^2W_1 &= 0; \\ W_2'' + 2\frac{D'}{D}W_2' - \beta^2W_2 &= 0, \end{aligned} \right\} \quad (6)$$

where $W=dW/dx$; $D(x)=(e^{-3x}-e^{-4x})^{1/2}$. If the expression $D(x)$ is replaced with function

$$D_1(x) = D_{01} \frac{\sqrt{x}}{x^2 + C_0}, \quad (7)$$

then it is possible, in the interval $x=0 \div 1$, at $D_{01}=0.21$; $C_0=0.2484$, to obtain a satisfactory match between the D and D_1 functions. In this case, as noted in work [16], it becomes possible to derive accurate solutions to equations (6) using the symmetry method, which take the following form:

$$\left. \begin{aligned} W_1 &= (x^2 + C_0) \left\{ \begin{aligned} &A \left[\alpha J_0(\alpha x) - \frac{2x}{x^2 + C_0} J_1(\alpha x) \right] + \\ &+ B \left[\alpha Y_0(\alpha x) - \frac{2x}{x^2 + C_0} Y_1(\alpha x) \right] \end{aligned} \right\}; \\ W_2 &= (x^2 + C_0) \left\{ \begin{aligned} &A_1 \left[\beta I_0(\beta x) - \frac{2x}{x^2 + C_0} I_1(\beta x) \right] - \\ &- B_1 \left[\beta K_0(\beta x) + \frac{2x}{x^2 + C_0} K_1(\beta x) \right] \end{aligned} \right\}, \end{aligned} \right\} \quad (8)$$

where $[J, Y, I, K]$ are the Bessel functions. The sum (8) is the desired general solution at which the values of constants A, B, A_1, B_1 depend on the boundary conditions for the plate in question. This solution, therefore, takes the following form:

$$W = W_1 + W_2 = (x^2 + C_0) \left\{ \begin{aligned} &A(P(x)) + B(Q(x)) + \\ &+ A_1(T(x)) - B_1(U(x)) \end{aligned} \right\}, \quad (9)$$

where

$$P(x) = \alpha J_0(\alpha x) - \frac{2x}{x^2 + C_0} J_1(\alpha x);$$

$$Q(x) = \alpha Y_0(\alpha x) - \frac{2x}{x^2 + C_0} Y_1(\alpha x);$$

$$T(x) = \beta I_0(\beta x) - \frac{2x}{x^2 + C_0} I_1(\beta x);$$

$$U(x) = \beta K_0(\beta x) + \frac{2x}{x^2 + C_0} K_1(\beta x),$$

$$\alpha^2 = +k_1^2;$$

$$\beta^2 = -k_2^2.$$

We find from derivatives

$$\left. \begin{aligned} W_1' &= -\alpha^2(x^2 + C_0)[AJ_1(\alpha x) + BY_1(\alpha x)]; \\ W_2' &= \beta^2(x^2 + C_0)[A_1I_1(\beta x) + B_1K_1(\beta x)] \end{aligned} \right\} \quad (10)$$

the following:

$$W' = (x^2 + C_0) \left[\begin{aligned} &-\alpha^2AJ_1(\alpha x) - \alpha^2BY_1(\alpha x) + \\ &+ \beta^2A_1I_1(\beta x) + \beta^2B_1K_1(\beta x) \end{aligned} \right]. \quad (11)$$

With a general solution (9) and its derivative (11), one can consider a series of problems on the oscillations of a plate of variable thickness of the parabolic profile under different boundary conditions. A plate with a point support is a particular case of the circular plate, fixed in the center, at $\rho=\rho_1$ if $\rho_1 \rightarrow 0$. The technique of moving from a general solution to the solution for such a plate is outlined in the following chapter 5.

5. Transforming a general solution to the form that meets the conditions on the support

Let the closing along the inner contour $\rho=\rho_1$ of the circular plate be rigid, then the boundary conditions should be fulfilled

$$(W)_{\rho=\rho_1} = 0; (W_\rho)_{\rho=\rho_1} = 0.$$

Given that $W_\rho = W_x x_\rho$, these conditions while moving to variable x , will be rewritten in the following form

$$(W)_{x=x_1} = 0; (W')_{x=x_1} = 0. \quad (12)$$

There are two cases if $x_1=0$. In the first case, if a plate is solid and there is no support in its center, then, at $x=0$, the plate deflection should be finite; therefore, in (9), it is necessary to adopt $B=0$ and $B_1=0$ because functions $Y_0(\alpha x)$, $Y_1(\alpha x)$, $K_0(\beta x)$, $K_1(\beta x)$ at $x \rightarrow 0$ tend to infinity. We shall, therefore, derive a general solution for a solid plate without a support in the center, which takes the following form

$$W = (x^2 + C_0) \left\{ \begin{aligned} &A \left[\alpha J_0(\alpha x) - \frac{2x}{x^2 + C_0} J_1(\alpha x) \right] + \\ &+ A_1 \left[\beta I_0(\beta x) - \frac{2x}{x^2 + C_0} I_1(\beta x) \right] \end{aligned} \right\}.$$

This case will not be considered here.

In the second case, when a plate is supported in the center (Fig. 1), it is obvious that $B \neq 0$ and $B_1 \neq 0$ and, therefore, it is necessary, when meeting conditions (12), to take into consideration the behavior of the Bessel functions in the vicinity of the point $x=0$.

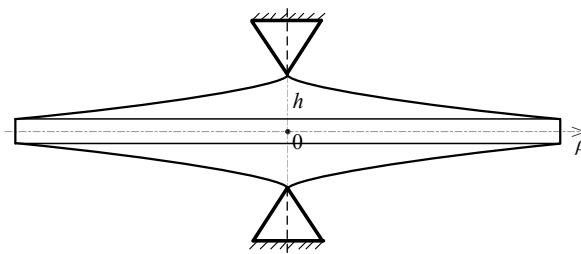


Fig. 1. Graphic representation of the plate with a point support

After introducing (9) and (11) to (12), we obtain, at $x=x_1=0$

$$\left. \begin{aligned} W(0) &= C_0(\alpha A + \beta A_1) + \\ &+ C_0 \lim_{x \rightarrow 0} [\alpha B Y_0(\alpha x) - \beta B_1 K_0(\beta x)] - 2G(x) = 0, \\ W'(0) &= C_0 \lim_{x \rightarrow 0} [-\alpha^2 B Y_1(\alpha x) + \beta^2 B_1 K_1(\beta x)] = 0, \end{aligned} \right\} \quad (13)$$

where

$$G(x) = \lim_{x \rightarrow 0} [x(BY_1(\alpha x) + B_1 K_1(\beta x))].$$

Taking into consideration the decomposition of Bessel functions into the power series [17], we shall obtain, for the limits included in (13):

$$\begin{aligned} &\lim_{x \rightarrow 0} [\alpha B Y_0(\alpha x) - \beta B_1 K_0(\beta x)] = \\ &= \alpha B \frac{2}{\pi} \left(\gamma + \ln \frac{\alpha}{2} \right) + \beta B_1 \left(\gamma + \ln \frac{\beta}{2} \right) + N(x), \end{aligned} \quad (14)$$

where $\gamma = 0.5772157$ is the Euler constant.

$$\begin{aligned} N(x) &= \lim_{x \rightarrow 0} \left[\frac{2}{\pi} \alpha B J_0(\alpha x) \ln x + \beta B_1 I_0(\beta x) \ln x \right]; \\ \lim_{x \rightarrow 0} \{ x [BY_1(\alpha x) + B_1 K_1(\beta x)] \} &= \\ &= -\frac{2}{\pi \alpha} B + \frac{1}{\beta} B_1 + R(x), \end{aligned} \quad (15)$$

here

$$\begin{aligned} R(x) &= \lim_{x \rightarrow 0} \left\{ \frac{2}{\pi} B J_1(\alpha x) x \ln x + \right. \\ &\left. + B_1 I_1(\beta x) x \ln x \right\}; \\ \lim_{x \rightarrow 0} [-\alpha^2 B Y_1(\alpha x) + \beta^2 B_1 K_1(\beta x)] &= \\ &= \lim_{x \rightarrow 0} \left\{ -\frac{2\alpha^2}{\pi} B \left[J_1(\alpha x) \ln x - \frac{J_0(\alpha x)}{\alpha x} \right] + V(x) \right\}, \end{aligned} \quad (16)$$

where

$$V(x) = \beta^2 B_1 \left[I_1(\beta x) \ln x + \frac{I_0(\beta x)}{\beta x} \right].$$

If one accepts

$$\frac{2\alpha}{\pi} B = -\beta B_1 \rightarrow B_1 = -\frac{2}{\pi} \frac{\alpha}{\beta} B, \quad (17)$$

then, since $J_0(0) = I_0(0) = 1$, expression (14) receives the resulting value

$$\begin{aligned} &\frac{2\alpha}{\pi} B \left(\gamma + \ln \frac{\alpha}{2} \right) - \frac{2\alpha}{\pi} B \left(\gamma + \ln \frac{\beta}{2} \right) = \\ &= \frac{2\alpha}{\pi} B \ln \frac{\alpha}{\beta} = -\beta B_1 \ln \frac{\alpha}{\beta}. \end{aligned} \quad (18)$$

The limit (15), given (17) and decompositions (17) for $J_1(\alpha x)$, $I_1(\beta x)$, as well as taking into consideration the limits

$$\lim_{x \rightarrow 0} x^n \ln x = 0$$

will, in turn, be finite and equal to the magnitude

$$\begin{aligned} &-\frac{2}{\pi \alpha} B + \frac{1}{\beta} B_1 = \\ &= -\frac{2\alpha}{\pi} B \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) = \beta B_1 \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} \right), \end{aligned} \quad (19)$$

and expression (16), additionally, taking into consideration the decompositions for $J_0(\alpha x)$, $I_0(\beta x)$ in the limit will equal zero, that is

$$\begin{aligned} &\lim_{x \rightarrow 0} B \frac{2\alpha}{\pi} \left[\frac{J_0 - I_0}{x} \right] = \\ &= \lim_{x \rightarrow 0} \frac{2\alpha}{\pi} B \left[\frac{-\alpha J_1 - \beta I_1}{1} \right] = \\ &= -\frac{2\alpha B}{\pi} (\alpha \cdot 0 + \beta \cdot 0) = 0. \end{aligned} \quad (20)$$

Taking into consideration the results from (18) to (20), we shall obtain, for a point support, a single boundary condition instead of two conditions (12)

$$W(0) = 0,$$

i. e.

$$\alpha A + \beta A_1 + SB = 0, \quad (21)$$

where

$$S = \frac{2\alpha}{\pi} \left[\ln \frac{\alpha}{\beta} + \frac{2}{C_0} \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) \right]. \quad (22)$$

Constant B is to be determined from (21)

$$B = -\frac{1}{S} (\alpha A + \beta A_1). \quad (23)$$

Following the substitution of (23) in (8) and (10), we obtain

$$\left. \begin{aligned} W_1 &= (L) \left\{ \begin{aligned} &A \left[\alpha \left(J_0 - \frac{\alpha}{S} Y_0 \right) - \frac{2x}{L} \left(J_1 - \frac{\alpha}{S} Y_1 \right) \right] - \\ &- A_1 \frac{\beta}{S} \left(\alpha Y_0 - \frac{2x}{L} Y_1 \right) \end{aligned} \right\}; \\ W_2 &= L \left\{ \begin{aligned} &-A \frac{2\alpha^2}{\pi \beta S} \left(\beta K_0 + \frac{2x}{L} K_1 \right) + \\ &+ A_1 \left[\beta \left[I_0 - \frac{2\alpha}{\pi S} K_0 \right] - \frac{2x}{L} \left(I_1 + \frac{2\alpha}{\pi S} K_1 \right) \right] \end{aligned} \right\}; \\ W_1' &= -L\alpha^2 \left[A \left(J_1 - \frac{\alpha}{S} Y_1 \right) - A_1 \frac{\beta}{S} Y_1 \right]; \\ W_2' &= L\beta^2 \left[A \frac{2\alpha^2}{\pi \beta S} K_1 + A_1 \left[I_1 + \frac{2\alpha}{\pi S} K_1 \right] \right], \end{aligned} \right\} \quad (24)$$

where $L = x^2 + C_0$.

Thus, the resulting equations (24) that meet conditions (21) make it possible to build a frequency equation, based only on the conditions at the edge $\rho = \rho_2$.

6. Deriving a frequency equation, computing the frequencies, and building the oscillation shapes of the plate with a free contour

If the contour $\rho=\rho_2$ is free, then, in the event of an asymmetrical deformation, the radial bending momenta and transverse forces on the contour are zero, that is

$$\begin{aligned} \left(W_{\rho\rho} + \frac{\nu}{\rho} W_{\rho} \right)_{\rho=\rho_2} &= 0; \\ \left(W_{\rho\rho\rho} + \frac{1}{\rho} W_{\rho\rho} - \frac{1}{\rho^2} W_{\rho} \right)_{\rho=\rho_2} &= 0. \end{aligned} \tag{25}$$

After the transition to variable $x=-\ln(1-\mu\rho)$, by completing the required transformations and a series of simplifications, we shall obtain, instead of (25), at $\nu=1/3$, the following conditions

$$\begin{aligned} \left[B^* W_1' + B^* W_2' - \alpha^2 W_1 + \beta^2 W_2 \right]_{x=x_2} &= 0; \\ \left[(A^* + \alpha^2) W_1' + (A^* - \beta^2) W_2' \right]_{x=x_2} &= 0, \end{aligned} \tag{26}$$

where

$$\begin{aligned} A^* &= \left[q + p \frac{3x^2 - C_0}{x(x^2 + C_0)} + \frac{3x^4 - 6C_0x^2 - C_0^2}{x^2(x^2 + C_0)^2} \right]_{x=x_2}; \\ B^* &= \left[p + \frac{3x^2 - C_0}{x(x^2 + C_0)} \right]_{x=x_2}; \\ q &= \left[1 + \frac{3e^x + 1}{3(e^x - 1)^2} \right]_{x=x_2}; \\ p &= \left[1 + \frac{1}{3(e^x - 1)} \right]_{x=x_2}. \end{aligned}$$

The substitution of (24) in (26) at $x=x_2$ leads to two equations relative to the desired constants A and A_1

$$\begin{cases} AL(x_2) + A_1L_1(x_2) = 0; \\ AL_2(x_2) + A_1L_3(x_2) = 0. \end{cases} \tag{27}$$

By equating the system's (27) determinant to zero, we obtain a frequency equation of the plate in question with a point support, that is

$$LL_3 - L_1L_2 = 0, \tag{28}$$

where

$$\begin{aligned} L &= \alpha^2 \left(-[1] + \frac{\alpha}{S}[2] + \frac{2\beta}{\pi S}[4] \right); \\ L_1 &= \beta \left(\frac{\alpha^2}{S}[2] + \beta[3] + \frac{2\alpha\beta}{\pi S}[4] \right); \\ L_2 &= \alpha^2 \left[a^* \left(-J_1 + \frac{\alpha}{S}Y_1 \right) + \frac{2\beta}{\pi S}b^*K_1 \right]; \end{aligned}$$

$$L_3 = \beta \left[\beta b^* \left(I_1 + \frac{2\alpha}{\pi S}K_1 \right) + \frac{\alpha^2 a^*}{S}Y_1 \right];$$

$$[1] = C^*J_1 + \alpha J_0; \quad [2] = C^*Y_1 + \alpha Y_0;$$

$$[3] = C^*I_1 + \beta I_0; \quad [4] = C^*K_1 - \beta K_0;$$

$$C^* = B^* - \frac{2x}{x^2 + C_0};$$

$$a^* = A^* + \alpha^2;$$

$$b^* = A^* - \beta^2;$$

$$J, Y = J(\alpha x), Y(\alpha x);$$

$$I, K = I(\beta x), K(\beta x).$$

It is easy to determine the amplitude coefficients from the system of equations (27)

$$\begin{aligned} \frac{A}{A_1} &= -\frac{L_1}{L} = -\frac{L_3}{L_2} = \\ &= -\frac{\beta}{\alpha} \left[1 + \frac{SJ_1 + MSI_1}{-SJ_1 + \alpha Y_1 + \frac{2\alpha}{\pi}MK_1} \right], \end{aligned} \tag{29}$$

where

$$M = \frac{\beta A^* - \beta^2}{\alpha A^* + \alpha^2}.$$

The oscillation shapes W , according to (24), will be determined from the following expression

$$W = (x^2 + C_0)A_1 \{ N_1 - N_2 - N_3 + N_4 \}, \tag{30}$$

where

$$N_1 = \frac{A}{A_1} \alpha J_0 + \beta I_0;$$

$$N_2 = \left(\frac{2x}{x^2 + C_0} \right) \left(\frac{A}{A_1} J_1 + I_1 \right);$$

$$N_3 = \frac{\alpha}{S} \left(\alpha \frac{A}{A_1} + \beta \right) \left(Y_0 + \frac{2}{\pi} K_0 \right);$$

$$N_4 = \frac{2x}{S(x^2 + C_0)} \left(\frac{A}{A_1} \alpha + \beta \right) \left(Y_1 - \frac{2}{\pi} \frac{\alpha}{\beta} K_1 \right).$$

Thus, equation (28) makes it possible to determine the natural frequencies of a free plate of variable thickness with a rigid point support. Expression (30) defines the functions of the plate deflections.

To illustrate the effectiveness of the constructed problem-solving algorithm, a circular plate was chosen, whose ratio of the limiting edge thicknesses $h(0)/h(\rho_2)=10.8$ at $\rho_2=0.5$, which corresponds to $\mu=1.39127$.

Based on the obtained estimation ratios, we determined the frequency numbers of plate λ_i as the solutions to the corresponding frequency equation (28) (Table 1).

In order to construct the first three oscillation shapes, one determines, using expression (30), after substituting the

frequency numbers λ_j ($j=1, 2, 3$) in it, the amplitude coefficients A/A_1 . A graphic representation of the deflections for the first three oscillation shapes is shown in Fig. 2–4.

Table 1

Natural plate frequencies and oscillation nodes coordinates

Oscillation shape number	Natural frequency	Coordinates of oscillation nodes ρ_0	Coordinates of oscillation antinodes ρ_{extri}	A/A_1
I	$\lambda_1=3.911329$	0	–	-59.283
II	$\lambda_2=6.635320$	0; 0.413	0.303	153.334
III	$\lambda_3=9.872506$	0; 0.307; 0.453	0.2; 0.396	-909.441
IV	$\lambda_4=13.334707$	0; 0.241; 0.375; 0.469	0.147; 0.318; 0.43	8.564×10^3
V	$\lambda_5=16.892550$	0; 0.198; 0.316; 0.409; 0.477	0.116; 0.263; 0.368; 0.448	-1.042×10^5

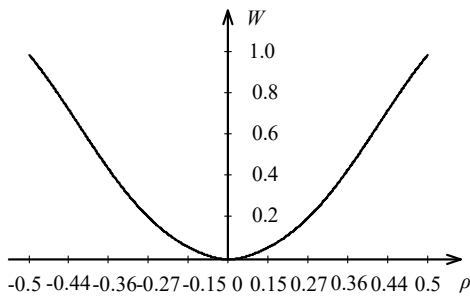


Fig. 2. Graphic representation of round plate deflections on the first oscillation shape

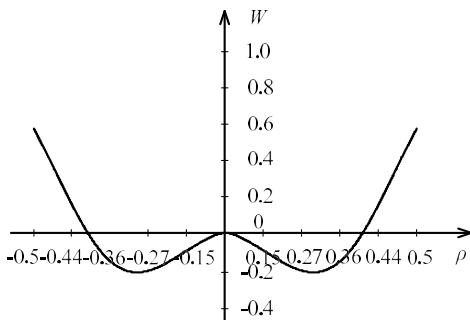


Fig. 3. Graphic representation of round plate deflections on the second oscillation shape

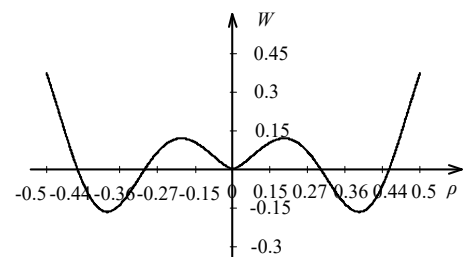


Fig. 4. Graphic representation of round plate deflections on the third oscillation shape

Note that, based on the classical theory of plates, dangerous radial stresses act in the zone of maximum deflections. It is known that the distribution of stresses on a support has a special character.

7. Discussion of results of solving the problem about a plate with a point support

The resulting general solution to differential equation (1) makes it possible to consider a series of specific problems on the oscillations of parabolic plates – circular and solid ones, with different types of contour fastening. This paper considers the ultimate case of a plate rigidly fixed on the inner contour ($\rho=\rho_1$), whose radius $\rho_1 \rightarrow 0$. There are two objectives for stating this type of problem. First, there are structures that demonstrate the “point” fastening; therefore, one must have an algorithm to analyze their oscillations. Second, it is often impossible to judge with certainty the scale of the fastening, but, given the relatively cumbersome way of obtaining a rigorous solution to the problem, the assumption about a “point” fastening is made. One can derive a solution for the case of point fastening, which passes the limit to the point support, if, by following conditions (12) and (26), one builds four equations regarding the integration constants A, B, A_1, B_1 . A rather cumbersome determinant of the constructed system of algebraic equations of the fourth order, equal to zero, is a frequency equation. Further, accepting the condition $\rho_1 \rightarrow 0$, it is necessary to run a mathematical analysis of the resulting equation, reducing it to the form containing the final terms. This path of transition to a point type of fastening is not productive, as it is still cumbersome because it is based on general boundary conditions. Another way is to transform the general solution (8) to the form that pre-meets the conditions on a rigid support. As a result of the transformations performed, we derived, instead of (8), a general form of the solution W as the sum of solutions (24) for a solid plate with a point support under any conditions on the contour $\rho=\rho_2$. This solution contains only two basic integration constants A and A_1 , so, in this paper, the frequency equation (for a free contour case) has been easily derived from the order II determinant. Such a problem-solving scheme is based on the assumption about the independence of frequencies on the magnitude of fastening radius ρ_1 if it is small compared to ρ_2 .

For a plate of constant thickness, it is believed that, if $\rho_1/\rho_2 < 0.2$, the fastening is of the point type. Based on the calculation results, in particular, from determining the coordinates ρ_0 of the first nodal circles S for five oscillation shapes, it can be assumed that the ratio $\rho_1/\rho_2 < 0.2$ remains valid for the plate of variable thickness. The basis for this conclusion is the comparison of ρ_0 for the plates of constant thickness [18] and the plate, examined in the given example, whose ratio of limit thicknesses $h(0)/h(0,5)=10.8$. The results of the comparison and the relative error δ of deviation in the parameters are given in Table 2.

Table 2

Comparison of parameter ρ_0 for the plates of constant thickness and the plate considered in this paper

Oscillation shape	I	II	III	IV	V
S , quantity	0	1	2	3	4
$\rho_0, h=\text{const}$	0	0.434	0.4525	0.464	0.4715
$\rho_0, h \neq \text{const}$	0	0.413	0.453	0.469	0.477
$\delta, \%$	0	-4.8	0	1	1.16

The location of nodal circles, given in Table 1, could in practice serve as a criterion for determining the character of fixation. If a plate is fixed in the center by some diameter,

then, by exciting the oscillations in it of one of the shapes (from II to V) and by measuring the radius R_0 of a first (from the edge) nodal circle, we compare the magnitude $R_0/0.5$ to ρ_0 , which should be approximately equal. If the relative value of this radius is not much different from, for example, $\rho_0=0.453$ for shape III, the fastening should be assumed to be of the point type; the oscillations should be analyzed based on the model of a plate with a point support.

The limitation of the current study stems from the algorithm to produce a general solution that requires lowering the order of the original differential equation. The factorization method, used for this purpose, while not universal, pre-restricts the form of functions at which it holds. These include parabolic functions.

The solution to the problem, reported in our paper, seems to be one of the possible cases of the theory being applied to the practical calculation of variable thickness plates. The theoretical basis of such calculations is the method of symmetry, developed and implemented as a general solution to equation (1).

Given this, in the future it is easy to consider, for this type of variable thickness plates, by directly using expressions (24), the free oscillations of a plate rigidly- or freely fixed along the contour with varying degrees of concaveness, determined from the parameter μ .

8. Conclusions

1. A general analytical solution to the IV-order differential equation has been derived for a problem on the free axisymmetric oscillations of a plate whose thickness $h=H_0(1-\mu\rho)^2$. This solution makes it possible to consider a series of problems about oscillations of parabolic plates – solid and circular ones, with different types of contour fastening, including a point support in the center.

2. The general solution has been transformed into the form that pre-meets the conditions on a rigid point support. The transform result is a simpler solution, which includes, instead of four desired integration constants, only two. Therefore, the frequency equation could be easily obtained from the second-order determinant under any conditions on the contour.

3. The frequency equation for a free-edged plate at $\mu=1.39127$ (the ratio of limit thicknesses is 10.8) has produced the first five eigenvalues λ_i . For λ_i at $i=1,2,3$ the oscillation shapes have been built as a graphic illustration. We have given the numerical values of amplitude ratios, the coordinates of oscillation antinodes and nodal circles in the form of relative radii for each of the five oscillation shapes ($i=1-5$). All numerical oscillation parameters could in practice be used to identify the characteristics and types of plate oscillations.

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