

*Розв'язано задачу про визначення закону руху фронту кристалізації і термомеханічного стану двофазного стержня у випадку взаємного впливу температурних і механічних полів. Для розв'язання задачі було використано наближений аналітичний метод, в сукупності з методом послідовних інтервалів і варіаційним принципом Гіббса. Цей метод має показати що «вигідніше» природі при заданих зовнішніх впливах – змінити температуру фіксованого елемента тіла або перевести цей елемент з одного агрегатного стану в інший. Саме такий підхід дозволив врахувати вплив температури на напружено-деформований стан в тілі і навпаки через закон руху межі розділу фаз, що визначається. Отримано співвідношення для визначення закону руху межі розділу фаз, температурного поля і напружено-деформованого стану в стержні. Результати представлені у вигляді графіків залежності температури і напружень від часу і координати.*

*Аналіз отриманих результатів показує, що зміна умов теплообміну з навколишнім середовищем і геометричних розмірів роблять визначальний вплив на процес кристалізації, а, отже, і на температурні і механічні поля. Основний результат полягає в наступному: розроблено наближений аналітичний метод і алгоритм розв'язання задачі термов'язкопружності для тіл, що ростуть (тіл з рухомою границею) при наявності фазового переходу з урахуванням теплообміну з навколишнім середовищем. На підставі розробленого методу закон руху границі розділу фаз, температурне поле і напружено-деформований стан визначаються в ході розв'язання так званої квазізв'язаної задачі термов'язкопружності. Отримано наближений аналітичний розв'язок, який може бути використаний в науково-дослідних і проєктних організаціях при моделюванні різних технологічних процесів, в машинобудуванні, металургії, ракетно-космічній техніці, в будівництві*

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# THE STRESSED-STRAINED STATE OF A ROD AT CRYSTALLIZATION CONSIDERING THE MUTUAL INFLUENCE OF TEMPERATURE AND MECHANICAL FIELDS

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## 1. Introduction

Recently, the linear theory of thermal conductivity can no longer meet the requirements of both new fields of technology and traditional industrial sectors, such as heat generation, machine building, and, especially, metallurgy. Modern practical tasks require taking into consideration the essential non-stationarity, heterogeneity, non-linearity, and other features to which mathematical methods developed in the classical (linear) theory of thermal conductivity are hardly applicable. Such problems can only be solved through the use of non-linear mathematical modeling.

It is known that thermomechanical processes are inherently non-linear. Examples include a series of problems related to modern technology. The aerodynamic heating and destruction of bodies when moving in the atmosphere are considered in [1]; high-temperature processes in rocket engines – in work [2]. Accounting for temperature stresses in the structural elements at high-temperature heat loads is necessary for the development of a new class of small solid-fuel rocket engines [3]. New technologies of thermocyclic destruction of rocks were considered in [4]; the radiation heating in metallurgical furnaces – in [5]. Paper [6] examines technologies that use phase transitions; the movements of elastic waves in variable-length ropes for machines that lift and lower weights using ropes – in [7].

The specified processes are associated with a significant change in heat-physical characteristics over a wide range of temperature changes. All these are non-linearities of the I, II, and III kinds; it is impossible to further improve modern technologies without considering them. Mathematical difficulties often force a researcher to approach the direct linearization of thermomechanics equations (thermal conductivity and thermoelasticity). In many cases, researchers turn to numerical methods that make it possible to derive approximate solutions to the non-linear problems of mechanics. In particular, work [8] considers the numerical modeling of nonlinear processes in the mechanics of continuous media; [9] solves a non-linear thermal problem in the presence of phase transitions. Paper [10] proposes a numerical procedure to solve the problem of deforming a polymeric crystallizing environment, taking into consideration large deformations. Work [11] reports the results of computer simulations of casting crystallization. Despite the versatility of numerical methods, the results of such studies are approximate, they often involve the linearization of equations and boundary conditions, and are not always convenient for analysis.

An effective way to study the non-linear problems of mechanics is the use of approximate-analytical approaches. The relevance of this scientific issue is predetermined by that the improvement of thermal technologies in order to preserve energy resources, the design of better structures of industrial machines, the choice of their optimal operational modes is an important industrial task. It is impossible to resolve it without careful and complete mathematical modeling of the crystallization process. This is due to the fact that experimental studies into the features of metal crystallization, for example, in the laypoint of a centrifugal machine are typically complicated (high melt temperatures, the presence of rotation, a gassed mold cavity, etc.).

The proposed scientific research should be carried out to ensure that their results could be used in research and design organizations in the simulation of various technological processes in machine building, metallurgy, rocket and space technology, construction, as well as in the training process.

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## 2. Literature review and problem statement

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Study [12] addresses the construction of research methods only for contact problems involving the elements of structures for casting, taking into consideration phase transitions. The authors consider an associated problem of thermoelasticity in the form of a system of differential equations in particular derivatives and inequalities, which comes down to variational inequalities. However, the issues related to determining the change law of an interphase boundary and taking into consideration the viscoelastic properties of material remained unresolved. The problem of a semi-space with microstructural transformations, exposed to the influence of thermal and mechanical pulses, was considered in [13]. The problem is solved numerically by the method of step-by-step implicit integration over time. There are quantitative estimates of temperature effects of thermostructural mechanical bonding. However, the issue of phase transition was not addressed at all. The initial and boundary-value problems describing the

thermomechanical behavior of alloys with a shape memory were investigated in work [14]. Some issues related to non-linearity problems in metallurgical thermomechanics were considered in [15]; the approaches to solving them were outlined. In addition, the cited paper notes the need to develop and construct analytical procedures to study the nonlinear thermal processes; however, no specific analytical methods were considered. Work [16] derived the defining equations for growing zero-thickness surfaces and their spatial discretization using a finite element method. Issues related to the reciprocal influence of mechanical and temperature fields were not addressed. Paper [17] tackles the construction of linear models for thin plates made from polymer gels. A multi-scale approach to modeling the hardening process in magnetic-sensitive polymer materials was proposed in [18]. Work [19] gives a one-dimensional formulation of the finite elements for temporal analysis of geometrically nonlinear frames associated with viscoelastic viscoplastic materials, including mechanical destruction. The issues of numerical modeling of the interaction among inclusions in a flat element of the elastic heterogeneous environment, depending on their orientation, shape, size, and rigidity characteristics, were considered in [20]. The safest options for their mutual location were established. In order to analyze significantly heterogeneous stresses [21], associated with the presence of foreign structures in bodies in the form of inclusions, cracks, pores, etc., the projection and iterative schemes of the implementation of network methods are effective [22].

Thus, one should note that the issue of determining the stressed-strained state of a body, which is in a phase transition, has not yet been considered or addressed comprehensively. Namely, there is no proposed approach, which would make it possible to determine not only the tense-deformed state of a body taking into consideration the mutual influence of temperature and mechanical fields but also to define, based on this relation, the motion law of a crystallization front.

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## 3. The aim and objectives of the study

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The aim of this study is to determine the stressed-strained state in a rod at crystallization, taking into consideration the mutual influence of temperature and mechanical fields.

To accomplish the aim, the following tasks have been set:

- to build an approximate analytical method and an algorithm to solve the quasi-related problem of thermoviscoelasticity for growing bodies in the presence of a phase transition, taking into consideration the heat exchange with the environment;
- to define, based on the constructed method, the motion law of an interphase boundary, a temperature field, and the stressed-strained state in a rod from the solution to the quasi-related problem of thermoviscoelasticity.

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## 4. The stressed-strained state of a rod at crystallization, taking into consideration the relatedness between the temperature and mechanical fields

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### 4. 1. Problem statement

Consider a rectilinear rod of length  $l$ . Introduce a Cartesian coordinate system  $(x_1, x_2, x_3)$  in an undeformed

configuration, with the  $x_3=x$  axis directed along the rod axis. One end of the rod  $x=0$  is tightly pinched, and the other,  $x=l$ , is free. Suppose that the rod substance exists in two phases – high-temperature and low-temperature. The behavior of a rod's material in the low-temperature phase is described by the equations of state of a non-uniformly aging viscoelastic body, and in the high-temperature – of an elastic body. Denote the temperature of a phase transition through  $\theta^0$ : at  $\theta_2 < \theta^0$ , the material's element is in the low-temperature phase, at  $\theta_1 > \theta^0$  – in the high-temperature one. For certainty, we believe that before a deformation, the substance of a rod in the low-temperature phase occupies region  $\omega_2(t) = \{l_0 \leq x \leq l\}$ , and the substance in the high-temperature phase – region  $\omega_1(t) = \{0 \leq x \leq l_0\}$ . The thermal interaction between the lateral and end surfaces of the low-temperature phase of the rod with the environment is characterized by convective heat release coefficients  $\alpha_{1,conv}$ ,  $\alpha_{2,conv}$  (respectively), and the radiation. In this case, the high-temperature phase is supposed to be heat-isolated.

At time point  $t=0$ , a longitudinal distributed compressing load of intensity  $P(t)$  is applied to the rod; the heat is released into the environment from the free end of the rod  $x=l$ . After this, part of the rod material enters the low-temperature phase (material buildup) so that its length changes according to the law  $x=a=a(t)$ .

It is required to define, under the assigned law of change in the external load  $P=P(t)$ , the motion law of an interphase boundary  $a=a(t)$ , the temperature  $\theta_i=\theta_i(t,x)$  ( $i=1,2$ ) and the stressed-strained state in the rod for the case of a single-axial strained state.

A mathematical statement of the formulated problem of thermoviscoelasticity for a rod, growing under the phase transition conditions, implies finding the enumerated unknowns from the joint solution to the following equations:

$$\begin{aligned} \varepsilon_i(t,x) &= \frac{\partial u_i(t,x)}{\partial x}, \\ \frac{\partial \sigma_i(t,x)}{\partial x} - \rho_i P(t) &= 0, \\ \sigma_i(t,x) &= E_i(t-\tau^*(x))\varepsilon_i(t,x) - \\ &- \int_{\tau^*(x)}^t \varepsilon_i(\tau,x) R(t-\tau^*(x),\tau-\tau^*(x)) d\tau, \quad (i=1,2), \end{aligned} \quad (1)$$

$$\begin{aligned} \rho_1 c_1 \frac{\partial \theta_1(t,x)}{\partial t} &= \frac{\partial}{\partial x} \left( \lambda_1 \frac{\partial \theta_1(t,x)}{\partial x} \right), \\ \rho_2 c_2 \frac{\partial \theta_2(t,x)}{\partial t} &= \frac{\partial}{\partial x} \left( \lambda_2 \frac{\partial \theta_2(t,x)}{\partial x} \right) + \Phi(\theta_2, \alpha_2), \end{aligned} \quad (2)$$

where  $E_i(t-\tau^*(x))$  is the elastically instantaneous deformation modulus;  $R(t-\tau^*(x), \tau-\tau^*(x))$  is the relaxation core of a viscoelastic material;  $\tau^*(x)$  is the moment when a body element enters a solid aggregate state;  $\rho_i, c_i, \lambda_i = \lambda_i(\theta_i), \alpha_i = \alpha_i(\theta)$  is the density, heat-conductivity, the thermal conductivity factor of the material, the reduced heat release factor [23], which characterizes the intensity of convective and radiant heat exchange with the environment;  $\theta_i(t,x)$  is the distribution of temperature in the corresponding phase of the rod;

$$\begin{aligned} \Phi(\theta_2, \alpha_2) &= -\frac{\alpha_{2,conv}}{h\lambda_2} [\theta_2(t_k,x) - \theta_{env}] - \\ &- \frac{\varepsilon_{12}\sigma_0}{h\lambda_2} [\theta_2^4(t_k,x) - \theta_{env}^4] = -\frac{\alpha_2}{h\lambda_2} [\theta_2(t_k,x) - \theta_{env}] \end{aligned}$$

the power of internal sources (stock) of heat in the low-temperature phase, which also takes into consideration the convective and radiant heat exchange between the side surface of the rod and the environment;  $\varepsilon_{12}$  is the degree of a material's blackness in a solid phase;  $\sigma_0$  is the Stefan-Boltzmann's constant;  $h=s/p$ ,  $s$  is the cross-sectional area of the rod;  $p$  is perimeter;  $\theta_{env}$  is the temperature of the environment.

In ratios (1), (2), and hereafter, the index "1" refers to the parameters of a substance in the high-temperature phase, the index "2" – in the low-temperature phase.

The boundary and initial conditions of the problem take the form

$$\begin{aligned} \sigma_2(t,l) &= 0, \quad u_1(t,0) = 0, \\ \sigma_1(t,a(t)) &= \sigma_2(t,a(t)), \\ u_1(t,a(t)) &= u_2(t,a(t)); \end{aligned} \quad (3)$$

$$\begin{aligned} \theta_1(t,0) &= f, \\ \theta_2(t,a(t)) &= \theta_1(t,a(t) - \Delta) = \theta^0, \end{aligned}$$

$$\begin{aligned} -\lambda_2 \frac{\partial \theta_2(t,x)}{\partial x} \Big|_{x=l} &= \alpha_{1,conv} (\theta_2(t,x) - \theta_{env}) \Big|_{x=l} + \\ &+ \varepsilon_{12}\sigma_0\theta_2^4(t,l) = \alpha_1 [\theta_2(t,l) - \theta_{env}], \\ -\lambda_1 \frac{\partial \theta_1(t,x)}{\partial x} \Big|_{x=a(t)} &= -\lambda_2 \frac{\partial \theta_2(t,x)}{\partial x} \Big|_{x=a(t)} + \rho_1 \mu \frac{da(t)}{dt}; \end{aligned} \quad (4)$$

$$\begin{aligned} \theta_1(t,x) \Big|_{t=0} &= \theta_1(0,x), \\ \theta_2(t,x) \Big|_{t=0} &= \theta_2(0,x), \end{aligned} \quad (5)$$

where  $\mu$  is the heat of the phase transition.

#### 4. 2. The method and algorithm to solve the problem

Ratios (1) to (5) are the statement of the thermoviscoelasticity problem for a growing two-phase rod in a closed form. The difference between this statement and those considered in [24] is that the thermal conductivity problem (2), (4), (5) takes into consideration the non-linearity of the first kind due to the dependence of the thermal-physical characteristics of material on temperature. One approach to solving non-linear thermal conductivity problems is the method of successive intervals [25], which implies splitting the entire process into finite intervals, in each of which the material's characteristics and the internal heat sources are constant. In such a statement, the non-linear problem of thermal conductivity is reduced to solving a set of linear problems with different initial and boundary conditions for each time interval. Therefore, to solve problem (1) to (5), it is proposed to use the approximate analytical method, suggested in [24], in combination with the method of successive intervals and a Gibbs variation principle in the following interpretation.

The time interval  $[0, t_{fin}]$ , within which we investigate the stressed-strained state in body  $\Omega$ , is divided by points  $t_k = k\Delta$ ,  $\Delta = t_{fin}/N$ ,  $k=0, 1, \dots, N$  into sub-intervals, so that the continuous gain process is replaced with the following discrete process. At moment  $t_k$ , the surface efforts  $f^{(k)}$ , massive forces  $F^{(k)}$  are applied to the body  $\Omega$ ; the amount of thermal energy equal to  $\omega^{(k)}$  is removed from the body. Let the quantities  $\bar{u}_*$  (displacement vector),  $\theta_*$  (temperature field), and  $a = a(t_k)$  (an interphase boundary position) fully describe the true state of the body at each time period. When external conditions change, the body exits the thermodynamic equilibrium  $(\bar{u}_*^{(k-1)}, \theta_*^{(k-1)}, a_*^{(k-1)})$ , the state it was in at moment  $t_k$ , and instantly enters a new equilibrium state  $(\bar{u}^{(k)}, \theta^{(k)}, a^{(k)})$ . In the time interval  $(t_k, t_{k+1})$ , a body  $\Omega$  is in the state of thermodynamic equilibrium  $(\bar{u}^{(k)}, \theta^{(k)}, a^{(k)})$ .

The state  $(\bar{u}^{(k)}, \theta^{(k)}, a^{(k)})$  is a kinematically possible equilibrium state of the body  $\Omega$  at moment  $t_k + 0$  (that is, at  $t > t_k$ ). The true state of the body  $\Omega$  (implemented in reality) at moment  $t_k + 0$  shall be denoted through  $(\bar{u}^*(k), \theta^*(k), a^*(k))$ .

The inner energy of body  $\Omega$  in the states  $(\bar{u}^*(k), \theta^*(k), a^*(k))$  and  $(\bar{u}^{(k)}, \theta^{(k)}, a^{(k)})$  is equal to  $V_*^{(k)}$  and  $V^{(k)}$ , respectively. These quantities are bound by the law of energy conservation

$$V^{(k)} = V_*^{(k-1)} + A^{(k)} - \omega^{(k)}, \tag{6}$$

where  $A^{(k)}$  is the work of external surface and mass forces

$$A^{(k)} = \int_{\partial\omega} \rho^0 \mathbf{F}^{(k)} (\bar{u}^{(k)} - \bar{u}_*^{(k-1)}) d\omega + \int_{\partial\omega} \mathbf{f}^{(k)} (\bar{u}^{(k)} - \bar{u}_*^{(k-1)}) d\partial\omega.$$

The true equilibrium state  $(\bar{u}^*(k), \theta^*(k), a^*(k))$  is characterized by the fact that it delivers the maximum value of a body entropy among all possible equilibrium states determined by equality (6).

In order to fulfill the principle formulated above and thus ensure that mechanical and temperature fields are interconnected, the problem of thermal conductivity (2), (4), (5) must first be solved at each time  $t_k$ . Its solution takes the following form:

$$\theta_1(t_k, x) = \frac{1}{\lambda_1} B (\theta^0 - \theta_{env}) \times \left[ \frac{\alpha_1 - B}{\alpha_1 - B - e^{2B_1(l-a(t_k))} (B + \alpha_1)} - \frac{\alpha_1 + B}{\alpha_1 + B - e^{-2B_1(l-a(t_k))} (\alpha_1 - B)} \right] x + f,$$

$$\theta_2(t_k, x) = \frac{(\alpha_1 - B)(\theta^0 - \theta_{env}) e^{B_1 x}}{e^{B_1 a(t_k)} [\alpha_1 - B] - e^{B_1(2l-a(t_k))} [B + \alpha_1]} - \frac{(\alpha_1 + B)(\theta^0 - \theta_{env}) e^{-B_1 x}}{e^{-B_1(2l-a(t_k))} [\alpha_1 - B] - e^{-B_1 a(t_k)} [\alpha_1 + B]} + \theta_{env},$$

where

$$B = \sqrt{\frac{\alpha_2 \lambda_2}{h}},$$

$$B_1 = \sqrt{\frac{\alpha_2}{\lambda_2 h}}.$$

Next, we shall determine the field of displacement from the solution to the elastic contact problem of two bodies  $\omega_1(t_k)$  and  $\omega_2(t_k)$ . In this case, let us assume that a substance in the liquid phase occupies region  $\omega_1(t_k) = \{0 \leq x \leq a(t_k)\}$ , in the solid phase  $-\omega_2(t_k) = \{a(t_k) \leq x \leq l\}$ .

In this case, the contact problem for two bodies  $\omega_1(t_k)$  and  $\omega_2(t_k)$  at moment  $t_k$  is reduced to solving the following system of equations (the bar means a derivative for coordinate  $x$ )

$$\begin{aligned} \sigma_1(t_k, x) &= E_1 u_1'(t_k, x), \\ \sigma_2(t_k, x) &= E_2 u_2'(t_k, x), \\ \sigma_1'(t_k, x) - \rho_1 P(t_k) &= 0, \\ \sigma_2'(t_k, x) - \rho_2 P(t_k) &= 0, \end{aligned} \tag{7}$$

under boundary conditions

$$\begin{aligned} \sigma_2(t_k, l) &= 0, \\ u_1(t_k, 0) &= 0, \\ \sigma_1(t_k, a(t_k)) &= \sigma_2(t_k, a(t_k)), \\ u_1(t_k, a(t_k)) &= u_2(t_k, a(t_k)). \end{aligned} \tag{8}$$

A solution to problem (7), (8) takes the following form [24]:

$$\sigma_2(t_k, x) = -\rho_2 P(t_k) (l - x); \tag{9}$$

$$\sigma_1(t_k, x) = -P(t_k) [\rho_2 [l - a(t_k)] + \rho_1 (a(t_k) - x)]; \tag{10}$$

$$u_1(t_k, x) = -\frac{P(t_k)}{E_1} \left[ \rho_2 (l - a(t_k)) x - \frac{\rho_1}{2} [(a(t_k) - x)^2 - a^2(t_k)] \right]; \tag{11}$$

$$\begin{aligned} u_2(t_k, x) &= \frac{\rho_2 P(t_k)}{2E_2} ((l - x)^2 - (l - a(t_k))^2) - \\ & - \frac{P(t_k) a(t_k)}{E_1} \left( \rho_2 (l - a(t_k)) + \frac{\rho_1 a(t_k)}{2} \right). \end{aligned} \tag{12}$$

Calculate the work of external forces  $A^{(k+1)}$  when moving from a thermodynamical equilibrium state  $(\bar{u}^{(k)}, \theta^{(k)}, a^{(k)})$  at moment  $t_k$  to a thermodynamical equilibrium state  $(\bar{u}^{(k+1)}, \theta^{(k+1)}, a^{(k+1)})$  at moment  $t_{k+1}$ :

$$\begin{aligned} A^{(k+1)} &= \int_0^{a(t_k)} \rho_1 P(t_{k+1}) (u_1(t_{k+1}, x) - u_1(t_k, x)) dx + \\ & + \int_{a(t_k)}^{a(t_{k+1})} \rho_2 P(t_{k+1}) u_2(t_{k+1}, x) dx - \\ & - \int_{a(t_k)}^{a(t_{k+1})} \rho_1 P(t_{k+1}) u_1(t_k, x) dx + \\ & + \int_{a(t_{k+1})}^l \rho_2 P(t_{k+1}) (u_2(t_{k+1}, x) - u_2(t_k, x)) dx. \end{aligned} \tag{13}$$

By substituting expressions (11), (12) in ratio (13) and integrating, we obtain



Determine the entropy  $S^{(k)}$  in a thermodynamic equilibrium state at time  $t_k$

$$S^{(k)} = \frac{V^{(k)} - A^{(k)} + \mu \rho_1 a(t_k) + \alpha_1 [\theta_2(t_k, l) - \theta_{env}]}{\theta^0} + \frac{\alpha_2 h^{-1}(l - a(t_k)) [\theta_2^m(t_k) - \theta_{env}]}{\theta^0}$$

The entropy  $S^{(k+1)}$  in a transition from the thermodynamic equilibrium state  $(\bar{u}^{(k)}, \theta^{(k)}, a^{(k)})$  at time  $t_k$  to the thermodynamic equilibrium state  $(\bar{u}^{(k+1)}, \theta^{(k+1)}, a^{(k+1)})$  at moment  $t_{k+1}$  will take the following form

$$S^{(k+1)} = \frac{V^{(k+1)} - A^{(k+1)} + \mu \rho_1 a(t_{k+1}) + \alpha_1 [\theta_2(t_{k+1}, l) - \theta_{env}]}{\theta^0} + \frac{\alpha_2 h^{-1}(l - a(t_{k+1})) [\theta_2^m(t_{k+1}) - \theta_{env}]}{\theta^0}$$

Then, in accordance with a Gibbs variation principle, the point  $a^{(k+1)} = a(t_k)$ , which characterizes the position of an interphase boundary at time  $t_{k+1}$ , delivers the maximum value of function  $F(a^{(k+1)}) = S^{(k+1)} - S^{(k)}$ . It follows from the condition for an extremum of the function that

$$\frac{dF(a^{(k+1)})}{da^{(k+1)}} = 0. \tag{19}$$

Substituting ratios (14), (18) and function  $\theta_2(t, x)$  into (19), we obtain

$$\begin{aligned} & \mu \rho_1 + \frac{\alpha_2}{h} [\theta_{env} - \theta_2^m(t_k)] + \rho_1 c_1 \Delta \theta_1 - \frac{\rho_2^2 P(t_k)}{2E_2} (l - a^{(k+1)})^2 + \frac{P^2(t_{k+1})}{E_1} \times \\ & \times \left\{ \rho_2^2 (l - a^{(k+1)})^2 - \rho_1 \rho_2 a^{(k+1)} (l - a^{(k+1)}) + \right. \\ & \left. + a^{(k+1)} \left[ -2\rho_2^2 (l - a^{(k+1)}) - \rho_1 \rho_2 \times \right. \right. \\ & \left. \left. \times (l - 2a^{(k+1)}) + \frac{2\rho_1^2 a^{(k+1)}}{3} \right] + \frac{\rho_1^2 (a^{(k+1)})^2}{3} \right\} + \alpha_1 (\theta^0 - \theta_{env}) B_1 \left\{ \frac{(B - \alpha_1) e^{B_1 l} F_3(a^{(k+1)})}{F_1^2(a^{(k+1)})} + \right. \\ & \left. + \frac{(\alpha_1 + B) e^{-B_1 l} F_4(a^{(k+1)})}{F_2^2(a^{(k+1)})} \right\} - \\ & - \rho_2 c_2 \left\{ -\theta_2^m(t_k) + \left[ \frac{\theta_{env} + \frac{(\alpha_1 - B)(\theta^0 - \theta_{env}) F_5(a^{(k+1)})}{F_1^2(a^{(k+1)})}}{(\alpha_1 + B)(\theta^0 - \theta_{env}) F_6(a^{(k+1)})} - \frac{1}{F_2^2(a^{(k+1)})} \right] \right\} - \\ & - \frac{\rho_1 P(t_{k+1})}{2E_1} \left\{ \frac{(a^{(k)})^2 P(t_{k+1})(\rho_2 - \rho_1) + \right. \\ & \left. + P(t_k) \left[ \rho_2 (l - a^{(k)}) (a^{(k+1)} + a^{(k)}) - \right. \right. \\ & \left. \left. - \frac{\rho_1}{6} \left( (a^{(k+1)})^2 - 2(a^{(k)})^2 \right) \right] \right\} + \\ & \left. + P(t_k) (a^{(k+1)} - a^{(k)}) \times \left[ \rho_2 (l - a^{(k)}) - \right. \right. \\ & \left. \left. \times \left[ -\frac{\rho_1}{3} (a^{(k+1)} - a^{(k)}) \right] \right] \right\} - \rho_2 P(t_{k+1}) \left\{ \frac{2a^{(k+1)} - l - a^{(k)} - \frac{P(t_{k+1}) \rho_2 l}{E_1} + \frac{2P(t_{k+1}) \rho_2 a^{(k+1)}}{E_1}}{E_1} - \right. \\ & \left. - \frac{P(t_{k+1}) \rho_1 a^{(k+1)}}{E_1} - \frac{\rho_2 P(t_{k+1}) (l - a^{(k+1)})^2}{2E_2} + \right. \\ & \left. + \frac{\rho_2}{E_2} \left[ \frac{P(t_{k+1}) (l - a^{(k+1)})^2}{3} + \frac{P(t_k) (l - a^{(k+1)})^2}{2} - \frac{(l - a^{(k+1)})^2}{3} \right] - \right. \\ & \left. + \frac{1}{E_1} \left[ \frac{P(t_k) a^{(k)} \left[ \frac{\rho_1 a^{(k)}}{2} + \rho_2 (l - a^{(k)}) \right] -}{-P(t_{k+1}) a^{(k+1)} \times} \right. \right. \\ & \left. \left. \times \left[ \frac{\rho_1 a^{(k+1)}}{2} + \rho_2 (l - a^{(k+1)}) \right] \right] \right\} + \\ & \left. + (l - a^{(k+1)}) \left[ \frac{\rho_2 (l - a^{(k+1)})}{3E_2} (2P(t_{k+1}) + P(t_k)) + \right. \right. \\ & \left. \left. + \frac{1}{E_1} \left( \frac{P(t_{k+1}) l \rho_2 +}{+ P(t_{k+1}) a^{(k+1)} [\rho_1 - 2\rho_2]} \right) \right] \right\} = 0, \tag{20} \end{aligned}$$

where

$$\begin{aligned}
 F_1(a^{(k+1)}) &= e^{B_1 a^{(k+1)}} (\alpha_1 - B) - e^{-B_1 (2l - a^{(k+1)})} (\alpha_1 + B), \\
 F_2(a^{(k+1)}) &= e^{-B_1 (2l - a^{(k+1)})} (\alpha_1 - B) - e^{B_1 a^{(k+1)}} (\alpha_1 + B), \\
 F_3(a^{(k+1)}) &= e^{B_1 a^{(k+1)}} (\alpha_1 - B) + e^{B_1 (2l - a^{(k+1)})} (\alpha_1 + B), \\
 F_4(a^{(k+1)}) &= e^{-B_1 (2l - a^{(k+1)})} (\alpha_1 - B) + e^{-B_1 a^{(k+1)}} (\alpha_1 + B), \\
 F_5(a^{(k+1)}) &= -2e^{2B_1 l} (B + \alpha_1) + \\
 &+ e^{B_1 (l + a^{(k+1)})} (\alpha_1 - B) + e^{B_1 (3l - a^{(k+1)})} (\alpha_1 + B), \\
 F_6(a^{(k+1)}) &= 2e^{-2B_1 l} (\alpha_1 - B) - \\
 &- e^{-B_1 (3l - a^{(k+1)})} (\alpha_1 - B) - e^{-B_1 (l + a^{(k+1)})} (\alpha_1 + B).
 \end{aligned}$$

Believing that the  $a(t), P(t), \theta_i(t, x)$  functions are continuous and are differentiated for  $t$  over the examined time period, we assume

$$\begin{aligned}
 a(t_{k+1}) &= a(t_k) + \dot{a}(t_k) \Delta + \dots, \\
 P(t_{k+1}) &= P(t_k) + \dot{P}(t_k) \Delta + \dots,
 \end{aligned} \tag{21}$$

where the ellipsis denotes terms of the  $\Delta^2$  order.

Substituting (21) into  $\theta_i(t, x)$  and (20), by moving to the limit at  $N \rightarrow \infty, \Delta \rightarrow 0, t_k \rightarrow t$ , we obtain

$$\begin{aligned}
 \theta_1(t, x) &= \frac{1}{\lambda_1} B (\theta^0 - \theta_{env}) \times \\
 &\times \left[ \frac{\alpha_1 - B}{\alpha_1 - B - e^{2B_1(l-a(t))} (B + \alpha_1)} - \frac{\alpha_1 + B}{\alpha_1 + B - e^{-2B_1(l-a(t))} (\alpha_1 - B)} \right] x + f,
 \end{aligned} \tag{22}$$

$$\begin{aligned}
 \theta_2(t, x) &= \frac{(\alpha_1 - B)(\theta^0 - \theta_{env}) e^{B_1 x}}{e^{B_1 a(t)} [\alpha_1 - B] - e^{B_1 (2l - a(t))} [B + \alpha_1]} - \\
 &- \frac{(\alpha_1 + B)(\theta^0 - \theta_{env}) e^{-B_1 x}}{e^{-B_1 (2l - a(t))} [\alpha_1 - B] - e^{-B_1 a(t)} [\alpha_1 + B]} + \theta_{env},
 \end{aligned} \tag{23}$$

$$\begin{aligned}
 \mu \rho_1 + \frac{\alpha_2}{h} [\theta_{env} - \theta_2^m(t)] + \rho_1 c_1 \Delta \theta_1 - \frac{\rho_2^2 P(t)}{2E_2} (l - a(t))^2 + \frac{P^2(t)}{E_1} & \left\{ \frac{\rho_2^2 (l - a(t))^2 - \rho_1 \rho_2 a(t) (l - a(t)) +}{3} + \frac{\rho_1^2 a(t)^2}{3} + a(t) \left[ \frac{-2\rho_2^2 (l - a(t)) -}{- \rho_1 \rho_2 (l - 2a(t)) + \frac{2\rho_1^2 a(t)}{3}} \right] \right\} +
 \end{aligned}$$

$$+ \alpha_1 (\theta^0 - \theta_{env}) B_1 \left\{ \frac{(B - \alpha_1) e^{B_1 l} F_3(a(t))}{F_1^2(a(t))} + \frac{(\alpha_1 + B) e^{-B_1 l} F_4(a(t))}{F_2^2(a(t))} \right\} - \rho_2 c_2 \left\{ \frac{-\theta_2^m(t) +}{\theta_{env} + \frac{(\alpha_1 - B)(\theta^0 - \theta_{env}) F_5(a(t))}{F_1^2(a(t))}} - \frac{(\alpha_1 + B)(\theta^0 - \theta_{env}) F_6(a(t))}{F_2^2(a(t))} \right\} -$$

$$- \frac{\rho_1 P^2(t) a(t)}{2E_1} \left\{ \rho_2 (2l - a(t)) - \frac{3}{2} \rho_1 a(t) \right\} - \rho_2 P(t) \{ a(t) - l \} = 0, \tag{24}$$

where functions

$$F_i(a(t)) = \lim_{\substack{t_k \rightarrow t \\ \Delta \rightarrow 0}} F_i(a^{(k+1)}).$$

Ratios (16), (22) to (24) are a system of non-linear equations to determine the temperature field in a rod and the position of an interphase boundary.

Transform equations (16), (24). To this end, differentiate equation (24) for  $t$ ; substitute  $\dot{\theta}_2^m(t)$  from (16) into the result. As a result, we shall have the following system of differential equations to determine the motion law of a crystallization front and the mean temperature in a rod:

$$\begin{aligned}
 \dot{a}(t) & \left\{ B_1 (\theta^0 - \theta_{env}) \left\{ \alpha_1 B_1 \left[ \frac{H_1}{F_1^3(a(t))} + \frac{H_2}{F_2^3(a(t))} \right] - \rho_2 c_2 \left[ \frac{(\alpha_1 - B) H_3}{F_1^3(a(t))} + \frac{(\alpha_1 + B) H_4}{F_2^3(a(t))} \right] \right\} - \right. \\
 & \left. \frac{\mu \rho_1 (\lambda_2 B_1^2 - \rho_2 c_2)}{\rho_2 c_2 (l - a(t))} + \frac{\rho_2^2 P(t) (l - a(t))}{E_2} + \right. \\
 & \left. + \frac{P^2(t)}{E_1} \left[ \frac{2\rho_1^2 a(t)}{3} - 2\rho_2^2 (l - a(t)) - \right. \right. \\
 & \left. \left. - \rho_1 \rho_2 (l - 2a(t)) + \right. \right. \\
 & \left. \left. + 2a(t) \left( \rho_2^2 + \rho_1 \rho_2 + \frac{\rho_1^2}{3} \right) - \right. \right. \\
 & \left. \left. - \rho_1 \rho_2 (l - a(t)) \right] - \right. \\
 & \left. - \rho_2 P(t) \right\} = \\
 & \left\{ \rho_2 (a(t) - l) - \frac{\rho_2^2 (l - a(t))^2}{2E_2} + \right. \\
 & \left. + \frac{2P(t)}{E_1} \left[ \frac{\rho_1^2 a^2(t)}{4} - 2\rho_1 \rho_2 a(t) \times \right. \right. \\
 & \left. \left. \times (l - a(t)) + \rho_2^2 (l - a(t))^2 - \right. \right. \\
 & \left. \left. - 2a(t) \rho_2^2 (l - a(t)) + \right. \right. \\
 & \left. \left. + \frac{a(t)}{2} \rho_1 \rho_2 (2l - a(t)) \right] \right\} + \\
 & + \frac{(\lambda_2 B_1^2 - \rho_2 c_2) (\alpha_1^2 - B^2) (\theta^0 - \theta_{env})}{\rho_2 c_2 (l - a(t))} \left\{ \frac{e^{-B_1 l}}{F_2(a(t))} - \frac{e^{B_1 l}}{F_1(a(t))} \right\}, \tag{25}
 \end{aligned}$$

where

$$\begin{aligned}
 H_1 &= (\alpha_1 - B)e^{B_1 l} \left[ e^{2B_1 a(t)} (\alpha_1 - B) + e^{2B_1 (2l-a(t))} \times \right. \\
 &\quad \left. \times (\alpha_1 + B) + 6e^{2B_1 l} (\alpha_1^2 - B^2) \right]; \\
 H_2 &= (\alpha_1 + B)e^{-B_1 l} \left[ e^{-2B_1 a(t)} (\alpha_1 + B) - e^{-2B_1 (2l-a(t))} \times \right. \\
 &\quad \left. \times (\alpha_1 - B) - 6e^{-2B_1 l} (\alpha_1^2 - B^2) \right]; \\
 H_3 &= \left\{ \left[ e^{B_1 (l+a(t))} [\alpha_1 - B] - e^{B_1 (3l-a(t))} [\alpha_1 + B] \right] \times \right. \\
 &\quad \left. \times F_1(a(t)) + 2F_5(a(t))F_3(a(t)) \right\}; \\
 H_4 &= \left\{ \left[ e^{-B_1 (3l-a(t))} [\alpha_1 - B] - e^{-B_1 (l+a(t))} [\alpha_1 + B] \right] \times \right. \\
 &\quad \left. \times F_2(a(t)) - 2F_6(a(t))F_4(a(t)) \right\}; \\
 \dot{\theta}_2^m(t) &= \frac{\mu \rho_1 \dot{a}(t)}{\rho_2 c_2 (l-a(t))} + \\
 &+ \frac{(\alpha_1^2 - B^2) (\theta^0 - \theta_{env})}{\rho_2 c_2 (l-a(t))} \left\{ \frac{e^{-B_1 l}}{F_2(a(t))} - \frac{e^{B_1 l}}{F_1(a(t))} \right\}. \quad (26)
 \end{aligned}$$

Once the function  $a=a(t)$  is determined, a temperature field in the rod is determined from formulae (22), (23), and the stressed-strained state – from formulae (9) to (12).

Thus, one can build the following algorithm to determine the stressed-strained state in a body  $\Omega$  during the phase transition. At moment  $t_k$ :

1) fix the interphase surface  $a(t_k)$ ; determine, from a solution to the stationary problem of thermal conductivity, the temperatures  $\theta_i^{(k)}(x)$  ( $i=1,2$ ) in a body  $\Omega$ ;

2) solve the contact problem for two bodies occupying the regions  $\omega_1(t_k)$  and  $\omega_2(t_k)$ , and determine the displacement field  $\bar{u}^{(k)}$ ;

3) for available values  $(\bar{u}^{(k)}, \theta^{(k)}, \gamma^{(k)})$ , determine the internal energy  $V^{(k)}$  and the entropy  $S^{(k)}$  of a body  $\Omega$ ;

4) from the first law of thermodynamics and the condition for a maximum of entropy  $S^{(k)}$  of the body  $\Omega$ , find the true position of an interphase boundary  $a^{(k)}$ ;

5) when one found the position of an interphase boundary surface  $a^{(k)}$  determine the true temperature  $\theta^{(k)}$  and the true displacement field  $\bar{u}^{(k)}$ ;

6) at the next time  $t_{k+1}$ , the initial temperature and the position of an interphase boundary correspond to those found at moment  $t_k$ , that is

$$\theta^{(k+1)} = \theta^{(k)},$$

$$a^{(k+1)} = a^{(k)}.$$

In this case, the proposed algorithm for solving the associated problem of thermoviscoelasticity takes into consideration the backstory of the entire process at each point of time  $t_k$  in the motion law of an interphase boundary, determined from a Gibbs variation principle.

### 5. Solving the quasi-related problem of thermoviscoelasticity for a rod at crystallization

To illustrate the applicability of the proposed method, let us consider specific examples: the crystallization (build-

up) of a rod with a heat-insulated side surface; the problem of building up a rod when interacting with the environment taking into consideration a phase transition.

#### 5.1. Solving the problem of crystallization (build-up) of a rod with a thermally insulated side surface

For the case of crystallizing a rod with a heat-insulated side surface ( $\alpha_{2,conv} = \alpha_2 = 0$ ) in problem (1) to (5) the expressions defining the motion law of an interphase boundary, as well as a temperature field in the solid phase, are simplified and take the following form

$$\begin{aligned}
 \dot{a}(t) &= \left\{ \begin{aligned} &2\rho_2^2 P(t)(l-a(t)) \left[ \frac{1}{E_2} - \frac{1}{E_1} \right] + \\ &+ \frac{P(t)}{E_1} \left[ \frac{2\rho_1^2 a(t)}{3} - \rho_1 \rho_2 (l-2a(t)) \right] + \\ &+ \frac{P^2(t)}{E_1} \left[ \frac{-2\rho_2^2 (l-a(t)) - \rho_1 \rho_2 (l-2a(t))}{3} + \right. \\ &\quad \left. + \frac{2\rho_1^2 a(t)}{3} + a(t) \left( 2\rho_2^2 + 2\rho_1 \rho_2 + \frac{2\rho_1^2}{3} \right) \right] + \\ &+ \rho_2 c_2 \frac{\lambda_2}{\alpha_1 \left( l + \frac{\lambda_2}{\alpha_1} - a(t) \right)^2} \times \\ &\times \left\{ \begin{aligned} &\theta^0 + \theta_{env} - \frac{1}{l + \frac{\lambda_2}{\alpha_1} - a(t)} \times \\ &\times \left[ \theta^0 \left( l + \frac{2\lambda_2}{\alpha_1} - a(t) \right) + \theta^m (l-a(t)) \right] \right\} + \\ &+ \frac{2\lambda_2 (\theta^0 - \theta^m)}{\left( l + \frac{\lambda_2}{\alpha_1} - a(t) \right)^3} + \frac{\mu \rho_1}{l-a(t)} \end{aligned} \right\} = \\
 \dot{P}(t) &= \left\{ \begin{aligned} &\rho_2^2 (l-a(t))^2 \left[ \frac{1}{E_2} - \frac{1}{E_1} \right] - \\ &- \frac{1}{E_1} \left[ \frac{\rho_1^2 a^2(t)}{3} - \rho_1 \rho_2 (l-a(t)) a(t) \right] - \\ &- \frac{2P(t)a(t)}{E_1} \times \\ &\times \left[ \begin{aligned} &-2\rho_2^2 (l-a(t)) - \\ &- \rho_1 \rho_2 (l-2a(t)) + \frac{2\rho_1^2 a(t)}{3} \end{aligned} \right] \end{aligned} \right\}, \\
 \theta_2(a(t), x) &= \frac{(\theta^0 - \theta_{env}) \left( l + \frac{\lambda_2}{\alpha_1} - x \right)}{l + \frac{\lambda_2}{\alpha_1} - a(t)} + \theta_{env}, \\
 \theta_2^m(a(t)) &= -\frac{\mu \rho_1}{\rho_2 c_2} \ln \frac{l-a(t)}{l-l_0} + 1700,
 \end{aligned}$$

and the stressed-strained state is determined from formulae (9) to (12).

The numerical results are obtained at the following values of the basic parameters:

$$l=0.5 \text{ m}, l_0=0.4 \text{ m}, E_2=1.4 \cdot 10^{10} \text{ kg/m}^2, \theta^0=1,765 \text{ K},$$

$$\theta_{env}=293 \text{ K}, f=1,800 \text{ K}, \mu=2.4 \text{ MJ/kg}, \rho_1=7,000 \text{ kg/m}^3,$$

$$\rho_2=7,800 \text{ kg/m}^3, c_1=837 \text{ J/(kg}\cdot\text{K)}, c_2=565 \text{ J/(kg}\cdot\text{K)},$$

$$\lambda_1=23.3 \text{ W/(m}\cdot\text{K)}, \lambda_2=48.1 \text{ W/(m}\cdot\text{K)}, \varepsilon_{12}=0.8;$$

they are shown in Fig. 1–6.

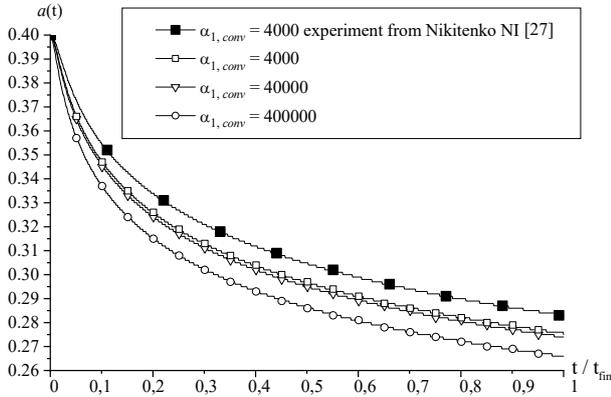


Fig. 1. The effect of rod surface cooling intensity on the motion law of an interphase boundary

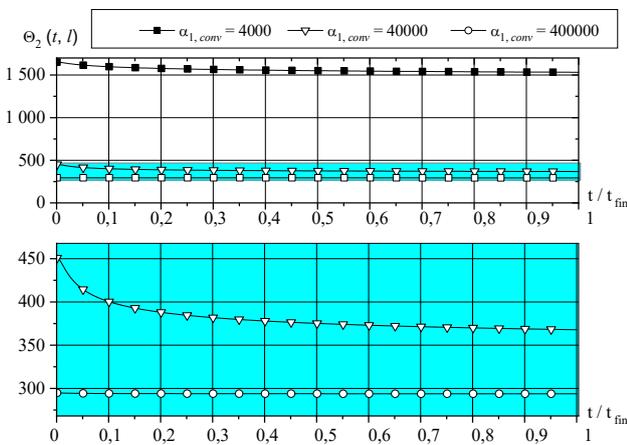


Fig. 2. The effect of cooling intensity on the temperature of the rod end surface. The bottom part of the chart illustrates the highlighted color-filled fragment

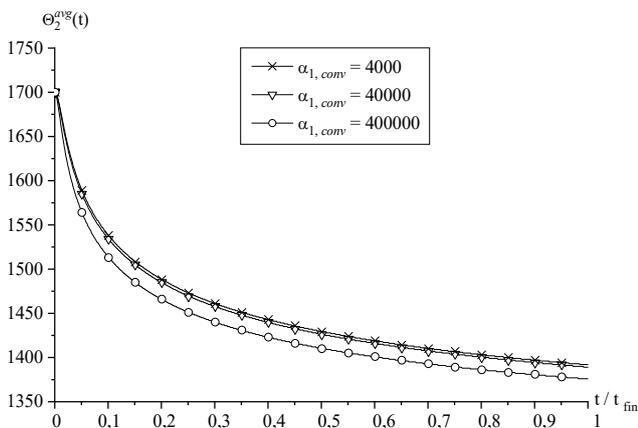


Fig. 3. Temporal dependence of the mean mass temperature of the solid phase for the rod of length  $l=0.5 \text{ m}$  ( $l_0=0.4 \text{ m}$ ) at different values of heat release factor

Fig. 1–3 show the motion law of an interphase boundary, a change in the temperature of the end surface of the rod over time, and temporal dependence of the mean mass solid phase temperature for the rod of length  $l=0.5 \text{ m}$ .

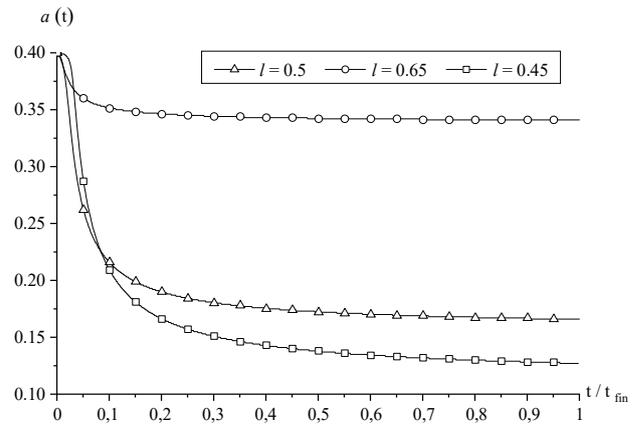


Fig. 4. The effect of rod length on the motion law of an interphase boundary

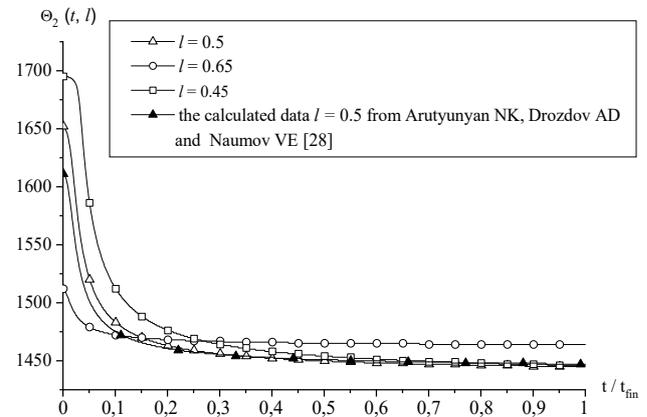


Fig. 5. The effect of rod length on the temperature of the end surface of the rod

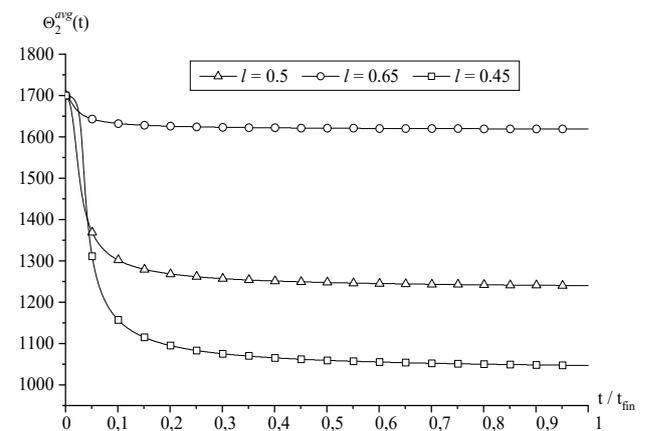


Fig. 6. Temporal dependence of the mean mass temperature of the solid phase of the rod at  $\alpha_{1,conv}=40 \cdot 10^3 \text{ W/(m}^2\cdot\text{K)}$  for different values of its length

Fig. 4–6 show the dependences  $a(t)$ ,  $\theta_2(t, l)$ ,  $\theta_2^m(t)$  at the heat release factor  $\alpha_{1,conv}=40 \cdot 10^3 \text{ W/(m}^2\cdot\text{K)}$  for a rod of varying lengths, but at  $l_0=0.4 \text{ m}$ .

**5. 2. Solving the problem of building up a rod in thermal interaction with the environment, taking into consideration the phase transition**

For the case of crystallization (build-up) of a rod in thermal interaction between the lateral and end surfaces and the environment taking into consideration the phase transition, the results of solving the equations (25), (26) are shown in Fig. 7, 8.

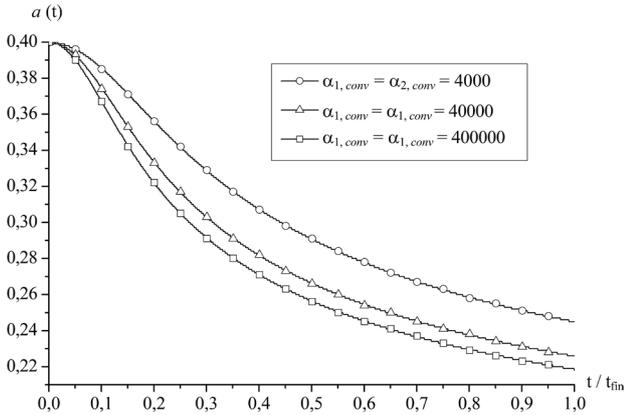


Fig. 7. The motion law of the growing rod crystallization front at different values of the heat release factor

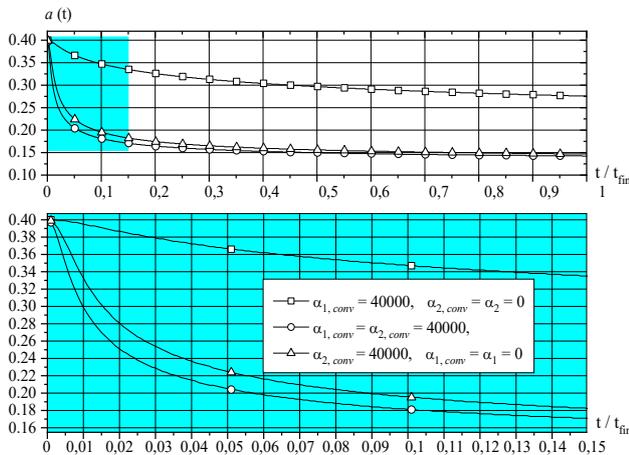


Fig. 8. The effect of conditions for heat exchange with the environment on the crystallization process in a rod. The bottom part of the chart illustrates the highlighted color-filled fragment

The results of the problem solving showed that both the geometric dimensions of a body and the conditions of heat exchange with the environment exert a significant impact on the hardening process.

**6. Discussion of results obtained in solving the quasi-related problem of thermoviscoelasticity for a rod at crystallization**

An analysis of our results demonstrates that cooling intensification, first, reduces the hardening time, and, second, increases the size of the solid phase at a fixed time value (Fig. 1).

Fig. 2 show that the increase in heat release factor leads to a sharp drop in the temperature  $\theta_2(t, l)$  at the initial mo-

ment in time. Thereafter, this temperature changes not so dramatically and, throughout the entire hardening process, it is a monotonously descending function. At high heat release factor values, the end surface is instantly cooled to a temperature close to ambient temperature and remains constant until the end of the process.

The charts shown in Fig. 4 demonstrate that changing the initial length of the solid phase has a significant impact on the crystallization process. This is confirmed by the results shown in Fig. 5, 6.

Thus, it has been established that both the geometric dimensions of a body and the conditions of heat exchange with the environment have a determining influence on the hardening process, which is consistent with known experimental and estimated data [27, 28].

Fig. 7 shows that if there is a heat exchange between the side surface and the environment, it significantly reduces the time of the hardening of the material.

The charts shown in Fig. 8 demonstrate that the heat exchange from the side of a rod has a greater impact on the crystallization process than that from the end surface, as evidenced by known experimental and estimated data [27, 28].

The results obtained are explained by the statement of the problem and the method of solving it.

One of the features of the proposed method is that the motion law of an interphase boundary is determined analytically from the joint solution to the contact problem and the problem of thermal conductivity, rather than set from the outside or determined from the experiment. The second feature is that the combined application of the method of successive intervals and a Gibbs variation principle makes it possible to take into consideration the mutual influence of mechanical and temperature fields not only on each other but also on the motion law of an interphase boundary.

All functions that are part of the defining ratios should be continuous.

It should be noted that the issue of solving the related (rather than quasi-related) problem of thermoviscoelasticity remains open. Under a classical statement, the boundness of the fields requires that either the temperature should be part of the mechanical equations of state or vice versa. This point has not yet been taken into consideration in this paper but it is the subject of further research.

One of the options for advancing the current study could tackle the need to take into consideration the dependence of heat-physical characteristics on time. However, it might be difficult to derive an analytical solution.

**7. Conclusions**

1. An approximate analytical method and an algorithm have been constructed for solving the quasi-related problem of thermoviscoelasticity for growing bodies in the presence of a phase transition considering heat exchange with the environment. This method makes it possible to assess the influence of both geometric and physical parameters of the problem on the formation of the stressed-strained state of a body taking into consideration the connection between thermomechanical fields, as well as analytically determine the motion law of a crystallization front.

2. Based on the developed method, two applied problems have been solved: the crystallization (build-up) of a rod with a heat-insulated side surface; the problem of building up a rod

when interacting with the environment, taking into consideration the phase transition. The results of solving these problems have shown that both the geometric dimensions of a body and the conditions of heat exchange with the environment have a significant impact on the hardening process. Namely, a change in the initial length of the solid phase affects the

process of crystallization and, as a result, the stressed-strained state of the body. The presence of heat exchange between a side surface and the environment significantly reduces the hardening time of a material. In this case, the heat exchange from the side surface of the rod has a greater impact on the crystallization process than from the end surface.

#### References

1. Dron, M., Dreus, A., Golubek, A., Abramovsky, Y. (2018). Investigation of aerodynamics heating of space debris object at reentry to earth atmosphere. 69th International Astronautical Congress, IAC-18-A6.1.5. Bremen, 7.
2. Yemets, V., Harkness, P., Dron, M., Pashkov, A., Worrall, K., Middleton, M. (2018). Autophagy Engines: Toward a Throttleable Solid Motor. *Journal of Spacecraft and Rockets*, 55 (4), 984–992. doi: <https://doi.org/10.2514/1.a34153>
3. Yemets, M., Yemets, V., Dron, M., Harkness, P., Worrall, K. (2018). Caseless throttleable solid motor for small Spacecraft. 69th International Astronautical Congress, IAC-18.C4.8-B4.5A.13. Bremen, 10.
4. Dreus, A. Y., Kozhevnikov, A. A., Liu, B., Sudakova, D. A. (2019). Approximate analytical model of rock thermal cyclical disintegration under convective cooling. *Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu*, 4, 42–47. doi: <https://doi.org/10.29202/nvngu/2019-4/5>
5. Opitz, F., Treffinger, P., Wöllenstein, J. (2017). Modeling of Radiative Heat Transfer in an Electric Arc Furnace. *Metallurgical and Materials Transactions B*, 48 (6), 3301–3315. doi: <https://doi.org/10.1007/s11663-017-1078-6>
6. Sudakov, A., Dreus, A., Ratov, B., Delikesheva, D. (2018). Theoretical bases of isolation technology for swallowing horizons using thermoplastic materials. *News of the National Academy of Sciences of the Republic of Kazakhstan, Series of Geology and Technical Sciences*, 2 (428), 72–80.
7. Syasev, A., Zelenskaya, T. (2015). Lengthwise movement dynamic boundary-value problem for trailing boundary ropes. *Metallurgical and Mining Industry*, 3, 283–287. Available at: [http://www.metaljournal.com.ua/assets/Journal/english-edition/MMI\\_2015\\_3/036%20Syasev.pdf](http://www.metaljournal.com.ua/assets/Journal/english-edition/MMI_2015_3/036%20Syasev.pdf)
8. Kravets, E. (2019). Determining the structure of a laminar detachable current in an open cavity. *Eastern-European Journal of Enterprise Technologies*, 6 (8 (102)), 28–37. doi: <https://doi.org/10.15587/1729-4061.2019.183811>
9. Kozhevnikov, A. A., Sudakov, A. K., Dreus, A. Yu., Lysenko, Ye. Ye. (2014). Study of heat transfer in cryogenic gravel filter during its transportation along a drillhole. *Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu*, 6, 49–54.
10. Kulikov, R. G., Kulikova, T. G. (2014). Numerical methods for solving the problem of polymer crystallizing media deformation taking into account finite deformations. *Computational Continuum Mechanics*, 7 (2), 172–180. doi: <https://doi.org/10.7242/1999-6691/2014.7.2.18>
11. Kul'bovskiy, I. K., Karelin, S. V., Ilyushkin, D. A. (2008). Komp'yuternoe modelirovanie protsessa kristallizatsii massivnykh otlivok vtulok tsilindrov sudovykh dizeley. *Vestnik BGTU*, 2 (18), 16–19.
12. Tkachuk, A. N. (2008). Issledovaniya termouprugih kontaktnykh zadach elementov press-form dlya lit'ya pod davleniem s uchedom fazovykh prevrashcheniy v otlivke. *Vestnik NTU «KhPI»*, 2, 144–158.
13. Senchenkov, I. K., Oksenchuk, N. D. (2013). An estimation of effects of thermostructural mechanical coupling under pulse loading of half-space. *Bulletin of Taras Shevchenko National University of Kyiv Series: Physics & Mathematics*, 3, 217–219.
14. Bonetti, E., Frémond, M., LExcellent, C. (2005). Global Existence and Uniqueness for a Thermomechanical Model for Shape Memory Alloys with Partition of the Strain. *Mathematics and Mechanics of Solids*, 11 (3), 251–275. doi: <https://doi.org/10.1177/1081286506040403>
15. Rao, A., Srinivasa, A. R. (2014). A three-species model for simulating torsional response of shape memory alloy components using thermodynamic principles and discrete Preisach models. *Mathematics and Mechanics of Solids*, 20 (3), 345–372. doi: <https://doi.org/10.1177/1081286514545917>
16. Holland, M. A., Kosmata, T., Goriely, A., Kuhl, E. (2013). On the mechanics of thin films and growing surfaces. *Mathematics and Mechanics of Solids*, 18 (6), 561–575. doi: <https://doi.org/10.1177/1081286513485776>
17. Paroni, R., Tomassetti, G. (2017). Linear models for thin plates of polymer gels. *Mathematics and Mechanics of Solids*, 23 (5), 835–862. doi: <https://doi.org/10.1177/1081286517698740>
18. Hossain, M., Chatzigeorgiou, G., Meraghni, F., Steinmann, P. (2015). A multi-scale approach to model the curing process in magneto-sensitive polymeric materials. *International Journal of Solids and Structures*, 69–70, 34–44. doi: <https://doi.org/10.1016/j.ijsolstr.2015.06.011>
19. Carniel, T. A., Muñoz-Rojas, P. A., Vaz, M. (2015). A viscoelastic viscoplastic constitutive model including mechanical degradation: Uniaxial transient finite element formulation at finite strains and application to space truss structures. *Applied Mathematical Modelling*, 39 (5–6), 1725–1739. doi: <https://doi.org/10.1016/j.apm.2014.09.036>
20. Gudramovich, V. S., Gart, É. L., Strunin, K. A. (2017). Modeling of the Behavior of Plane-Deformable Elastic Media with Elongated Elliptic and Rectangular Inclusions. *Materials Science*, 52 (6), 768–774. doi: <https://doi.org/10.1007/s11003-017-0020-z>
21. Hart, E. L., Hudramovich, V. S. (2016). Projection-iterative modification of the method of local variations for problems with a quadratic functional. *Journal of Applied Mathematics and Mechanics*, 80 (2), 156–163. doi: <https://doi.org/10.1016/j.jappmath-mech.2016.06.005>
22. Hart, E. L., Hudramovich, V. S. (2014). Projection-Iterative Schemes for the Realization of the Finite-Element Method in Problems of Deformation of Plates with Holes and Inclusions. *Journal of Mathematical Sciences*, 203 (1), 55–69. doi: <https://doi.org/10.1007/s10958-014-2090-x>

23. Lykov, A. V. (1967). Teoriya teploprovodnosti. Moscow: Vysshaya shkola, 560.
24. Syasev, A. V. (2001). Priblizhennyi analiticheskiy metod rascheta rastushchih tel s uchetom fazovogo perehoda. Visnyk Dnipropetr. un-tu. Seriya: Mekhanika, 1 (5), 125–137.
25. Karnauhov, V. G. (1982). Svyazannye zadachi termovyazkouprugosti. Kyiv: Naukova dumka, 250.
26. Lyubov, B. Ya. (1975). Teoriya kristallizatsii v bol'shii obemah. Moscow: Nauka, 256.
27. Nikitenko, N. I. (1978). Issledovanie protsessov teplo- i massoobmena metodom setok. Kyiv: Naukova dumka, 213.
28. Arutyunyan, N. H., Drozdov, A. D., Naumov, V. E. (1987). Mehanika rastushchih vyazkouprugoplasticheskikh tel. Moscow: Nauka, 472.

*У статті виокремлено особливості застосування загальних рівнянь математичної фізики еліптичного типу в задачах моделювання специфічних явищ взаємодії електромагнітних полів з елементами і частинками неоднорідного дисперсного середовища. Такі явища мають місце в установках сепарації органічної та мінеральної сировини або електромагнітної обробки зерна, насіння та ін. Зазначене є актуальним, оскільки звичайний підхід до формулювання математичних моделей у наведених задачах, що базується переважно на диференціальних рівняннях теорії поля у спрощеній формі, не завжди адекватно відображає фізичну сутність згаданих явищ. Тому він обмежує можливості поглибленого дослідження впливу багатьох факторів, які обумовлюють кінцеві результати процесів сепарації та електромагнітної обробки (ЕМО). В роботі запропоновано альтернативний підхід, побудований на використанні інтегральних рівнянь теорії поля, який базується на концепції первинних і вторинних джерел поля і дозволяє значно зменшити порядок системи рівнянь при чисельній реалізації алгоритмів розв'язання задач ЕМО та загальний обсяг потрібних обчислювальних ресурсів. При такому підході виявляються доступними для обрахування локальні параметри поля у взаємодії з окремими частинками та їхнього впливу одне на одне. Зазначений аспект є суттєвим для визначення технологічних характеристик виробничих установок ЕМО. Представлена математична модель, на відміну від поширених спрощених підходів до визначення параметрів поля і пондеромоторних сил, що діють у полі на частинки речовини, адекватно відображає фізичні закономірності розподілу потенціалів і напруженості електричного поля реальних зарядів та індукованих джерел. Завдяки цьому вона наочно відтворює механізм формування головних складових механічних зусиль, що діють на поляризоване тіло з боку електричного поля в цілому, через щільність елементарних зусиль, які справляє поле на поверхневі заряди, що індукуються в діелектричних тілах у зоні дії полів. Така математична модель є універсальним і компактним інструментом аналізу, проектування та оптимізації різноманітних установок і пристроїв, у яких використовується електричне поле та його електромагнітна взаємодія зі середовищем і окремими тілами*

*Ключові слова: електромагнітна обробка, дисперсні матеріали, математичне моделювання, електричне поле, частинки, речовина, сила*

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# FEATURES OF MATHEMATICAL MODELING OF ELECTROMAGNETIC PROCESSING OF BULK MATERIALS

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## 1. Introduction

In recent years, in agricultural production for the purification, separation and processing of cereals, seeds, fruits and other bulk materials, technologies using electric and magnetic fields have spread [1–5]. On a large amount of

experimental data, the effectiveness of the effect of an artificial electromagnetic field for stimulating the vital and growth processes of plants has been proved [4, 5]. Therefore, the use of electrophysical methods of processing biological objects is a good alternative to using it in agricultural production to increase crop yields of chemicals and genetically