Сформульовано і розглянуто проблему формування системи правил виконання бінарних операцій над нечіткими числами. Поставлену задачу розв'язано з нечіткими числах (L–R)-типу з компактним носієм. Така постановка задачі обумовлена простотою аналітичного опису цих чисел, що дозволяє однозначно задати нечітке число набором значень його параметрів. Ця обставина стосовно до чисел (L–R)-типу дає можливість звести шукані правила виконання операцій над нечіткими числами до правил виконання простих арифметичних операцій над їх параметрами. Встановлено, що у великій кількості цитованих робіт наводяться співвідношення, що описують правила виконання операцій над нечіткими числами (L–R)-типу, які містять помилки. Крім того, у всіх випадках відсутнє обґрунтування цих правил.

З метою побудови коректної системи правил нечіткої арифметики запропоновано сукупність метаправил, що визначають принципи побудови та структуру правил виконання операцій. З використанням цієї сукупності метаправил розроблено і описано систему правил виконання основних арифметичних операцій (додавання, віднімання, множення, ділення). При цьому для правил множення і ділення наведені різні правила залежно від положення носіїв чисел, що беруть участь в операції, по відношенню до нуля. Запропонована система правил забезпечує можливість коректного розв'язання безлічі практичних задач, в яких вихідні дані визначені нечітко. Ця система правил над нечіткими числами з компактним носієм поширена випадок з нескінченним носієм. Відповідний підхід реалізується двокроковою процедурою. Визначено переваги та недоліки цього підходу

Ключові слова: нечіткі числа (L–R)-типу, компактний носій, правила виконання арифметичних операцій

UDC 519.681
DOI: 10.15587/1729-4061.2020.203590

PERFORMING ARITHMETIC OPERATIONS OVER THE (L—R)-TYPE FUZZY NUMBERS

L. Raskin
Doctor of Technical Sciences,
Professor, Head of Department*
O. Sira
Doctor of Technical Sciences, Professor*
E-mail: topology@ukr.net
*Department of Distributed Information Systems and Cloud Technologies
National Technical University «Kharkiv Polytechnic Institute»
Kyrpychova str., 2,
Kharkiv, Ukraine, 61002

Received date 30.04.2020
Accepted date 19.05.2020
Published date 30.06.2020

1. Introduction

Practical tasks related to the analysis and synthesis of systems, decision theory, management theory, are resolved under conditions of uncertainty. Methods from a probability theory or a fuzzy set theory are used to describe this uncertainty. Probability theory is a strict axiomatic theory. However, it is this circumstance that limits the scope of its application for many practical problems. The fact is that the fundamental concepts of this theory, such as the distribution of random probabilities, can be strictly defined if the mechanism of forming these random quantities does not change over multiple observations. In reality, the conditions for the formation of observed quantities can vary significantly, resulting in the inadequacy of the results derived from them. Less demanding is the description of uncertainty in terms of fuzzy set theory [1]. This theory is a step towards bringing together the impeccable accuracy of classical mathematics and the pervasive inaccuracy of the real world [2].

2. Literature review and problem statement

The objects of fuzzy mathematics are fuzzy numbers. As in any mathematical theory, the success of solving practical problems in terms of fuzzy mathematics is determined by the correctness of the rules for performing operations over the objects of this theory. These rules are introduced as follows [3, 4].

Consider an arbitrary binary operation on fuzzy numbers x_1 and x_2 with membership functions $\mu_1(x_1)$, $\mu_2(x_2)$. Introduce the * symbol of an arbitrary binary arithmetic queue (addition, subtraction, multiplication, division). Any such operation assigns the numbers x_1 and x_2 a certain result z. We also introduce the «reverse» operation ⊗, which assigns its second element to the result of composition z and one of its elements (for example, x_1) its second element.

Set the variants for a binary arithmetic operation:

$$z = x_1 * x_2 = x_1 + x_2;\ \ z = x_1 * x_2 = x_1 - x_2; \qquad (1)$$

$$z = x_1 * x_2 = x_1 x_2;\ \ z = x_1 * x_2 = \frac{x_1}{x_2}.$$

Then, by using the «reverse» operation, we obtain, accordingly:

$$x_2 = z \otimes x_1 = z - x_1;\ \ x_2 = z \otimes x_1 = x_1 - z; \qquad (2)$$

$$x_2 = z \otimes x_1 = \frac{z}{x_1};\ \ x_2 = z \otimes x_1 = \frac{x_1}{z}.$$

The membership function of the result of a binary composition $z = x_1 * x_2$ is determined by ratio [5–7]:

$$\mu(z) = \int_{-\infty}^{\infty} \mu_1(t)\mu_2(z \otimes t)\mathrm{d}t. \quad (3)$$

In particular, for the summation operation, the corresponding membership function of the fuzzy number z takes the following form:

$$\mu(z) = \int_{-\infty}^{\infty} \mu_1(t)\mu_2(z - t)\mathrm{d}t. \quad (4)$$

Similarly, for the operations of subtraction, multiplication, and division, the membership functions operations corresponding to the result of the operation execution are determined by the following ratios:

$$\mu(z) = \int_{-\infty}^{\infty} \mu_1(t)\mu_2(t - z)\mathrm{d}t, \quad (5)$$

$$\mu(z) = \int_{-\infty}^{\infty} \mu_1(t)\mu_2\left(\frac{z}{t}\right)\mathrm{d}t, \quad (6)$$

$$\mu(z) = \int_{-\infty}^{\infty} \mu_1(t)\mu_2\left(\frac{t}{z}\right)\mathrm{d}t. \quad (7)$$

The downside of these rules of arithmetic operations is that the fuzzy numbers resulting from their implementations are not normalized. This shortcoming is taken into consideration in [8]. In this case, the membership functions, determined by considering (4) to (7), are normalized by the corresponding maximum value. Then, in a general case, we obtain:

$$\tilde{\mu}(z) = \left[\max_z \{\mu(z)\}\right]^{-1} \int_{-\infty}^{\infty} \mu_1(t)\mu_2(z \otimes t)\mathrm{d}t. \quad (8)$$

It is clear that the difficulty of performing computational operations in accordance with (8) depends on the form of the membership functions of the elements of composition x_1 and x_2, in specific cases, it can be unacceptably high. Therefore, many works [9–12] consider simplifying the procedure for arithmetic operations on fuzzy numbers. In this case, fuzzy numbers of different types are proposed to be used as operands: sigmoidal [12], interval [13], trapezoidal [14, 15] numbers. However, the low adequacy of these variants for describing uncertainty has led to the development of a more informative unified description of the membership functions of fuzzy numbers in the form of so-called (L–R)-type functions [1–8]. The membership function of such a fuzzy number x takes the form:

$$\mu_{LR}(x) = \begin{cases} L\left(\dfrac{m - x}{\alpha}\right), & \text{if } x < m, \\ R\left(\dfrac{x - m}{\beta}\right), & \text{if } x \geq \bar{m}, \end{cases} \quad (9)$$

where m is the mode of a fuzzy number x, α, β are the left and right coefficients of fuzziness $(\alpha > 0, \beta > 0)$, $L(t)$, $R(t)$ are the arbitrary functions that are not ascending over a set of non-negative numbers, and $L(0) = R(0) = 1$.

According to (9), the (L–R)-type fuzzy number can be unequivocally defined by the three parameters $< m, \alpha, \beta >$. Note that the asymmetric nature of the membership functions of the (L–R)-type fuzzy numbers provides a possibility of obtaining good-quality approximations for a wide class of specific membership functions. In this case, a universal and extremely simple way of formally describing the membership functions of the (L–R)-type numbers initiated the development of a specific technology for performing the simplest of operations (addition, subtraction, multiplication, division) over these numbers using only the parameters of these numbers. As a result, a large number of works have emerged that each has its own rules for performing operations over the (L–R)-type fuzzy numbers. Significant differences among them are due to the need to analyze known results and to construct a unified approach to the formation of the justified set of rules for implementing arithmetic operations over the (L–R)-type numbers. The relevant results are given below.

Let the (L–R)-type fuzzy numbers x_1 and x_2 be set by the membership functions:

$$\mu_1(x_1) = \begin{cases} L_1\left(\dfrac{m_1 - x_1}{\alpha_1}\right), & \text{if } x_1 \leq m_1, \\ R\left(\dfrac{x_1 - m_1}{\beta_1}\right), & \text{if } x_1 > m_1; \end{cases} \quad (10)$$

$$\mu_2(x_2) = \begin{cases} L_2\left(\dfrac{m_2 - x_2}{\alpha_2}\right), & \text{if } x_2 \leq m_2, \\ R_2\left(\dfrac{x_2 - m_2}{\beta_2}\right), & \text{if } x_2 > m_2. \end{cases} \quad (11)$$

Thus, these two fuzzy numbers are defined by sets $x_1 = < m_1, \alpha_1, \beta_1 >$, $x_2 = < m_2, \alpha_2, \beta_2 >$. In accordance with this, the fuzzy number z, resulting from the operation $z = x_1 * x_2$, is described by the set $< m, \alpha, \beta >$, where the numbers m, α, β are determined by the rules corresponding to the * operation. Consider the known rules for performing arithmetic operations (addition, subtraction, multiplication, and division). A large number of publications provide their own rules for performing operations on fuzzy numbers. We shall choose some of the most cited ones from those papers, containing the main results related to this issue.

Studies [16, 17] propose the following set of rules for performing operations over the (L–R) fuzzy numbers.

Addition $z = x_1 + x_2$.

$$< m, \alpha, \beta > = < m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2 >. \quad (12)$$

Subtraction $z = x_1 - x_2$.

$$< m, \alpha, \beta > = < m_1 - m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2 >. \quad (13)$$

Multiplication $z = x_1 \cdot x_2$.

$$< m, \alpha, \beta > = < m_1 \cdot m_2, m_1 \cdot \alpha_2 + m_2 \cdot \alpha_1, m_1 \cdot \beta_2 + m_2 \cdot \beta_1 >. \quad (14)$$

Ratio (13) (a subtraction operation) is wrong. It is clear that each fuzzy coefficient of the number z that sets the parameters for membership function (9) should depend simultaneously on the degree of blurriness in both the reduced and subtracted. Ratio (14) also produces an inaccurate result. For simple reasons, it is clear that the values of the fuzzy coefficients α and β in the result of the multiplication of fuzzy numbers should be higher than the fuzzy coefficients of the efficients. However, if the modal values of the efficients are small, the fuzzy result obtained according to (14) may

be inexplicably small. In particular, if the modal values m_1 and m_2 are close to zero, then the result, even with the high vagueness of the efficients, would be an almost clear number, which, of course, is not the case.

Work [18] reports the following rules for operation execution. The rule of addition repeats (12).

Subtraction $z = x_1 - x_2$.

$$< m,\alpha,\beta > = < m_1 - m_2,\ \alpha_1 - \alpha_2,\ \beta_1 - \beta_2 >. \quad (15)$$

The proposed rule (15) of the subtraction operation makes the erroneous ratio (13) absurd. According to this formula, the blurring level of the result of the subtraction for any two blurred, to any degree (but almost equally), reduced and subtracted would be almost zero, which contradicts common sense.

The multiplication rule repeats error (14).

In work [19], the addition rule repeats (12).

Subtraction $z = x_1 - x_2$.

$$< m,\alpha,\beta > = < m_1 - m_2,\ \alpha_1 + \beta_2,\ \alpha_2 + \beta_1 >. \quad (16)$$

Multiplication $z = x_1 \cdot x_2$.

$$< m,\alpha,\beta > = \left\langle \begin{matrix} m_1 \cdot m_2,\ m_1 \cdot \beta_1 + m_2 \cdot \alpha_1 - \\ -\alpha_1 \cdot \beta_1,\ m_1 \cdot \beta_2 + m_2 \cdot \alpha_2 + \alpha_2 \cdot \beta_2 \end{matrix} \right\rangle. \quad (17)$$

Division $z = \dfrac{x_1}{x_2}$.

$$< m,\alpha,\beta > = < \frac{m_1}{m_2},\ \frac{m_1 \cdot \beta_2 + m_2 \cdot \alpha_1}{m_2(m_2 + \beta_2)},\ \frac{m_2 \cdot \alpha_1 + m_1 \cdot \beta_1}{m_2(m_2 - \beta_1)} >. \quad (18)$$

Rule (17) makes an unsuccessful attempt to improve the consideration of importance of the efficients blurriness. As is shown below, the reported formula is not accurate. The proposed ratio (18) to describe the results of division is also not accurate. The denominator of the third component in formula (18) should include a parameter that describes the fuzziness of the divider rather than the divisible in fraction $\dfrac{x_1}{x_2}$.

In the set of rules for the execution of operations, proposed in [20, 21], the rules of addition and subtraction repeat the previous (12) and (16).

Multiplication $z = x_1 \cdot x_2$.

$$< m,\alpha,\beta > = < m_1 \cdot m_2,\ m_1 \cdot \alpha_2 + m_2 \cdot \alpha_1,\ m_1 \cdot \beta_2 + m_2 \cdot \beta_1 >. \quad (19)$$

Division $z = \dfrac{x_1}{x_2}$.

$$< m,\alpha,\beta > = < \frac{m_1}{m_2},\ \frac{m_1 \cdot \beta_2 + m_2 \cdot \alpha_1}{m_2^2},\ \frac{m_2 \cdot \alpha_2 + m_2 \cdot \beta_1}{m_2^2} >. \quad (20)$$

Ratio (19) repeats error (14). Rule (20) for the division of fuzzy numbers, as is shown below, contains inaccuracies in determining the left and right fuzzy coefficients.

Thus, a brief analysis of the known sets of rules for performing arithmetic operations over the ($L-R$)-type fuzzy numbers suggests that there is not any substantiated system of rules for operating on such numbers, which requires continued research.

3. The aim and objectives of the study

The aim of this study is to develop a sound system of fuzzy arithmetic rules.

To accomplish the aim, the following tasks have been set:

– to devise a unified, scientifically-substantiated approach to the wording of rules for performing arithmetic operations on the ($L-R$)-type fuzzy numbers;

– to implement the devised approach and to form with its use a system of rules for the execution of fuzzy arithmetic operations.

4. Rules for performing operations over the ($L-R$)-type fuzzy numbers

The desired approach to the formation of a system of rules for performing arithmetic operations over the ($L-R$)-type fuzzy numbers should be simple, naturally interpreted, and general in character, regardless of the type of operation. In accordance with this, we shall introduce a set of metarules for forming the rules for the execution of operations over the ($L-R$)-type fuzzy numbers.

The proposed metarules for compiling the rules for performing operations over the ($L-R$)-type fuzzy numbers structurally employ their following fundamental feature: the membership functions of such numbers are clearly determined by the numerical values of their parameters (a mode and the fuzzy coefficients). The rules for performing operations over the ($L-R$)-type fuzzy numbers should define algebra, that is, meet the generally accepted requirement: for any pair of fuzzy numbers from the ($L-R$)-type numbers class, no operation involving these numbers should exclude the operation result from this class. Let M_{LR} be a set of the ($L-R$)-type fuzzy numbers with a finite carrier and a pair of fuzzy numbers x_1 and x_2 belong to this set, that is, $x_1 \in M_{LR}$, $x_2 \in M_{LR}$.

Introduce *, an arbitrary operation on fuzzy numbers belonging to M_{LR}. Then, the fuzzy number z, obtained as a result of this operation, is equal to

$$z = x_1 * x_2 \in M_{LR}, \quad (21)$$

describe in the standard way:

$$x_1 = < m_1,\alpha_1,\beta_1 >,\ \ x_2 = < m_2,\alpha_2,\beta_2 >,\ \ z = < m,\alpha,\beta >. \quad (22)$$

Taking into consideration (21), (22), we shall introduce the following metarules for the formation of fuzzy arithmetic rules.

Metarule 1. The modal value m of the fuzzy number z, derived from the execution of (21), depends only on the modal values m_1 and m_2 of numbers x_1 and x_2, that is:

$$m = m_1 * m_2. \quad (23)$$

Metarule 2. The left fuzzy coefficient α of the number z is defined as the difference between the modal value m of the z number and the minimum possible value of the result of the operation execution, that is:

$$\alpha = (m_1 * m_2) - \min_{x_1, x_2}(x_1 * x_2). \quad (24)$$

Metarule 3. The right fuzzy coefficient β is defined as the difference between the maximum possible value of the ope-

ration execution result and the modal value of the z number, that is:

$$\beta = \max_{x_1, x_2}(x_1 * x_2) - (m_1 * m_2). \quad (25)$$

Thus, the introduced metarules 1–3 clearly set a formal procedure for calculating the parameters of the membership function of the fuzzy number resulting from performing arithmetic operations over the $(L–R)$-type fuzzy numbers. We shall use these metarules to consistently describe the rules for performing basic operations over the $(L–R)$-type fuzzy numbers with a finite carrier.

Introduce $x_1 = <m_1, \alpha_1, \beta_1>$, $x_2 = <m_2, \alpha_2, \beta_2>$. Next, in line with (23) to (25), write down the rules for performing arithmetic operations over numbers x_1 and x_2, accompanying them with explanatory drawings.

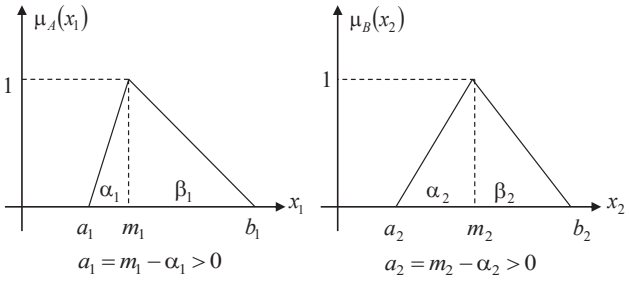

Fig. 1. The membership functions of operands

Addition $z = x_1 + x_2$; $z = <m, \alpha, \beta>$:

$$\min(x_1 + x_2) = (m_1 - \alpha_1) + (m_2 - \alpha_2) =$$
$$= (m_1 + m_2) - (\alpha_1 + \alpha_2),$$
$$\max(x + y) = (m_1 + \beta_1) + (m_2 + \beta_2) =$$
$$= (m_1 + m_2) + (\beta_1 + \beta_2),$$

then

$$m = m_1 + m_2; \ \alpha = \alpha_1 + \alpha_2; \ \beta = \beta_1 + \beta_2. \quad (26)$$

Subtraction $z = x_1 - x_2$; $z(m, \alpha, \beta)$:

$$\min z = (m_1 - \alpha_1) - (m_2 + \beta_2) = (m_1 - m_2) - (\alpha_1 + \beta_2),$$
$$\max z = (m_1 + \beta_1) - (m_2 - \alpha_2) = (m_1 - m_2) + (\beta_1 + \alpha_2),$$

Then

$$m = m_1 - m_2; \ \alpha = \alpha_1 + \beta_2; \ \beta = \beta_1 + \alpha_2. \quad (27)$$

The rules for multiplication and division operations vary depending on the position of the x_1 and x_2 number carriers relative to zero. We shall give these rules for different variants of ratios between m_1, α_1, β_2 and m_2, α_2, β_2.

Variant 1.

Multiplication $z = x_1 \cdot x_2$; $z = <m, \alpha, \beta>$; $m = m_1 \cdot m_2$.

$$\min z = (m_1 - \alpha_1) \cdot (m_2 - \alpha_2) = m_1 m_2 - m_1 \alpha_2 - m_2 \alpha_1 + \alpha_1 \alpha_2,$$
$$\max z = (m_1 + \beta_1) \cdot (m_2 + \beta_2) = m_1 m_2 + m_1 \beta_2 + m_2 \beta_1 + \beta_1 \beta_2,$$
$$\alpha = m_1 m_2 - (m_1 m_2 - m_1 \alpha_2 - m_2 \alpha_1 + \alpha_1 \alpha_2) =$$
$$= m_1 \alpha_2 + m_2 \alpha_1 - \alpha_1 \alpha_2, \quad (28)$$
$$\beta = m_1 m_2 + m_1 \beta_2 + m_2 \beta_1 + \beta_1 \beta_2 - m_1 m_2 =$$
$$= m_1 \beta_2 + m_2 \beta_1 + \beta_1 \beta_2. \quad (29)$$

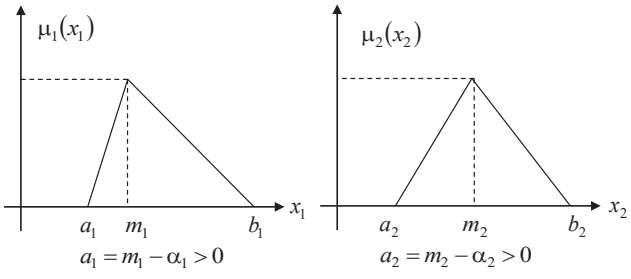

Fig. 2. Performing operations. Variant 1

Division $z = \frac{x_1}{x_2} = <\frac{m_1}{m_2}; \alpha; \beta>$; $m = \frac{m_1}{m_2}$.

$$\min z = \frac{m_1 - \alpha_1}{m_2 + \beta_2}, \quad \max z = \frac{m_1 + \beta_1}{m_2 - \alpha_2},$$

$$\alpha = \frac{m_1}{m_2} - \frac{m_1 - \alpha_1}{m_2 + \beta_2} =$$

$$= \frac{m_1 m_2 + m_1 \beta_2 - m_1 m_2 + m_2 \alpha_1}{m_2 (m_2 + \beta_2)} = \frac{m_2 \alpha_1 + m_1 \beta_2}{m_2 (m_2 + \beta_2)}, \quad (30)$$

$$\beta = \frac{m_1 + \beta_1}{m_2 - \alpha_2} - \frac{m_1}{m_2} =$$

$$= \frac{m_1 m_2 + m_2 \beta_1 - m_1 m_2 + m_1 \alpha_2}{m_2 (m_2 - \alpha_2)} = \frac{m_1 \alpha_2 + m_2 \beta_1}{m_2 (m_2 - \alpha_2)}. \quad (31)$$

Variant 2.

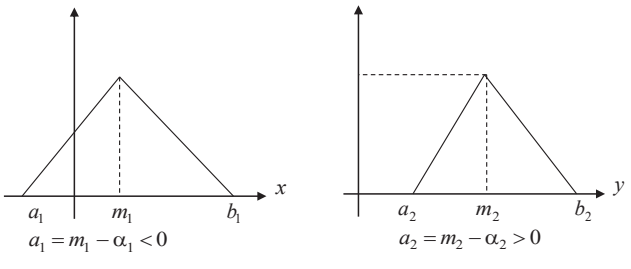

Fig. 3. Performing operations. Variant 2

Multiplication $z = x_1 \cdot x_2$; $m = m_1 \cdot m_2$.

$$\min z = (m_1 - \alpha_1) \cdot (m_2 + \beta_2) = m_1 m_2 + m_1 \beta_2 - m_2 \alpha_1 - \alpha_1 \beta_2,$$

$$\max z = (m_1 + \beta_1) \cdot (m_2 + \beta_2) = m_1 m_2 + m_1 \beta_2 + m_2 \beta_1 + \beta_1 \beta_2,$$

$$\alpha = m_1 m_2 - (m_1 m_2 + m_1 \beta_2 - m_2 \alpha_1 - \alpha_1 \beta_2) =$$
$$= m_2 \alpha_1 - m_1 \beta_2 + \alpha_1 \beta_2, \quad (32)$$

$$\beta = m_1 m_2 + m_1 \beta_2 + m_2 \beta_1 + \beta_1 \beta_2 - m_1 m_2 =$$
$$= m_1 \beta_2 + m_2 \beta_1 + \beta_1 \beta_2. \quad (33)$$

Division $z=\frac{x_1}{x_2}=<m;\alpha;\beta>;\ \ m=\frac{m_1}{m_2}.$

$$\min z=\frac{m_1-\alpha_1}{m_2+\beta_2},\ \ \max z=\frac{m_1+\beta_1}{m_2-\alpha_2}.$$

The result of the division operation, in this case, repeats the above result.

Variant 3.

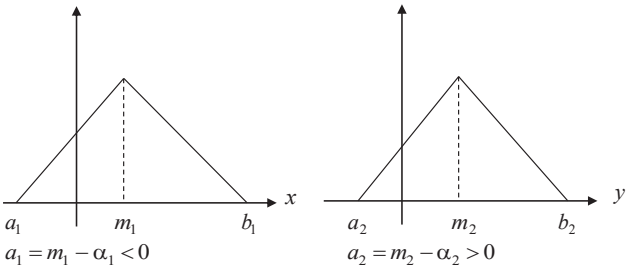

Fig. 4. Performing operations. Variant 3

Multiplication $z=x_1\cdot x_2;\ \ m=m_1\cdot m_2.$

$$\min z=\min\left\{\begin{matrix}(m_1-\alpha_1)(m_2+\beta_2);\\(m_2-\alpha_2)(m_1+\beta_1)\end{matrix}\right\}=$$

$$=\min\left\{\begin{matrix}(m_1m_2+m_1\beta_2-m_2\alpha_1-\alpha_1\beta_2);\\(m_1m_2+m_2\beta_1-m_1\alpha_2-\alpha_2\beta_1)\end{matrix}\right\}.$$

$$\max z=\max\left\{\begin{matrix}(m_1-\alpha_1)(m_2-\alpha_2);\\(m_1+\beta_1)(m_2+\beta_2)\end{matrix}\right\}=$$

$$=\max\left\{\begin{matrix}(m_1m_2-m_1\alpha_2-m_2\alpha_1+\alpha_1\alpha_2);\\(m_1m_2+m_1\beta_2+m_2\beta_1+\beta_1\beta_2)\end{matrix}\right\}.$$

$$\alpha=m_1m_2-\min\{(m_1m_2+m_1\beta_2-m_2\alpha_1-\alpha_1\beta_2);$$

$$(m_1m_2+m_2\beta_1-m_1\alpha_2-\alpha_2\beta_1)\};\qquad(34)$$

$$\beta=\max\left\{\begin{matrix}(-m_1\alpha_2-m_2\alpha_1+\alpha_1\alpha_2)\times\\\times(m_1\beta_2+m_2\beta_1+\beta_1\beta_2)\end{matrix}\right\}.\qquad(35)$$

The division operation in this variant is impossible as the carrier of the divider covers zero.

Thus, we have devised the system of rules for performing arithmetic operations over the ($L-R$)-type fuzzy numbers.

5. Discussion of results obtained in the development of the fuzzy number arithmetic rule system

This paper proposes a strictly based system of rules for performing arithmetic operations on fuzzy numbers. These rules, when necessary, take into consideration the location of the carriers of fuzzy arguments of the operation performed relative to zero. The proposed analytical ratios have been obtained to execute rules over the ($L-R$)-type fuzzy numbers with a compact carrier. For other cases, if, for example, the membership functions of the fuzzy arguments are Gaussian, then these rules would produce an approximate result. The system of rules for performing operations over fuzzy numbers with an infinite carrier, corresponding to such a situation, can be obtained as follows. Operations are performed in two stages. In the first stage, for each of the arguments of the implemented rule, their membership functions are cut on a certain set of levels. To this end, for the appropriate pair of the membership functions:

$$\mu_1(x_1)=\begin{cases}L_1\left(\frac{m_1-x_1}{\alpha_1}\right), & \text{if } x_1\le m_1,\\ R_1\left(\frac{x_1-m_1}{\beta_1}\right), & \text{if } x_1>m_1;\end{cases}$$

$$\mu_2(x_2)=\begin{cases}L_2\left(\frac{m_2-x_2}{\alpha_2}\right), & \text{if } x_2\le m_2,\\ R_2\left(\frac{x_2-m_2}{\beta_2}\right), & \text{if } x_2>m_2.\end{cases}$$

and for the selected value of the d level, the following equations are solved:

$$L_K\left(\frac{m_K-x_K}{\alpha_K}\right)=d,\ \ R_k\left(\frac{x_k-m_k}{\beta_k}\right)=d,\ \ k=1,2.\qquad(38)$$

The roots of these equations $l_k(d), r_k(d), k=1,2$, for arguments x_1 and x_2 form intervals:

$$I_1^{(d)}(x_1)=\left[l_1(d), r_1(d)\right],\ \ I_2^{(d)}(x_2)=\left[l_2(d), r_2(d)\right].\qquad(39)$$

The resulting intervals (39) in the second stage are used to calculate the d-level interval, which is the result of the * operation performed over the fuzzy numbers, set by the intervals $I_1^{(d)}(x_1)$ and $I_2^{(d)}(x_2)$. In this case, we shall obtain $I^{(d)}(z)=I_1^{(d)}(x_1)*I_2^{(d)}(x_2)$. The corresponding operation is performed according to the rules introduced above. The described steps are performed for each of the d-levels. The undeniable advantage of the described approach is the possibility to restore the operation over the original fuzzy numbers at any predefined accuracy. The obvious drawback is the laboriousness of obtaining the result, associated with the need to multiply solve the set of equations (38), not necessarily easily solved, which naturally limits the scope of application of this approach.

Possible areas of further research include the development of an expanded system of rules for any binary algebraic operations in order to create a formal basis for problem solving, analysis, and fuzzy system optimization under uncertain conditions [22, 23]; the construction of rules for performing operations over fuzzy numbers of the second type.

6. Conclusions

1. We have devised a scientifically-substantiated approach to form a system of rules of fuzzy numbers arithmetic whose axiomatic basis is a set of metarules.

2. Using this approach, a strictly based system of rules has been built for performing arithmetic operations over the ($L-R$)-type fuzzy numbers with a compact carrier. The proposed rule system has been expanded to perform operations on fuzzy numbers with a non-finite carrier. In this case, the initial problem is reduced to a set of simple problems with a compact carrier.

References

1. Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8 (3), 338–353. doi: https://doi.org/10.1016/s0019-9958(65)90241-x
2. Kaufman, A. (1975). Theory of Fuzzy subsets. Academic Press, 432.
3. Dubois, D., Prade, H. (1978). Operations on fuzzy numbers. International Journal of Systems Science, 9 (6), 613–626. doi: https://doi.org/10.1080/00207727808941724
4. Nahmias, S. (1978). Fuzzy variables. Fuzzy Sets and Systems, 1 (2), 97–110. doi: https://doi.org/10.1016/0165-0114(78)90011-8
5. Dubois, D., Prade, H. (1988). Théorie des possibilités: application à la représentation des connaissances en informatique. Masson, Paris.
6. Dubois, D., Prade, H. (1987). Analysis of Fuzzy Information. Mathematics and logic. Vol. 1. CRC Press, Boca Raton, 3–39.
7. Kaufman, A., Gupta, Madan M. (1985). Introduction to Fuzzy Arithmetic Theory and Applications. Van Nostrand Reinhold Company, 351.
8. Raskin, L., Sira, O. (2008). Nechetkaya matematika. Kharkiv: Parus, 352.
9. Musa, I. M. (2015). Investigation of Basic Concepts of Fuzzy Arithmetic. Eastern Mediterranean University, 49.
10. Akther, T., Ahmad, S. U. (1970). A computational method for fuzzy arithmetic operations. Daffodil International University Journal of Science and Technology, 4 (1), 18–22. doi: https://doi.org/10.3329/diujst.v4i1.4350
11. Keskar, A. G. (2016). Presentation on Fuzzy Arithmetic. VNIT-Nagpur, 86.
12. Garg, H. (2017). Some arithmetic operations on the generalized sigmoidal fuzzy numbers and its application. Granular Computing, 3 (1), 9–25. doi: https://doi.org/10.1007/s41066-017-0052-7
13. Vahidi, J., Rezvani, S. (2013). Arithmetic Operations on Trapezoidal Fuzzy Numbers. Journal of Nonlinear Analysis and Application, 2013, 1–8. doi: https://doi.org/10.5899/2013/jnaa-00111
14. Seresht, N. G., Fayek, A. R. (2018). Fuzzy Arithmetic Operations: Theory and Applications in Construction Engineering and Management. Fuzzy Hybrid Computing in Construction Engineering and Management, 111–147. doi: https://doi.org/10.1108/978-1-78743-868-220181003
15. Gerami Seresht, N., Fayek, A. R. (2019). Computational method for fuzzy arithmetic operations on triangular fuzzy numbers by extension principle. International Journal of Approximate Reasoning, 106, 172–193. doi: https://doi.org/10.1016/j.ijar.2019.01.005
16. Rotshteyn, A. P. (1999). Intellektual'nye tehnologii identifikatsii: nechetkie mnozhestva, geneticheskie algoritmy, neyronnye seti. Vinnytsia: UNIVERSUM-Vinnytsia, 295.
17. Alim, A., Johora, F. T., Babu, S., Sultana, A. (2015). RETRACTED: Elementary Operations on L-R Fuzzy Number. Advances in Pure Mathematics, 05 (03), 131–136. doi: https://doi.org/10.4236/apm.2015.53016
18. Grinyaev, Yu. V. (2008). Teoriya nechetkih mnozhestv. Tomsk, 196.
19. Asady, B., Akbari, M., Keramati, M. A. (2011). Ranking of fuzzy numbers by fuzzy mapping. International Journal of Computer Mathematics, 88 (8), 1603–1618. doi: https://doi.org/10.1080/00207160.2010.518757
20. Leonenkov, A. V. (2003). Nechetkoe modelirovanie v srede Matlab i fuzzyTech. Sankt-Peterburg: BHV – Peterburg, 736.
21. Ibragimov, V. A. (2010). Elementy nechetkoy matematiki. Baku: AGNA, 394.
22. Raskin, L., Sira, O. (2016). Fuzzy models of rough mathematics. Eastern-European Journal of Enterprise Technologies, 6 (4 (84)), 53–60. doi: https://doi.org/10.15587/1729-4061.2016.86739
23. Raskin, L., Sira, O. (2016). Method of solving fuzzy problems of mathematical programming. Eastern-European Journal of Enterprise Technologies, 5 (4 (83)), 23–28. doi: https://doi.org/10.15587/1729-4061.2016.81292