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EXECUTION OF ARITHMETIC OPERATIONS INVOLVING THE SECOND-ORDER FUZZY NUMBERS

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The need to improve the adequacy of conventional models of the source data uncertainty in order to undertake research using fuzzy mathematics methods has led to the development of natural improvement in the analytical description of the fuzzy numbers' membership functions. Given this, in particular, in order to describe the membership functions of the three-parametric fuzzy numbers of the (L-R)-type, the modification implies the following. It is accepted that these functions' parameters (a modal value, the left and right fuzzy factors) are not set clearly by their membership functions. The numbers obtained in this way are termed the second-order fuzzy numbers (bi-fuzzy). The issue, in this case, is that there are no rules for operating on such fuzzy numbers. This paper has proposed and substantiated a system of operating rules for a widely used and effective class of fuzzy numbers of the (L-R)-type whose membership functions' parameters are not clearly defined. These rules have been built as a result of the generalization of known rules for operating on regular fuzzy numbers. We have derived analytical ratios to compute the numerical values of the membership functions of the fuzzy results from executing arithmetic operations (addition, subtraction, multiplication, division) over the second-order fuzzy numbers. It is noted that the resulting system of rules is generalized for the case when the numbers-operands' fuzziness order exceeds the second order. The examples of operations execution over the second-order fuzzy numbers of the (L-R)-type have been given

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1. Introduction

The bi-fuzzy numbers (the second-order fuzzy numbers) were introduced in [1] as a natural generalization of the regular fuzzy numbers. The fundamental novelty of the technology that formally defines these numbers is that the parameters of their membership functions are themselves fuzzy with their own membership functions. An important positive feature of the emerging complexity of the structure of mathematical notation of fuzzy numbers is the arising possibility to radically improve the adequacy of the uncertainty models of objects in the external world and within their functioning environment. The resulting formalism significantly expands the space of the mathematical models of real-world objects operating under conditions of hierarchical uncertainty. The two-stage character of the analytical description of the membership functions' parameters of the second-order fuzzy numbers makes it possible to more accurately and strictly determine the real inaccuracy of the conventional descriptions of objects and their functioning environment. However, the practical usefulness and effectiveness of using this new field of fuzzy mathematics are largely limited by the lack of the algebra of operations on the corresponding fuzzy numbers, which requires further research.

2. Literature review and problem statement

Following the pioneering work [1], papers [2, 3], which addressed the theory of fuzzy sets, focused on the prospect of expanding the fuzzy mathematics arsenal by using the second-order fuzzy numbers and the relevance of the study in this area. The issues of building the second-order fuzzy numbers' membership functions and the need to form the axiomatics for these numbers are discussed in [4, 5]. Papers [5, 6] provide examples of building the second-order fuzzy numbers' membership functions and discuss the emerging opportunities to improve the adequacy of system models under the conditions of uncertainty. Studies [7–9] address the use of the second-order fuzzy numbers in solving fuzzy logic problems. Contemporary works [10, 11] discuss the issues associated with the possible application of the second-order fuzzy numbers in regression analysis. In recent years, papers [12–15] have explored the possibilities of practical use of second-order fuzzy numbers with an interval membership function. The authors note the simplicity to form appropriate membership functions, but it is noted that the models that arise are not adequate enough. In studies [16, 17], the application area of interval numbers expands toward the development of models for the non-linear management tasks.

Thus, the following important first steps have been taken in known studies on the second-order fuzzy numbers. The idea of constructing fuzzy numbers above the first order has been formulated, and the methods of building the second-order fuzzy numbers' membership functions have been considered. Practical examples of numerous problems have been given, in which the use of these numbers significantly improves the adequacy of mathematical models. However, these problems are solved at the simplest possible interval level of the description of the membership function parameters. It is clear that the practical usefulness of the results obtained in this case is very relative.

A brief analysis of known works has made it possible to outline a problematic issue – the lack of general rules for operations involving the second-order fuzzy numbers. A solution to this problem in its most general form is hardly feasible. Below, this problem is solved for an important particular case where fuzzy numbers have a membership function of the (*L*-*R*)-type.

This choice is clear: the membership functions of such numbers are similarly and clearly defined by a set of their numerical characteristics. In particular, the membership function of a three-parametric number of the (*L*-*R*)-type is assigned by the set $<m, \alpha, \beta>$, where m is the modal value of a fuzzy number, α and β are, respectively, the left and right fuzziness factors. If this number is bi-fuzzy, the parameters m, α, β are the fuzzy numbers, defined by the following sets:

$$<m_m,\alpha_m,\beta_m>,\ <m_\alpha,\alpha_\alpha,\beta_\alpha>,\ <m_\beta,\alpha_\beta,\beta_\beta>.$$

This natural and easy-to-interpret notation is hereafter used in the development of rules for the execution of operations on the second-order fuzzy numbers of the (*L*-*R*)-type.

3. The aim and objectives of the study

The aim of this study is to build a sound system of rules for the execution of arithmetic operations involving the second-order fuzzy numbers. Devising these rules would make it possible to move from a decorative description of problem statements to actually solve them in order to obtain specific results under the conditions of clearly defined source data. In this case, the computational solving process is implemented by executing a clearly described sequence of arithmetic operations.

To accomplish the aim, the following tasks have been set:

– to build a system for forming the rules of executing arithmetic operations involving the second-order fuzzy numbers;

– to define the formula ratios for computing the parameters of the membership functions of the results from executing binary arithmetic operations involving the second-order fuzzy numbers.

4. Building a system of rules for executing arithmetic operations involving the second-order fuzzy numbers of the (*L*-*R*)-type

Let the membership functions of two fuzzy numbers of the (*L*-*R*)-type be set by the threesomes of their parameters' values:

$$x_1=\langle m_1,\alpha_1,\beta_1\rangle,\quad x_2=\langle m_2,\alpha_2,\beta_2\rangle.$$

Introduce a binary operation

$$Z=x_1\otimes x_2=\langle m,\alpha,\beta\rangle.$$

In accordance with the rules from [18] for the arithmetic of fuzzy numbers of the (*L*-*R*)-type, the formulae for calculating the parameters of the membership function of the fuzzy result of main operations take the following form:

– addition

$$m=m_1+m_2,\quad \alpha=\alpha_1+\alpha_2,\quad \beta=\beta_1+\beta_2; \tag{1}$$

– subtraction

$$m=m_1-m_2,\quad \alpha=\alpha_1+\beta_2,\quad \beta=\beta_1+\alpha_2; \tag{2}$$

– multiplication

$$m=m_1\cdot m_2,\quad \alpha=m_1\cdot\alpha_2+m_2\cdot\alpha_1-\alpha_1\cdot\alpha_2,$$

$$\beta=m_1\cdot\beta_2+m_2\cdot\beta_1+\beta_1\cdot\beta_2; \tag{3}$$

– division

$$m=\frac{m_1}{m_2},\quad \alpha=\frac{m_2\cdot\alpha_1+m_1\cdot\beta_2}{m_2\left(m_2+\beta_2\right)},\quad \beta=\frac{m_1\cdot\alpha_2+m_2\cdot\beta_1}{m_2\left(m_2-\alpha_2\right)}. \tag{4}$$

In these ratios, the parameters of the membership functions of the results of arithmetic operations are found from the computations based on formulae (1) to (4) involving regular numbers. The character of these ratios allows them to be summarized in the following way. If, according to the idea from [1], the fuzzy numbers' membership function parameters are the fuzzy numbers themselves, this can be used constructively. Assume that the parameters m_1, α_1, β_1, m_2, α_2, β_2 of the membership functions of the fuzzy numbers x_1 and x_2 are the fuzzy numbers themselves with membership functions defined by the following sets

$$m_1=\left\langle m_{m_1},\alpha_{m_1},\beta_{m_1}\right\rangle,\quad \alpha_1=\left\langle m_{\alpha_1},\alpha_{\alpha_1},\beta_{\alpha_1}\right\rangle,$$

$$\beta_1=\left\langle m_{\beta_1},\alpha_{\beta_1},\beta_{\beta_1}\right\rangle, \tag{5}$$

$$m_2=\left\langle m_{m_2},\alpha_{m_2},\beta_{m_2}\right\rangle,\quad \alpha_2=\left\langle m_{\alpha_2},\alpha_{\alpha_2},\beta_{\alpha_2}\right\rangle,$$

$$\beta_2=\left\langle m_{\beta_2},\alpha_{\beta_2},\beta_{\beta_2}\right\rangle. \tag{6}$$

Then the parameters of computation results according to (1) to (4) involving the fuzzy numbers described in (5), (6) become fuzzy numbers, and the results of these operations would determine the second-order fuzzy numbers. Thus, a system is proposed that forms the rules for the execution of arithmetic operations on fuzzy numbers whose order is higher than the first order. In this case, the parameters of the fuzzy numbers' membership functions of the assigned order in general ratios (1) to (4) are used to calculate the parameters of the membership functions of the next, higher order.

Calculate the parameters of the fuzzy numbers' membership functions resulting from the execution of x_1+x_2, x_1-x_2, $x_1 \cdot x_2$, x_1/x_2, operations by using (1) to (6).

Define the following system of rules for the execution of arithmetic operations involving the bi-fuzzy numbers of the (*L*-*R*)-type.

Addition.

According to (1), taking into consideration (5), (6), it is necessary to find the parameters of the membership functions of fuzzy numbers m_1+m_2, $\alpha_1+\alpha_2$, $\beta_1+\beta_2$.

We shall perform the appropriate operations.

$$m=m_1+m_2=\left\langle m_{m_1}+m_{m_2},\ \alpha_{m_1}+\alpha_{m_2},\ \beta_{m_1}+\beta_{m_2}\right\rangle,$$

$$\alpha=\alpha_1+\alpha_2=\left\langle m_{\alpha_1}+m_{\alpha_2},\ \alpha_{\alpha_1}+\alpha_{\alpha_2},\ \beta_{\alpha_1}+\beta_{\alpha_2}\right\rangle. \tag{7}$$

$$\beta=\beta_1+\beta_2=\left\langle m_{\beta_1}+m_{\beta_2},\ \alpha_{\beta_1}+\alpha_{\beta_2},\ \beta_{\beta_1}+\beta_{\beta_2}\right\rangle.$$

The resulting ratios (7) assign the membership functions of the parameters m, α, β for the result of adding the fuzzy numbers x_1 and x_2. The parameters of these membership functions are defined through the parameters of the membership functions of the fuzzy terms $x_1=\langle m_1,\alpha_1,\beta_1\rangle$, $x_2=\langle m_2,\alpha_2,\beta_2\rangle$, set by ratios (5), (6). Thus, the result of the addition of two bi-fuzzy numbers x_1 and x_2 is also a bi-fuzzy number.

Subtraction.

In accordance with (2), to calculate the parameters of the membership function of the fuzzy result from executing a subtraction operation, one needs to find the parameters of the membership functions of the fuzzy numbers m_1-m_2, $\alpha_1+\beta_2$, $\beta_1+\alpha_2$.

Perform the appropriate operations.

$$m=m_1-m_2=\left\langle m_{m_1}-m_{m_2},\ \alpha_{m_1}+\beta_{m_2},\ \beta_{m_1}+\alpha_{m_2}\right\rangle,$$

$$\alpha=\alpha_1+\beta_2=\left\langle m_{\alpha_1}+m_{\beta_2},\ \alpha_{\alpha_1}+\alpha_{\beta_2},\ \beta_{\alpha_1}+\beta_{\beta_2}\right\rangle, \tag{8}$$

$$\beta=\beta_1+\alpha_2=\left\langle m_{\beta_1}+m_{\alpha_2},\ \alpha_{\beta_1}+\alpha_{\alpha_2},\ \beta_{\beta_1}+\beta_{\alpha_2}\right\rangle.$$

Multiplication.

Using (3), compute the parameters of the membership function of the fuzzy number x_1x_2 taking into consideration the fuzziness of the parameters of the membership function of each of the multipliers defined from (5) and (6). To this end, one needs to consistently calculate the membership functions of the following products of the fuzzy numbers m_1m_2, $m_1\alpha_2$, $m_2\alpha_1$, $\alpha_1\alpha_2$, $m_1\beta_2$, $m_2\beta_1$, $\beta_1\beta_2$.

Perform the appropriate operations.

Introduce

$$m_1m_2=\left\langle\begin{matrix} m_{m_1}m_{m_2};\ m_{m_1}\alpha_{m_2}+m_{m_2}\alpha_{m_1}-\alpha_{m_1}\alpha_{m_2};\\ m_{m_1}\beta_{m_2}+m_{m_2}\beta_{m_1}+\beta_{m_1}\beta_{m_2}\end{matrix}\right\rangle; \tag{9}$$

$$m_1\alpha_2=\left\langle\begin{matrix} m_{m_1}m_{\alpha_2};\ m_{m_1}\alpha_{\alpha_2}+m_{\alpha_2}\alpha_{m_1}-\alpha_{m_1}\alpha_{\alpha_2};\\ m_{m_1}\beta_{\alpha_2}+m_{\alpha_2}\beta_{m_1}+\beta_{m_1}\beta_{\alpha_2}\end{matrix}\right\rangle; \tag{10}$$

$$m_2\alpha_1=\left\langle\begin{matrix} m_{m_2}m_{\alpha_1};\ m_{m_2}\alpha_{\alpha_1}+m_{\alpha_1}\alpha_{m_2}-\alpha_{m_2}\alpha_{\alpha_1};\\ m_{m_2}\beta_{\alpha_1}+m_{\alpha_1}\beta_{m_2}+\beta_{m_2}\beta_{\alpha_1}\end{matrix}\right\rangle; \tag{11}$$

$$\alpha_1\alpha_2=\left\langle\begin{matrix} m_{\alpha_1}m_{\alpha_2};\ m_{\alpha_1}\alpha_{\alpha_2}+m_{\alpha_2}\alpha_{\alpha_1}-\alpha_{\alpha_1}\alpha_{\alpha_2};\\ m_{\alpha_1}\beta_{\alpha_2}+m_{\alpha_2}\beta_{\alpha_1}+\beta_{\alpha_1}\beta_{\alpha_2}\end{matrix}\right\rangle; \tag{12}$$

$$m_1\beta_2=\left\langle\begin{matrix} m_{m_1}m_{\beta_2};\ m_{m_1}\alpha_{\beta_2}+m_{\beta_2}\alpha_{m_1}-\alpha_{m_1}\alpha_{\beta_2};\\ m_{m_1}\beta_{\beta_2}+m_{\beta_2}\beta_{m_1}+\beta_{m_1}\beta_{\beta_2}\end{matrix}\right\rangle; \tag{13}$$

$$m_2\beta_1=\left\langle\begin{matrix} m_{m_2}m_{\beta_1};\ m_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{m_2}-\alpha_{m_2}\alpha_{\beta_1};\\ m_{m_2}\beta_{\beta_1}+m_{\beta_1}\beta_{m_2}+\beta_{m_2}\beta_{\beta_1}\end{matrix}\right\rangle; \tag{14}$$

$$\beta_1\beta_2=\left\langle\begin{matrix} m_{\beta_1}m_{\beta_2};\ m_{\beta_1}\alpha_{\beta_2}+m_{\beta_2}\alpha_{\beta_1}-\alpha_{\beta_1}\alpha_{\beta_2};\\ m_{\beta_1}\beta_{\beta_2}+m_{\beta_2}\beta_{\beta_1}+\beta_{\beta_1}\beta_{\beta_2}\end{matrix}\right\rangle. \tag{15}$$

Next, according to (3), one needs to find the parameters of the membership functions of the following fuzzy numbers:

$$m_1\alpha_2+m_2\alpha_1;\quad m_1\alpha_2+m_2\alpha_1-\alpha_1\alpha_2;$$

$$m_1\beta_2+m_2\beta_1;\quad m_1\beta_2+m_2\beta_1+\beta_1\beta_2.$$

Perform the appropriate operations.

$$m_1\alpha_2+m_2\alpha_1=\left\langle\begin{matrix} m_{m_1}m_{\alpha_2}+m_{m_2}m_{\alpha_1};m_{m_1}\alpha_{\alpha_2}+\\ +m_{\alpha_2}\alpha_{m_1}+m_{m_2}\alpha_{\alpha_1}+m_{\alpha_1}\alpha_{m_2}-\\ -\alpha_{m_1}\alpha_{\alpha_2}-\alpha_{m_2}\alpha_{\alpha_1};m_{m_1}\beta_{\alpha_2}+\\ +m_{\alpha_2}\beta_{m_1}+m_{m_2}\beta_{\alpha_1}+\\ +m_{\alpha_1}\beta_{m_2}+\beta_{m_1}\beta_{\alpha_2}+\beta_{m_2}\beta_{\alpha_1}\end{matrix}\right\rangle;$$

$$\alpha=m_1\alpha_2+m_2\alpha_1-\alpha_1\alpha_2=$$

$$=\left\langle\begin{matrix} m_{m_1}m_{\alpha_2}+m_{m_2}m_{\alpha_1}-m_{\alpha_1}m_{\alpha_2};m_{m_1}\alpha_{\alpha_2}+\\ +m_{\alpha_2}\alpha_{m_1}+m_{m_2}\alpha_{\alpha_1}+m_{\alpha_1}\alpha_{m_2}-\alpha_{m_1}\alpha_{\alpha_2}-\\ -\alpha_{m_2}\alpha_{\alpha_1}-m_{\alpha_1}\alpha_{\alpha_2}-m_{\alpha_2}\alpha_{\alpha_1}+\alpha_{\alpha_1}\alpha_{\alpha_2};\\ m_{m_1}\beta_{\alpha_2}+m_{\alpha_2}\beta_{m_1}+m_{m_2}\beta_{\alpha_1}+m_{\alpha_1}\beta_{m_2}+\\ +\beta_{m_1}\beta_{\alpha_2}+\beta_{m_2}\beta_{\alpha_1}-m_{\alpha_1}\beta_{\alpha_2}-m_{\alpha_2}\beta_{\alpha_1}-\\ -\beta_{\alpha_1}\beta_{\alpha_2}-\alpha_{\alpha_1}\alpha_{\alpha_2}\end{matrix}\right\rangle; \tag{16}$$

$$m_1\beta_2+m_2\beta_1=\left\langle\begin{matrix} m_{m_1}m_{\beta_2}+m_{m_2}m_{\beta_1};m_{m_1}\alpha_{\beta_2}+\\ +m_{\beta_2}\alpha_{m_1}+m_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{m_2}-\\ -\alpha_{m_1}\alpha_{\beta_2}-\alpha_{m_2}\alpha_{\beta_1};m_{m_1}\beta_{\beta_2}+\\ +m_{\beta_2}\beta_{m_1}+\beta_{m_1}\beta_{\beta_2}+\\ +m_{m_2}\beta_{\beta_1}+m_{\beta_1}\beta_{m_2}+\beta_{m_2}\beta_{\beta_1}\end{matrix}\right\rangle;$$

$$\beta=m_1\beta_2+m_2\beta_1+\beta_1\beta_2=$$

$$=\left\langle\begin{matrix} m_{m_1}m_{\beta_2}+m_{m_2}m_{\beta_1}+m_{\beta_1}m_{\beta_2};m_{m_1}\alpha_{\beta_2}+\\ +m_{\beta_2}\alpha_{m_1}+m_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{m_2}-\alpha_{m_1}\alpha_{\beta_2}-\\ -\alpha_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{\beta_2}+m_{\beta_2}\alpha_{\beta_1}-\alpha_{\beta_1}\alpha_{\beta_2};\\ m_{m_1}\beta_{\beta_2}+m_{\beta_2}\beta_{m_1}+\beta_{m_1}\beta_{\beta_2}+m_{m_2}\beta_{\beta_1}+\\ +m_{\beta_1}\beta_{m_2}+\beta_{m_1}\beta_{\beta_1}+m_{\beta_1}\beta_{\beta_2}+m_{\beta_2}\beta_{\beta_1}+\beta_{\beta_1}\beta_{\beta_2}\end{matrix}\right\rangle. \tag{17}$$

Thus, formulae (11), (16), (17) set parameters for the membership functions of the result of multiplying two bi-fuzzy numbers.

Division.

Determine a membership function of the bi-fuzzy number, resulting from the division of two fuzzy numbers whose membership functions' parameters are defined fuzzily by their membership functions (5), (6). To this end, according to (4), one first needs to find the parameters of the membership functions of the following fuzzy numbers m_1/m_2, $m_2\alpha_1$, $m_1\beta_2$, $m_2+\beta_2$, $m_2(m_2+\beta_2)$, $m_1\alpha_2$, $m_2\beta_1$, $m_1\alpha_2+m_2\beta_1$, $m_2(m_2-\alpha_2)$, $\dfrac{m_2\alpha_1+m_1\beta_2}{m_2(m_2+\beta_2)}$, $\dfrac{m_1\alpha_2+m_2\beta_1}{m_2(m_2-\alpha_2)}$, which are the components in ratios (4).

Perform the appropriate operations.

$$m=\frac{m_1}{m_2}=$$

$$=\left\langle\frac{m_{m_1}}{m_{m_2}};\frac{m_{m_2}\cdot\alpha_{m_1}+m_{m_1}\cdot\beta_{m_2}}{m_{m_2}\left(m_{m_2}+\beta_{m_2}\right)};\frac{m_{m_1}\cdot\alpha_{m_2}+m_{m_2}\cdot\beta_{m_1}}{m_{m_2}\left(m_{m_2}-\alpha_{m_2}\right)}\right\rangle. \tag{18}$$

The membership functions' parameters for the fuzzy numbers $m_1\alpha_2$, $m_2\alpha_1$, $m_1\beta_2$, $m_2\beta_1$ were found earlier;

they are described, accordingly, by formulae (10), (11), (13), (14). Then

$$m_2\alpha_1+m_1\beta_2=\left\langle\begin{array}{l} m_{m_2}m_{\alpha_1}+m_{m_1}m_{\beta_2};m_{m_2}\alpha_{\alpha_1}+\\ +m_{\alpha_1}\alpha_{m_2}-\alpha_{m_2}\alpha_{\alpha_1}+m_{m_1}\alpha_{\beta_2}+\\ +m_{\beta_2}\alpha_{m_1}-\alpha_{m_1}\alpha_{\beta_2};m_{m_2}\beta_{\alpha_1}+\\ +m_{\alpha_1}\beta_{m_2}+m_{\alpha_1}\beta_{m_2}+\beta_{m_2}\beta_{\alpha_1}+\\ +m_{m_1}\beta_{\beta_2}+m_{\beta_2}\beta_{m_1}+\beta_{m_1}\beta_{\beta_2} \end{array}\right\rangle; \qquad (19)$$

$$m_2+\beta_2=\left\langle m_{m_2}+m_{\beta_2};\alpha_{m_2}+\alpha_{\beta_2};\beta_{m_2}\beta_{\beta_2}\right\rangle; \qquad (20)$$

$$m_2\left(m_2+\beta_2\right)=$$
$$=\left\langle\begin{array}{l} m_{m_2}\left(m_{m_2}+m_{\beta_2}\right);m_{m_1}\left(\alpha_{m_2}+\alpha_{\beta_2}\right)+\\ +\left(m_{m_2}+m_{\beta_2}\right)\alpha_{m_2}-\beta_{m_2}\left(\beta_{m_2}+\beta_{\beta_2}\right);\\ m_{m_1}\left(\beta_{m_2}+\beta_{\beta_2}\right)+\left(m_{m_2}+m_{\beta_2}\right)\beta_{m_2}+\\ +\beta_{m_2}(\beta_{m_2}+\beta_{\beta_2}) \end{array}\right\rangle; \qquad (21)$$

$$m_1\alpha_2+m_2\beta_1=\left\langle\begin{array}{l} m_{m_1}m_{\alpha_2}+m_{m_2}m_{\beta_1};m_{m_1}\alpha_{\alpha_2}+\\ +m_{\alpha_2}\alpha_{m_1}-\alpha_{m_1}\alpha_{\alpha_2}+m_{m_2}\alpha_{\beta_1}+\\ +m_{\beta_1}\alpha_{m_2}-\alpha_{m_2}\alpha_{\beta_1};m_{m_1}\beta_{\alpha_2}+\\ +m_{m_2}\beta_{m_1}+\beta_{m_1}\beta_{\alpha_2}+m_{m_2}\beta_{\beta_1}+\\ +m_{\beta_1}\beta_{m_2}+\beta_{m_2}\beta_{\beta_1} \end{array}\right\rangle; \qquad (22)$$

$$m_2-\alpha_2=\left\langle m_{m_2}-m_{\alpha_2};\alpha_{m_2}+\beta_{\alpha_2};\beta_{m_2}+\alpha_{\alpha_2}\right\rangle; \qquad (23)$$

$$m_2\left(m_2-\alpha_2\right)=$$
$$=\left\langle\begin{array}{l} m_{m_2}\left(m_{m_2}-m_{\alpha_2}\right);m_{m_2}\left(\alpha_{m_2}+\beta_{\alpha_2}\right)+\\ +\left(m_{m_2}-m_{\alpha_2}\right)\alpha_{m_2}-\alpha_{m_2}\left(\alpha_{m_2}+\beta_{\alpha_2}\right);\\ m_{m_2}\left(\beta_{m_2}+\alpha_{\alpha_2}\right)+\left(m_{m_2}-m_{\alpha_2}\right)\beta_{m_2}+\\ +\beta_{m_2}\left(\beta_{m_2}+\beta_{\beta_2}\right) \end{array}\right\rangle. \qquad (24)$$

Thus, we have derived the required intermediate ratios to compute the membership functions of the fuzzy components included in formulae (4) to calculate the parameters of the fuzzy result of the division. In order to simplify the ratios obtained, we shall introduce the following designations

$$m=m_1/m_2,\quad \alpha=m_2\cdot\alpha_1+m_1\cdot\beta_2,$$

$$b=m_2\left(m_2+\beta_2\right),$$

$$c=m_1\alpha_2+m_2\beta_1,\quad d=m_2\left(m_2-\alpha_2\right).$$

In this case, according to (4), the numerical values m_1/m_2, a/b, c/d set the parameters of the membership function of the desired fuzzy result of executing the operation x_1/x_2.

The membership functions' parameters of the fuzzy numbers m, a, b, c, d are set by formulae (18), (19), (21), (22), (24). To describe these membership functions, we use a standard approach:

$$m=\left\langle m_m,\alpha_m,\beta_m\right\rangle,\quad a=\left\langle m_a,\alpha_a,\beta_a\right\rangle,\quad b=\left\langle m_b,\alpha_b,\beta_b\right\rangle,$$

$$c=\left\langle m_c,\alpha_c,\beta_c\right\rangle,\quad d=\left\langle m_d,\alpha_d,\beta_d\right\rangle.$$

Then, in accordance with (4). we obtain

$$\frac{a}{b}=\left\langle\frac{m_a}{m_b};\frac{m_b\cdot\alpha_a+m_a\cdot\beta_b}{m_b\left(m_b+\beta_b\right)};\frac{m_a\cdot\alpha_b+m_b\cdot\beta_a}{m_b\left(m_b+\alpha_b\right)}\right\rangle; \qquad (25)$$

$$\frac{c}{d}=\left\langle\frac{m_c}{m_d};\frac{m_d\cdot\alpha_c+m_c\cdot\beta_d}{m_d\left(m_d+\beta_c\right)};\frac{m_c\cdot\alpha_d+m_d\cdot\beta_c}{m_c\left(m_d+\alpha_c\right)}\right\rangle. \qquad (26)$$

Formulae (18), (25), (26) set the parameters of the membership function of the bi-fuzzy result of dividing x_1 by x_2.

Thus, the ratios obtained determine the numerical values of the membership functions' parameter of the fuzzy results of arithmetic operations involving the fuzzy numbers x_1 and x_2. In all cases, these results are bi-fuzzy.

Consider an example.

Set the fuzzy numbers x_1 and x_2 based on the values of the membership functions:

$$x_1=\left\langle m_1,\alpha_1,\beta_1\right\rangle=\left\langle 80;8;12\right\rangle;$$

$$x_2=\left\langle m_2,\alpha_2,\beta_2\right\rangle=\left\langle 40;4;6\right\rangle.$$

Perform the arithmetic operations involving x_1 and x_2 in accordance with (1) to (4):

– addition: $Z=x_1+x_2$.

$$m=m_1+m_2=80+40=120,$$

$$\alpha=\alpha_1+\alpha_2=8+4=12,$$

$$\beta=\beta_1+\beta_2=12+6=18;$$

– subtraction: $Z=x_1-x_2$.

$$m=m_1-m_2=80-40=40,$$

$$\alpha=\alpha_1+\beta_2=8+6=14,$$

$$\beta=\beta_1+\alpha_2=12+4=16;$$

– multiplication: $Z=x_1\cdot x_2$.

$$m=m_1\cdot m_2=80\cdot 40=3{,}200,$$

$$\alpha=m_1\alpha_2+m_2\alpha_1-\alpha_1\alpha_2=80\cdot 4+40\cdot 8-8\cdot 4=608,$$

$$\beta=m_1\beta_2+m_2\beta_1+\beta_1\beta_2=80\cdot 6+40\cdot 12+6\cdot 12=1{,}032.$$

– division: $Z=\dfrac{x_1}{x_2}$.

$$m=\frac{m_1}{m_2}=\frac{40}{20}=2;$$

$$\alpha=\frac{m_2\alpha_1+m_1\beta_2}{m_2\left(m_2+\beta_2\right)}=\frac{40\cdot 8+80\cdot 6}{40\left(40+6\right)}=0.43;$$

$$\beta=\frac{m_1\alpha_2+m_2\beta_1}{m_2\left(m_2-\alpha_2\right)}=\frac{40\cdot 4+40\cdot 12}{40\left(40-4\right)}=0.555.$$

The actions performed are illustrated in Fig. 1–6.

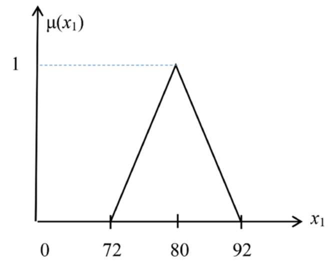

Fig. 1. Membership function x_1

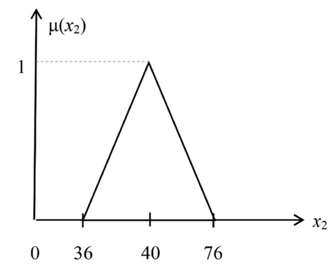

Fig. 2. Membership function x_2

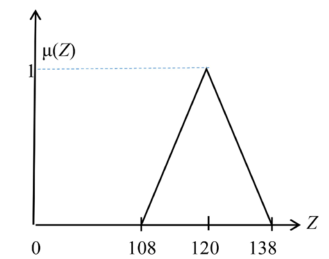

Fig. 3. Membership function $Z=x_1+x_2$

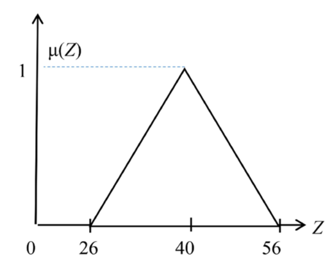

Fig. 4. Membership function $Z=x_1-x_2$

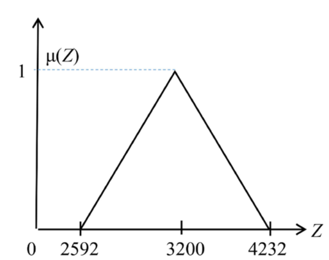

Fig. 5. Membership function $Z=x_1x_2$

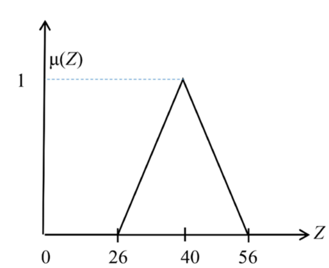

Fig. 6. Membership function $Z=x_1/x_2$

Next, let the membership functions' parameters of the fuzzy numbers x_1 and x_2 are set in a bi-fuzzy fashion:

$$m_1=\langle m_{m_1},\alpha_{m_1},\beta_{m_1}\rangle=\langle 80;\,0;\,0\rangle,$$

$$\alpha_1=\langle m_{\alpha_1},\alpha_{\alpha_1},\beta_{\alpha_1}\rangle=\langle 8;\,2;\,8\rangle,$$

$$\beta_1=\langle m_{\beta_1},\alpha_{\beta_1},\beta_{\beta_1}\rangle=\langle 12;\,4;\,6\rangle,$$

$$m_2=\langle m_{m_2},\alpha_{m_2},\beta_{m_2}\rangle=\langle 40;\,0;\,0\rangle,$$

$$\alpha_2=\langle m_{\alpha_2},\alpha_{\alpha_2},\beta_{\alpha_2}\rangle=\langle 4;\,1;\,3\rangle,$$

$$\beta_2=\langle m_{\beta_2},\alpha_{\beta_2},\beta_{\beta_2}\rangle=\langle 6;\,2;\,4\rangle.$$

We shall perform arithmetic operations involving the bi-fuzzy numbers x_1 and x_2 using the source data (27).

Addition.

We shall compute the parameters of the membership function of the bi-fuzzy result from the addition $Z=x_1+x_2$ in line with formulae (7).

$$m=m_1+m_2=\langle m_{m_1}+m_{m_2},\alpha_{m_1}+\alpha_{m_2},\beta_{m_1}+\beta_{m_2}\rangle=$$
$$=\langle 80+40;\,0;\,0\rangle=\langle 120;\,0;\,0\rangle;$$

$$\alpha=\alpha_1+\alpha_2=\langle m_{\alpha_1}+m_{\alpha_2},\ \alpha_{\alpha_1}+\alpha_{\alpha_2},\ \beta_{\alpha_1}+\beta_{\alpha_2}\rangle=$$
$$=\langle 8+4;\,2+1;\,6+3\rangle=\langle 12;\,3;\,9\rangle;$$

$$\beta=\beta_1+\beta_2=\langle m_{\beta_1}+m_{\beta_2},\ \alpha_{\beta_1}+\alpha_{\beta_2},\beta_{\beta_1}+\beta_{\beta_2}\rangle=$$
$$=\langle 12+6;\,4+2;\,6+4\rangle=\langle 18;\,6;\,10\rangle.$$

Subtraction.

We shall compute the parameters of the membership function of the bi-fuzzy result from the subtraction $Z=x_1-x_2$ in line with formula (8).

$$m=m_1-m_2=\langle m_{m_1}-m_{m_2},\alpha_{m_1}+\beta_{m_2},\beta_{m_1}+\alpha_{m_2}\rangle=$$
$$=\langle 80-40;\,0;\,0\rangle=\langle 40;\,0;\,0\rangle;$$

$$\alpha=\alpha_1+\beta_2=\langle m_{\alpha_1}+m_{\beta_2},\alpha_{\alpha_1}+\alpha_{\beta_2},\ \beta_{\alpha_1}+\beta_{\beta_2}\rangle=$$
$$=\langle 8+6;\,2+2;\,6+4\rangle=\langle 14;4;\,10\rangle;$$

$$\beta=\beta_1+\alpha_2=\langle m_{\beta_1}+m_{\alpha_2},\alpha_{\beta_1}+\alpha_{\alpha_2},\beta_{\beta_1}+\beta_{\alpha_2}\rangle=$$
$$=\langle 12+4;\,4+1;\,6+3\rangle=\langle 16;5;\,9\rangle.$$

Multiplication.

We shall compute the parameters of the membership functions of the bi-fuzzy result from the multiplication $Z=x_1x_2$ in line with formulae (9), (16), (17).

$$m=m_1m_2=\left\langle \begin{matrix} m_{m_1}m_{m_2};\ \ m_{m_1}\alpha_{m_2}+m_{m_2}\alpha_{m_1}- \\ -\alpha_{m_1}\alpha_{m_2};\, m_{m_1}\beta_{m_2}+m_{m_2}\beta_{m_1}+\beta_{m_1}\beta_{m_2} \end{matrix} \right\rangle=$$
$$=\langle m_m,\ \alpha_m,\ \beta_m\rangle=\langle 3{,}200;\ 0;\ 0\rangle;$$

$$\alpha=\left\langle\begin{array}{l} m_{m_1}m_{\alpha_2}+m_{m_2}m_{\alpha_1}-m_{\alpha_1}m_{\alpha_2};\ m_{m_1}\alpha_{\alpha_2}+ \\ +m_{\alpha_2}\alpha_{m_1}+m_{m_2}\alpha_{\alpha_1}+m_{\alpha_1}\alpha_{m_2}-\alpha_{m_1}\alpha_{\alpha_2}- \\ -\alpha_{m_2}\alpha_{\alpha_1}-m_{\alpha_1}\alpha_{\alpha_2}-m_{\alpha_2}\alpha_{\alpha_1}+\alpha_{\alpha_1}\alpha_{\alpha_2}; \\ m_{m_1}\beta_{\alpha_2}+m_{\alpha_2}\beta_{m_1}+m_{m_2}\beta_{\alpha_1}+m_{\alpha_1}\beta_{m_2}+ \\ +\beta_{m_1}\beta_{\alpha_2}+\beta_{m_2}\beta_{\alpha_1}-m_{\alpha_1}\beta_{\alpha_2}- \\ -m_{\alpha_2}\beta_{\alpha_1}-\beta_{\alpha_1}\beta_{\alpha_2} \end{array}\right\rangle=$$

$$=\langle m_\alpha,\ \alpha_\alpha,\ \beta_\alpha\rangle=\langle 608;\ 146;\ 480\rangle;$$

$$\beta=\left\langle\begin{array}{l} m_{m_1}m_{\beta_2}+m_{m_2}m_{\beta_1}+m_{\beta_1}m_{\beta_2};\ m_{m_1}\alpha_{\beta_2}+ \\ +m_{\beta_2}\alpha_{m_1}+m_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{m_2}-\alpha_{m_1}\alpha_{\beta_2}- \\ -\alpha_{m_2}\alpha_{\beta_1}+m_{\beta_1}\alpha_{\beta_2}+m_{\beta_2}\alpha_{\beta_1}-\alpha_{\beta_1}\alpha_{\beta_2}; \\ m_{m_1}\beta_{\beta_2}+m_{\beta_2}\beta_{m_1}+\beta_{m_1}\beta_{\beta_2}+m_{m_2}\beta_{\beta_1}+ \\ +m_{\beta_1}\beta_{m_2}+\beta_{m_1}\beta_{\beta_1}+m_{\beta_1}\beta_{\beta_2}+ \\ +m_{\beta_2}\beta_{\beta_1}+\beta_{\beta_1}\beta_{\beta_2} \end{array}\right\rangle=$$

$$=\langle m_\beta,\ \alpha_\beta,\ \beta_\beta\rangle=\langle 1{,}032;\ 520;\ 716\rangle.$$

Division.

We shall compute the parameters of the membership functions of the bi-fuzzy result from the division operation in line with formulae (18), (25), (26). By omitting the bulky computations, we shall give the ultimate results.

$$\frac{x_1}{x_2}=\langle m,\alpha,\beta\rangle.$$

In this case,

$$m=\langle m_m,\alpha_m,\beta_m\rangle=\langle 2;0;0\rangle,$$

$$\alpha=\langle m_\alpha,\alpha_\alpha,\beta_\alpha\rangle=\langle 0.348;\ 0.128;\ 0.319\rangle,$$

$$\beta=\langle m_\beta,\alpha_\beta,\beta_\beta\rangle=\langle 0.625;\ 0.166;\ 0.149\rangle.$$

Solving is over. Thus, the results of the execution of arithmetic operations involving the second-order fuzzy numbers are again the fuzzy numbers of the same order.

5. Discussion of results of building a system of the rules for executing operations involving the bi-fuzzy numbers of the (*L*-*R*)-type

The natural generalization of the canonical definition of fuzzy numbers, which led to the emergence of the second-order fuzzy numbers, gave rise to the issue associated with the absence of algebra for the bi-fuzzy numbers. In this case, when studying the real objects of the physical world, only a superficial, qualitative analysis of them is at hand. Moreover, even if the object under study can be quantified, the absence of algebra for the fuzzy numbers whose values set the state of the object excludes the possibility to manage that state. This paper has examined the simplest part of this problem – the algebra of arithmetic operations involving bi-fuzzy numbers of the (*L*-*R*)-type. The choice of this very field of the algebra of bi-fuzzy numbers is associated with wide and effective use of them in solving many practical problems. This is a consequence of the simplicity of the (*L*-*R*)-type number description as a set of the values of their membership functions' parameters. We have considered a case where the membership functions' parameters of the fuzzy numbers of the (*L*-*R*) type are themselves set by fuzzy numbers of the (*L*-*R*)-type. This work has proposed the rules for executing the arithmetic operations involving the resulting second-order fuzzy numbers. The formulae have been derived to compute the parameters of the membership function of the results of the execution of addition (7), subtraction (8), multiplication (11), (16), (17), and division (18), (25), (26) operations involving the bi-fuzzy numbers of the *(L*-*R*)-type.

Note that the resulting system of rules for performing arithmetic operations involving numbers of the (*L*-*R*)-type is easy to summarize. Introduce the following definition. A fuzzy number of the *n*-th order denotes a fuzzy number whose membership function's parameters are the fuzzy numbers of the (*n*–1)-th order. According to this definition, a standard fuzzy number has a membership function whose parameters are standard numbers. Next, the fuzzy number of the second order has a membership function whose parameters are the fuzzy numbers of the first order; a fuzzy number of the third order has a membership function whose parameters are the second-order fuzzy numbers, etc. It is clear that the recurrent application of the proposed rules makes it possible to obtain an analytical description of the results of the execution of arithmetic operations involving fuzzy numbers of any order.

The possible areas of further research are related to the generalization of the proposed approach in order to build rules for the execution of other algebraic operations: raising a bi-fuzzy number to a bi-fuzzy power, taking the logarithm of a bi-fuzzy number on a bi-fuzzy base, etc. In addition, it is interesting to construct methods for solving specific problems (solving equations and systems of equations, optimization, etc.) for the case where the original data are bi-fuzzy, etc. The task of constructing the bi-fuzzy models of inaccurate numbers is also of great practical interest.

6. Conclusions

1. A system of rules has been built for performing arithmetic operations (addition, subtraction, multiplication, division) for the fuzzy numbers of the (*L*-*R*)-type of the second order. This system is axiomatic and makes it possible to unambiguously obtain a complete and consistent set of rules for operating on fuzzy numbers whose order is higher than the first order.

2. The formulae have been proposed to compute the parameters of the membership functions of the results of executing arithmetic operations involving the second-order fuzzy numbers of the (*L*-*R*)-type. Using these rules makes it possible to solve in practice the numerous problems of analysis and synthesis of the systems for which the fuzziness of parameter setting exceeds the first order.

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