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The study was performed to analyze the flux of energy of internal gravitational-capillary waves in a two-layer hydrodynamic liquid system with finite layer thicknesses. The problem was considered for an ideal incompressible fluid in the field of gravity as well as taking into account the forces of surface tension. The problem was formulated in a dimensionless form for small values of the coefficient of nonlinearity. The dispersion of the gravitational-capillary progressive waves was studied in detail depending on the coefficient of surface tension and the ratio of layer densities. It was proved that with the increase in the wavenumber, the group velocity begins to pass ahead of the phase velocity and their equality occurs at the minimum of the phase velocity. Dependence of the total average energy flux on the wavenumber (wavelength) and thickness of the liquid layers was calculated and graphically analyzed for different values of physical quantities, in particular, density and the coefficient of surface tension. It follows from the analysis that the energy flux of gravitational internal waves increases to a certain maximum value with an increase in the thickness of the lower layer and then approaches a certain limit value. For capillary waves, the energy flux of internal waves is almost independent of the thickness of the lower layer. It was also shown that the average energy flux for gravitational waves at a stable amplitude is almost independent of the wavelength. On the contrary, for capillary waves, the energy flux increases sharply with an increase in the wavenumber.

The results of the analysis of the energy flux of internal progressive waves make it possible to qualitatively assess physical characteristics in the development of environmental technologies that use internal undulatory motions in various aquatic environments as a source of energy

Keywords: energy flux, internal progressive waves, two-layer hydrodynamic system, anomalous dispersion

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DEPENDENCE OF THE INTERNAL WAVE ENERGY FLUX ON THE PARAMETERS OF A TWO-LAYER HYDRODYNAMIC SYSTEM

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1. Introduction

Internal waves are undulatory motions in a stratified fluid. Stratification means that, for example, ocean water is divided into layers of different densities which increase with depth. Stratification is caused by various natural phenomena: storms, melting of ice, heating of upper water layers, changes in ocean salinity, etc. It is at the boundary of layers

with different densities that a whole class of different wave phenomena occurs.

The relevance of studying such phenomena is determined by the necessity of developing environmental technologies that use internal waves as a source of energy in various aquatic environments. Also, such a study will be useful in creating devices for damping internal waves where they are harmful. In this case, the mechanical en-

ergy of undulatory motion which is transferred through a cross-section per unit time, that is, the energy flux, is an important characteristic. Growth or reduction of energy flux of internal waves also creates a significant impact on natural processes in water bodies.

2. Literature review and problem statement

Laboratory experiments with internal separated waves in ice-covered water were conducted in [1]. Internal separated waves propagating in a stably stratified two-layer liquid in which upper boundary condition varies from open water to ice were investigated. A force induced by such waves on the surface is capable to carry ice in horizontal direction and measures of turbulent dissipation of kinetic energy under the ice can be compared with data on the contact surface. Moreover, in cases where ice falls into a pycnowedge, interaction with the ice edge can result in the ice rupture or even to its destruction as a result of this process. The results suggest that interaction between separated waves and sea ice may become an important mechanism of the dissipation of energy of single waves in the Arctic Ocean. However, a case of the presence of permanent hard crumbs corresponding to the case when the ice layer has a large area compared to the water thickness was not considered in [1]. This was determined by the need to conduct field experiments.

The aim of [2] implied establishing the existence of solutions for single waves in two classes of two-layer systems that simulate the propagation of internal waves. More precisely, the Businesque's dispersion system and a long-wave system were considered. No attention was paid in this work to the study of capillary waves in two-layer hydrodynamic systems.

An overview of theoretical, numerical, and experimental results of studies of structure and dynamics of weakly nonlinear internal waves in the ocean was presented in [3]. Data accumulated in the last 40 years since an approximate equation called Ostrovsky's equation was obtained in 1978 were used. Connection of this equation with other known wave equations, integration of Ostrovsky's equation, and a condition for the existence of stationary separated waves and separated waves in a shell were discussed. Adiabatic dynamics of Corteweg de Vries solitons in the presence of fluid rotation was described. Mutual influence of ocean inhomogeneity and rotation on the dynamics of separated waves was considered. The universality of the Ostrovsky's equation with respect to waves in other media was pointed out. This study did not consider solitons arising from nonlinear Schrödinger's equation.

Features of the separated wave dynamics within Ostrovsky's equation with variable coefficients relative to the surface and internal waves in the ocean with variable bottom topography were presented in [4]. For separated waves moving to a beach, attenuation effect can be suppressed by the effect of rotation. Two main examples of the ocean bottom profile were analyzed in detail and confirmed by direct numerical simulations. One of them is a bottom with a constant slope and the other is a specific bottom profile that provides a separate wave with a constant amplitude. Estimates with real ocean parameters have shown that the predicted effects of stable dynamics of solitons in a coastal zone may occur, in particular, for inland waves.

An experimental study of internal single waves of altitude and depression type generated by gravitational

collapse was performed in [5]. The gravitational collapse in stratified fluids can form internal separated waves. Observations of waves of this type and dispersion have made it possible to understand in detail the conditions that generate their stable propagation and properties. This study compared the results of such experiments with the wave profile, phase velocity, characteristic frequency, and velocity of the induced flow calculated using nonlinear theories. In addition, the ability of nonlinear theories to predict accurately characteristics of stable waves generated by gravitational collapse was analyzed. The results show that such waves are generated by the evolution of a vortex that develops from vertical shear motion in the region of mixed fluid density. These large-amplitude waves are generally consistent with extended KdV (eKdV) or Miyata-Choi-Camassa (MCC) theories. Energy characteristics of undulatory motion were not studied because of the difficulty of obtaining numerical and analytical results.

The study [6] addresses the interaction of an internal separated wave on the contact surface of the two-layer liquid and a free surface wave in the upper part of the upper layer. This approach is based on the theory of potential flux because internal waves are associated with large Reynolds numbers. Potential flows in the upper and lower layers were modeled using the method of multi-domain boundary elements (MDBEM). The calculation model was confirmed by experimental results for the profile and velocity of the internal wave. MDBEM is suitable for modeling waves in a stratified liquid system with low and high density. The wave velocity was compared with an approximate analytical theory for various ratios of the liquid density of the two layers. In addition, amplitude, velocity, and profile of the surface wave induced by the internal wave were investigated. There was a shift of the free surface as opposed to the shift of the contact surface, and the amplitude of the surface wave increased with a jump in the amplitude and density. The surface wave induced by the internal wave can be a soliton-like wave propagating at a constant velocity. This study did not consider the behavior of the lower layer in the interaction of internal and surface waves. Also, no attention was paid to the study of energy transfer by solitons in the studied system.

An analytical model related to internal waves in the presence of a tension in a two-layer system bounded by a rigid floating plate and a rigid bottom was presented in [7] based on Businesque's equations. General equations and boundary conditions of motion of internal waves at interphase tension were described. Detailing of Bussinesque's equations is related to a linear tension with two depth parameters. These parameters describe heights in the upper and lower layers of liquids. Detailing of the equations was obtained on the basis of the development of velocity potentials into a power series with a dispersion effect. For simplicity, one-dimensional linearized Bussinesque's equations and a dispersion relation with linear tension were obtained. Amplitudes of internal waves of the first and second-order with linear tension were obtained based on the perturbation method. Also, subharmonic interactions of internal waves of the second order were obtained. Accuracy of the analytical result of displacement of the internal wave was checked by comparing it with the calculation of hydrodynamic models without the introduction of interfacial tension. It was observed that profiles of the waves and their peak amplitudes were well consistent. In addition, a linearized dispersion ratio was compared with that recently obtained. The influence of interfacial tension

on properties of internal waves was studied by analyzing the dispersion ratios, velocities of liquid particles, harmonic waves of the first and second order. It was established that smaller values of surface tension and wave number lead to a stronger influence of the second order on the profile of the internal wave. No attention was paid to the study of capillary waves and the effects connected with them.

More than 500 sets of internal waves in the Strait of Georgia (British Columbia, Canada) were analyzed in [8], based on a large number of satellite images. The spatial and temporal distribution of internal waves in the central region of the strait was discussed using statistical analysis. Possible sources of the observed internal waves were divided into three categories depending on their different directions of propagation and geographical location:

1) interaction between narrow channels south of the strait and tidal currents causing the formation of waves that propagate mainly to the east and north;

2) interaction between tidal currents and the terrain near Cape Roberts causing mainly waves propagating to the west;

3) excitation by the river train, mainly near the Fraser River mouth which leads to the formation of mainly western waves along the direction of the river train.

The relation between the appearance of internal waves in remote sensing images and the level of wind or tide was also discussed. It was established that most of the observed internal waves occur at low tide. However, because of the influence of the river, internal waves propagating to the east, are rarely met near the river mouth at the lowest tide. In addition, internal waves are more easily detected by remote sensing images in the summer season due to lower wind speeds than in the winter season. Therefore, the seasonal distribution of internal waves in remote sensing images may not fully reflect the real situation in the area under study. Finally, it was found by combining the measured data on the site and original data of the model that the Benjamin-Ono equation satisfactorily models characteristic parameters of the studied internal waves. The study did not pay attention to the analysis of the interaction of internal waves.

The dynamics of completely nonlinear internal waves in continuously stratified fluids were studied in [9]. The study was conducted using a new theory of simple waves (or the Riemann waves) with topographic effects of the 2nd order. Unlike previous completely nonlinear theories of internal waves, the proposed theory can be applied to large changes in depth over long distances, provided that the local slopes of the bottom are relatively small. The division of “adiabatic” and “diabatic” topographic effects is an essential step in theoretical development. Because the Riemann variable (or invariant) remains constant along the characteristic, this extends the applicability of the method based on characteristic curves from a flat bottom to gentle slopes. Second-order diabatic topographic effects were then added to the adiabatic wave solution using the perturbation method. The study did not pay attention to taking into account the energy effects of the interaction of such internal waves.

The phenomenon of the appearance of resonance between long and short waves was investigated in [10] for internal waves in a liquid stratified by density. This can result in killer waves that are modeled as special pulsating modes. Properties of these killer waves, such as polarity, amplitude, and stability were investigated, and it was shown that they are critically dependent on density stratification

and mode selection. Three examples were considered: a two-layer liquid, a stratified liquid with constant frequency and a case of variable frequency. It was shown that maximum displacements should not be limited by the fixed ratio of the background plane wave. In addition, there are no restrictions on signs of nonlinearity and variance, or on any requirements to the fluid depth. All these features contrast sharply with the characteristics of the wave packet developing on the water of finite depth and controlled by the nonlinear Schrödinger's equation. The amplitude of these internal killer waves usually increases when the change in density in the layered or stratified fluid is smaller. Numerical modeling was performed using baseband modes as initial conditions for estimating the stability of the killer waves.

Propagation of internal separated waves in systems of three different types in a two-layer fluid flux was studied in [11]. A mathematical model based on the generalized Corteweg de Vries's equation with a variable coefficient was used. The basic equation was then solved numerically. Numerical calculations have shown that the internal separated waves are adiabatically deformed on a slowly increasing slope.

Nonlinear interactions between surface and internal gravitational waves in a two-layer system were studied in [12]. The study was conducted using explicit nonlinear evolution equations of the second order in the Hamiltonian form. The study was focused on modulation of surface waves using a group resonance mechanism which corresponds to an almost resonant interaction between a long internal wave and short surface waves. Numerical solutions were consistent with laboratory measurements of local amplitude and wave inclination. They confirmed that surface modulation becomes significant when group velocity of surface waves coincides with the phase velocity of the internal wave as predicted by the theory of linear modulation. However, it was shown that following a sufficient increase in the envelope amplitude, the surface and internal waves begin to exchange with energy through almost resonant interactions. This is essential for an accurate description of long-term modulation of surface waves by an internal wave. Numerical solutions for the density ratio close to unity were presented for ocean applications. It was found that significant energy exchange occurs through primary and sequential resonant interactions. This study did not analyze cases where the density ratio is not close to unity.

The problem of wave energy transfer at short intervals was investigated in [13]. The problem of collision of the Rossby waves with a barrier having two small gaps was considered. In the case where waves collide with a single-gap barrier, very little wave energy passes through the barrier. In the case of two or more gaps, the linear theory suggests that the barrier may be surprisingly ineffective in blocking the transmission of the Rossby wave energy. However, the theory neglects viscosity and nonlinear effects in pools and gaps. To investigate these effects, the results of a series of laboratory experiments were presented. It was concluded that large-scale waves can pass through a barrier with two gaps. However, although the linear theory covers large-scale flow structures, viscosity and nonlinearity significantly affect the flux along boundaries and near the barrier gaps.

The influence of the presence of underwater mountains on the generation of internal waves that propagate far from the underwater mountain and are further destroyed was investigated in [14]. The relative importance of these processes was studied for idealized isolated underwater mountains

(with a characteristic width and height) in a homogeneous barotropic flux. The results show that vortex processes energetically dominate in the energy flux of the internal wave in this range of parameters. The internal wave field was specially studied and divided into stable and nonstationary waves. It was established that the energy flux of a leeward wave cannot be explained by existing analytical theories. This study did not pay attention to capillary waves and their energy characteristics.

The energy of a close field that arose by a turbulent trace of motion of a sphere in the liquid was investigated in [15]. Modeling was performed for the initial Reynolds number and the internal Froude number which were determined using body-based scales. It was found that the internal wave radiation is an important absorber of kinetic energy in the trace.

The influence of a gravitational wave on a floating round permeable elastic membrane was investigated in [16]. The study was conducted using the linear wave theory in both homogeneous and two-layer fluxes. Integral representation of a plane wave in a form of Bessel and Hankel functions was used to study the effect of a porous structure on the attenuation of wave energy in a far zone. The traction force acting on the membrane, the membrane deviation, amplitude of the waves, distribution of the flux around the structure and scatter of energy in the far zone were investigated. The study was conducted for three different boundary conditions: a free edge, a partially fixed edge, and a fixed edge. The study has shown that the amplitude of surface waves on the leeward side of the structure is significantly reduced in the presence of a floating porous elastic membrane. In addition, the membrane having a fixed edge dissipates more wave energy than in other cases. The study did not consider the case when the membrane area is much larger than the liquid depth.

The energy of undulatory motion in a two-layer hydrodynamic system “a layer with a hard bottom – a layer with a free surface” was investigated in [17]. Dependences of the energy of surface and internal undulatory motion on geometric and physical parameters of the studied system were analyzed. Contribution of taking into account the second approximation in the study of the total energy of the system was assessed. The validity of the obtained results was checked. However, this paper considered a model that will poorly describe energy processes in the propagation of internal waves in liquids covered with, for example, a layer of ice.

The article [18] presents a numerical analysis of the behavior of mechanical and internal energies of fluid in time during generation, propagation, and absorption of gravitational waves. Particular attention was paid to the analysis of energy transmitted to the liquid as a work performed by a movable hard cover on a free surface. A comparison of the evolution of energy components over time for a case of reflection of waves from a flat bottom ending in a vertical wall was presented. Cases of absorption of the waves generated by damages in an inclined bottom caused by waves were separately considered. The energy analysis was used later in the study of characteristics of the waves interacting with a breakwater. Temporal evolutions of individual energy components were analyzed separately for two areas: the wave propagation region and the breakwater region. Thus, a concrete analysis was devoted to determine the amount of energy scattered only due to the presence of a breakwater. In this case, the issue of energy dissipation in the presence of a solid cover on the liquid surface was not considered.

Propagation of internal wave energy in a three-layer hydrodynamic liquid system “a layer with a hard bottom – a layer – a layer with a cover” was studied in [19]. The problem statement was done in a dimensionless form. The first three linear problems were obtained using the method of multi-scale development. For the first approximation problem, the first-order dispersion equation was derived and two pairs of independent solutions were obtained. The dependence of total energy on the wavenumber and thickness at different values of physical parameters, in particular, the thickness of the upper and lower layers, density, and surface tension was graphically illustrated and analyzed. A limit case was considered in which the obtained results were compared with the calculation of the energy transmitted by the wave in a hydrodynamic system “a liquid layer – a layer with a cover” which was studied earlier. The transition of this system to a two-layer system was graphically depicted. The data obtained in the study of the problem of propagation of internal wave energy in a liquid three-layer hydrodynamic system can be used in the study of similar ocean areas. The study results can be used in the development of appropriate technologies and devices that use the energy of water waves or convert it into electricity.

The review of the studies on the propagation of internal waves in layered structures and analysis of their energy characteristics shows that this problem is relevant today. Studies of the energy characteristics of wave propagation are conducted for various models. However, this issue has not yet been finally resolved for a two-layer liquid. Therefore, it is advisable to study the energy flux carried by internal waves in two-layer hydrodynamic systems.

3. The aim and objectives of the study

The study objective was to calculate the energy flux of gravitational and capillary internal waves depending on the physical parameters of a two-layer hydrodynamic system of finite thickness. This will enable modeling and evaluation of physical parameters of layered systems in the development of environmental technologies that use internal undulatory motions in various aquatic environments as a source of energy.

To achieve this objective, the following tasks were set:

- perform analysis of the dispersion equation depending on the system parameters;
- obtain ratios for the phase and group velocities of gravitational-capillary internal waves and compare them;
- calculate the average energy flux of internal waves for typical parameters of a two-layer liquid.

4. Mathematical statement of the problem of undulatory motions in a two-layer fluid

A hydrodynamic system with a boundary between layers of different densities was considered. This boundary can be considered as a membrane of zero thickness which creates surface tension. The waves having properties materially determined by the forces of surface tension are called capillary waves. They can be excited either by direct perturbation of the interface or a nonlinear transformation on the gravitational wave crests. Capillary waves create small-scale distortions in the surface and significantly affect the processes

of reflection and scattering of electromagnetic and acoustic waves by the water surface, in particular, the optical properties of water.

Therefore, the problem of propagation of two-dimensional waves on the surface of a non-viscous incompressible fluid under the influence of gravity and surface tension forces was considered. It is assumed that the waves propagate along the x -axis and the contact surface $\eta(x, t)$ and vertical z -axis are tilted in a direction opposite to the gravity direction (Fig. 1).

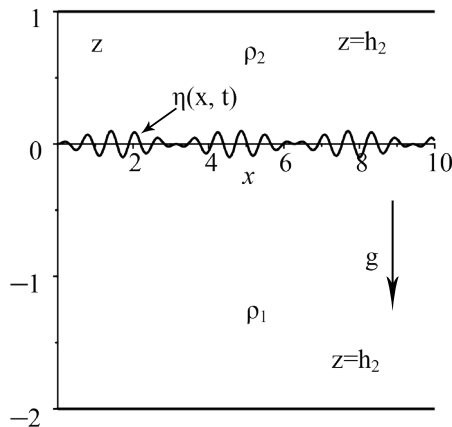


Fig. 1. A two-layer liquid

Mathematical model of the problem of propagation of wave packets along the surface of contact of two liquid layers $\Omega_1 = \{(x, z), |x| < \infty, -h_1 < z < 0\}$ (density ρ_1) and $\Omega_2 = \{(x, z), |x| < \infty, 0 < z < h_2\}$ (density ρ_2) in dimensionless quantities introduced by the layer thickness h_2 , characteristic wavelength L , maximum deviation a of the lower contact surface and acceleration of free fall g , takes the following form:

$$\frac{\partial^2 \varphi_j}{\partial x^2} + \frac{\partial^2 \varphi_j}{\partial z^2} = 0 \text{ in } \Omega_j, (j=1,2), \tag{1}$$

$$\frac{\partial \eta}{\partial t} - \frac{\partial \varphi_j}{\partial z} = -\alpha \frac{\partial \varphi_j}{\partial x} \frac{\partial \eta}{\partial x} \text{ at } z = \alpha \eta(x, t), \tag{2}$$

$$\frac{\partial \varphi_1}{\partial t} - \rho \frac{\partial \varphi_2}{\partial t} + (1-\rho)\eta + \frac{1}{2}\alpha \left[(\nabla \varphi_1)^2 - (\nabla \varphi_2)^2 \right] - T \left[1 + \alpha^2 \left(\frac{\partial \eta}{\partial x} \right)^2 \right]^{-3/2} \frac{\partial^2 \eta}{\partial x^2} = 0, \tag{3}$$

at $z = \alpha \eta(x, t)$,

$$\frac{\partial \varphi_1}{\partial z} = 0 \text{ at } z = -h_1, \tag{4}$$

$$\frac{\partial \varphi_2}{\partial z} = 0 \text{ at } z = h_2, \tag{5}$$

where $\alpha = a/h_2$ is the nonlinearity coefficient; $\varphi_i (i=1, 2)$ are velocity potentials in liquid media; η is the deviation of the contact surface; $\rho = \rho_2/\rho_1$ is the ratio of densities of the liquids; T is surface tension.

Using the method of multiscale developments to the third order,

$$\eta(x, t) = \sum_{n=1}^3 \alpha^{n-1} \eta_n(x_0, x_1, x_2, t_0, t_1, t_2) + O(\alpha^3), \tag{6}$$

$$\begin{aligned} \varphi_j(x, z, t) = \\ = \sum_{n=1}^3 \alpha^{n-1} \varphi_{jn}(x_0, x_1, x_2, z, t_0, t_1, t_2) + O(\alpha^3), \quad j=1, 2, \end{aligned} \tag{7}$$

where $x_k = \alpha^k x$ and $t_k = \alpha^k t$ ($k=0, 1, 2$), statements were obtained for the first three approximations of the problem, and studies were conducted in the first two approximations.

5. Analysis of the dispersion ratio due to the physical parameters of the system.

Here are the ratios required for the further study. There are solutions in the first (linear) approximation:

$$\eta_1 = A \exp(i\theta) + \bar{A} \exp(-i\theta), \tag{8}$$

$$\varphi_{11} = -i\omega k^{-1} \left[A \exp(i\theta) - \bar{A} \exp(-i\theta) \right] \frac{\text{ch}(k(h_1+z))}{\text{sh}(kh_1)}, \tag{9}$$

$$\varphi_{21} = i\omega k^{-1} \left[A \exp(i\theta) - \bar{A} \exp(-i\theta) \right] \frac{\text{ch}(k(h_2-z))}{\text{sh}(kh_2)}, \tag{10}$$

where A is the envelope of the wave packet; $\theta = kx_0 - \omega t_0$, $k = 2\pi/\lambda$ is the wavenumber of the center of the wave packet; ω is the frequency of the center of the wave packet. The following dispersion equation for capillary waves was obtained by substituting the solutions into the last equation of the system:

$$\omega^2 = \frac{(1-\rho + Tk^2)k}{\text{cth}(kh_1) + \rho \text{cth}(kh_2)}. \tag{11}$$

Fig. 2 shows the graph of dependence $\omega = \omega(k)$ for the following parameters of the hydrodynamic system: $h_2=1, h_1=2, \rho=0.5, T=0.1$. All values are given in a dimensionless form.

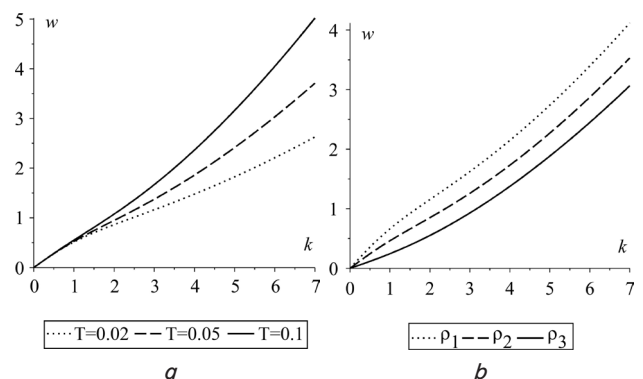


Fig. 2. Dispersion diagrams for a two-layer hydrodynamic system: a – for values $T \in \{0.02, 0.05, 0.10\}$ at $\rho=0.5$; b – for values $\rho \in \{0.3, 0.5, 0.9\}$ at $T=0.1$

As can be seen from the nature of the dispersion diagrams, first, the curve behaves proportionally to $k^{1/2}$, and then it bends and changes the law of dependence by $k^{3/2}$. This indicates that the gravitational effects are decisive at sufficiently large wavelengths and the main role is played by capillary forces at short lengths. Moreover, the greater the coefficient of surface tension, the closer the inflection

point to the origin (Fig. 2, *a*). Also, the ratio of the density of the layers creates a significant effect on the balance of gravitational and capillary effects: an increase in the density ratio increases dispersion which is a consequence of the influence of the surface tension forces (Fig. 2, *b*).

6. Phase and group velocities of progressive internal waves

The phase and group velocities of capillary-gravitational waves are respectively equal to the following:

$$c = \frac{\omega}{k} = \sqrt{\frac{1 - \rho + Tk^2}{k(c\text{th}(kh_1) + \rho c\text{th}(kh_2))}} \tag{12}$$

$$c_g = \frac{d\omega}{dk} = \frac{2Tk^2 + \frac{\omega^2}{k} [c\text{th}kh_1 - kh_1(1 - c\text{th}^2kh_1) + \rho(c\text{th}kh_2 - kh_2(1 - c\text{th}^2kh_2))]}{2\omega(c\text{th}kh_1 + \rho c\text{th}kh_2)} \tag{13}$$

It is with the group velocity that the energy of the undulatory motion of internal waves is transferred. Analysis of the graphs of the dependence of the phase and group velocities on the wavenumber (Fig. 3) gives grounds for several interesting conclusions. In the absence of the surface tension forces, the phase velocity is greater than the group velocity and their values decrease with an increase in *k* (Fig. 3, *a*). However, the phase velocity is greater than group one in the presence of surface tension for small wavenumbers, that is, for large wavelengths, which is characteristic of gravitational waves. However, as the wave number increases, the velocities decrease to a certain minimum. Further, growth begins and the group velocity becomes greater than the phase one because it grows more intensely (Fig. 3, *a, b*). Characteristically, the intersection of the velocity graphs always occurs at a minimum phase velocity. An increase in the coefficient of surface tension leads to the fact that the group velocity begins to outpace the phase one (anomalous dispersion) for smaller wavenumbers and the decrease in phase and group velocities becomes less pronounced (Fig. 3, *d*).

The process of outrunning the phase velocity by the group one can be demonstrated analytically. If the wave number is large enough, then the dispersion ratio can be written in a form valid for capillary waves:

$$\omega^2 = \frac{Tk^3}{c\text{th}(kh_1) + \rho c\text{th}(kh_2)} \tag{14}$$

Respectively, the phase and group velocities for capillary waves in a two-layer liquid are

$$c = \sqrt{\frac{Tk}{c\text{th}(kh_1) + \rho c\text{th}(kh_2)}} \tag{15}$$

$$c_g = \omega'_k = \frac{1}{2\omega} \times \left(\frac{3Tk^2(c\text{th}(kh_1) + \rho c\text{th}(kh_2)) + Tk^3 \left(\frac{h_1}{sh^2(kh_1)} + \frac{\rho h_2}{sh^2(kh_2)} \right)}{(c\text{th}(kh_1) + \rho c\text{th}(kh_2))^2} \right) \tag{16}$$

Some transformations can give a ratio that connects the phase and group velocities of the capillary waves:

$$c_g = \frac{3}{2}c + \frac{\sqrt{Tk^3} \left(\frac{h_1}{sh^2(kh_1)} + \frac{\rho h_2}{sh^2(kh_2)} \right)}{2(c\text{th}(kh_1) + \rho c\text{th}(kh_2))^{\frac{3}{2}}} \tag{17}$$

Hence, the capillary waves have anomalous dispersion. Thus, in contrast to the gravitational waves, the energy carried by the capillary waves moves faster than the phase velocity of these waves.

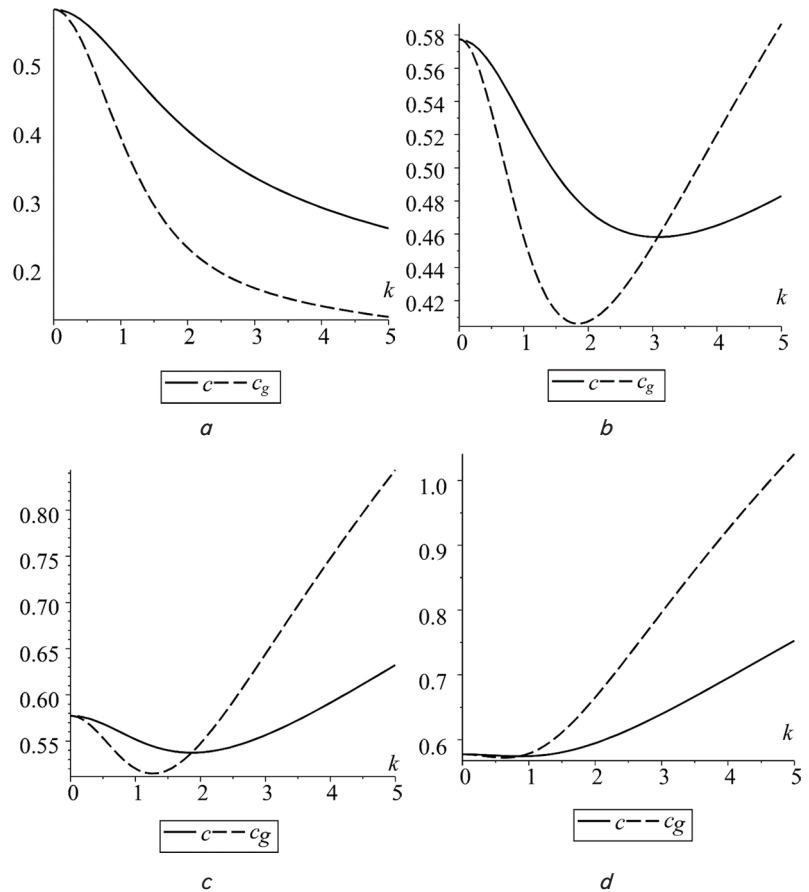


Fig. 3. Comparison of the dependence of values of phase and group velocities on the wavenumber for different values of the surface tension coefficient: *a* – for *T*=0; *b* – for *T*=0.05; *c* – for *T*=0.1; *d* – for *T*=0.15

7. Discussion of the results obtained in the study of energy flux of internal waves in a two-layer liquid

The energy flux will be meant the total mechanical energy of undulatory motion transferred through a fixed surface

S per unit time. The concept of energy flux will apply to the time intervals τ much larger than the wave period. Therefore, we are talking about the average flow of undulatory motion. If the velocity potential of a liquid system is known, then the energy flux of the undulatory motion can be calculated by the following formula [17]:

$$\Pi = -\frac{\rho}{\tau} \int_t^{t+\tau} dt \int_S \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial n} dS. \tag{18}$$

For a two-layer hydrodynamic system in a flat form, the flow of energy transferred through each layer can be rewritten as follows:

$$\Pi_2 = -\frac{1}{\tau} \int_t^{t+\tau} dt \int_0^{h_2} \frac{\partial \varphi_{21}}{\partial t} \frac{\partial \varphi_{21}}{\partial x} dz, \tag{19}$$

$$\Pi_1 = -\frac{\rho}{\tau} \int_t^{t+\tau} dt \int_{-h}^0 \frac{\partial \varphi_{11}}{\partial t} \frac{\partial \varphi_{11}}{\partial x} dz. \tag{20}$$

The limits of integration over z are taken here for the case of a small nonlinearity coefficient $\alpha = a/h_2$. The total flow of energy passing through the cross-section of the hydrodynamic system is equal to $\Pi = \Pi_1 + \Pi_2$.

Fig. 4 shows the dependence of the total energy Π_1, Π_2, Π on the thickness of the lower layer h_1 for the following system parameters $h_2=1, T=0.1, \rho=0.5, A=0.1$.

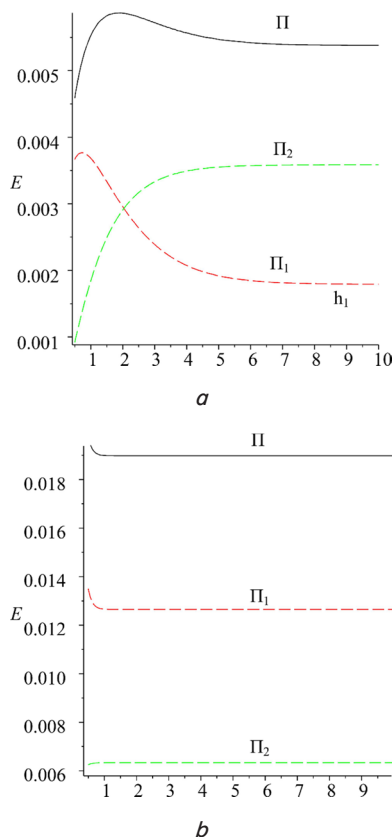


Fig. 4. Dependence of the energy flux on the thickness of the lower layer for gravitational and capillary waves: a – for $k=0.5$; b – for $k=5$

Fig. 4, a illustrates this dependence for gravitational waves ($k=0.5$). It follows from the graph that as the thick-

ness of the lower layer h_1 increases, the graph of total flow increases to some maximum value. Further, it falls to a certain limit. The flow that carries the energy of the lower layer behaves similarly. The flow of the upper layer first increases rapidly and then asymptotically approaches some limit value. For capillary waves in Fig. 4, b ($k=5$), the energy flux of internal waves almost does not depend on the thickness of the lower layer. This can be explained by the small range of capillary forces.

Fig. 5, a shows graphs of the dependence of the flows Π_1 and Π_2 in the lower and the upper layers and the total flow Π on the wavenumber $k=2\pi/\lambda$ for $\rho=0.7$. It is clear that the flux for gravitational waves is almost unchanged, the energy flux for capillary waves increases sharply with an increase in the wavenumber (a decrease in the wavelength). Fig. 5, b shows the dependence of the total energy fluxes on the wavenumber at different values of the ratio of densities of the layers $\rho \in \{0.5, 0.7, 0.9\}$. The total flux of internal waves increases with a decrease in the ratio of densities of the layers. This is because the internal waves require less energy to be excited in a hydrodynamic system at a slight difference in densities.

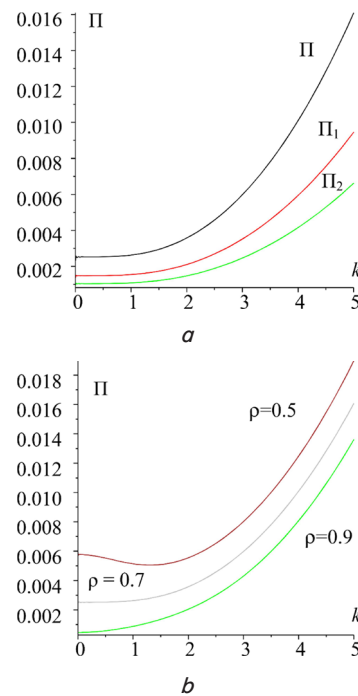


Fig. 5. Dependence of flux on the wave number k : a – fluxes Π_1, Π_2, Π at $\rho=0.7$; b – total flux Π at different values of the ratio of densities of the layers

Fig. 6 shows the dependence of the total average flux on the wave number k at different values of the coefficient of surface tension T . If the coefficient of surface tension $T=0$ (Fig. 6, a), the energy flux of internal waves slowly decreases and approaches some limit value.

At small values of the coefficient of surface tension for gravitational waves, the flux also decreases to a certain minimum and then increases quite rapidly when the capillary forces begin to act. For large values of the coefficient of surface tension, the total flux first increases slightly, and then the growth rate increases with an increase in the wavenumber (Fig. 6, b).

The obtained results fully correspond to the physical nature of the modeled object, namely the hydrodynamic

system “layer – layer”. Such a system is quite sensitive to the changes in its physical parameters, especially when the ratio of densities of the layers is close to unity. The method used in this problem is applicable only to weakly linear waves, that is, to the waves whose amplitude is small in comparison with the wavelength and layer thicknesses. It should be noted that this study considered an ideal fluid, so it did not take into account the effects associated with viscosity and energy dissipation. However, the analysis results are quite applicable to real liquids as a qualitative analysis as well as a basis for experiments.

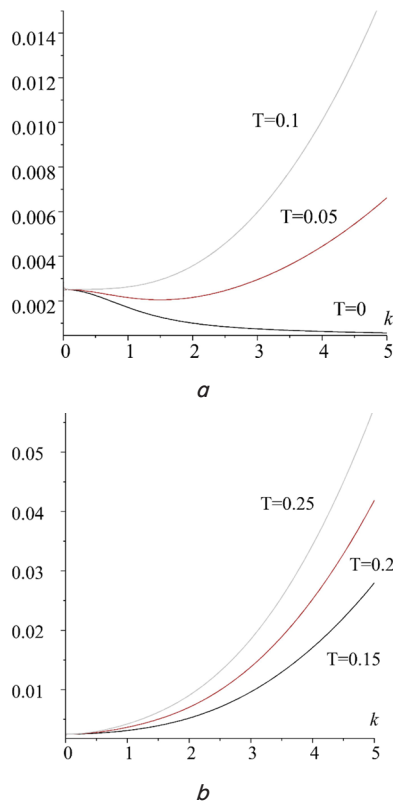


Fig. 6. Dependence of the energy flux on the wave number k at different values of surface tension: for values $T \in \{0, 0.05, 0.10\}$ at $\rho=0.9$ (a); for values $T \in \{0.15, 0.20, 0.25\}$ at $\rho=0.9$ (b)

This method of flux calculation can be further extended for complex undulatory motions. In particular, it can be applied to nonlinear waves of different types: wave packets of different structures, classical solitons, envelope solitons, etc. There may be difficulties in finding the appropriate energy integrals with boundaries that hardly depend on time. Here, it is better to apply modern systems of computer algebra that

make it possible to carry out a large volume of calculations with sufficient accuracy.

8. Conclusions

1. The study of dispersion of internal waves in a two-layer system shows that there are gravitational effects (the dispersion law is proportional to $k^{1/2}$) for the parameters $h_2=1$, $h_1=2$, $\rho=0.5$, $T=0.1$ at small wave numbers ($k < 1$). The main role is played by the capillary forces at small lengths ($k > 1$). Moreover, if the coefficient of surface tension is large enough (for example, $T=0.15$), the earlier $k > 1.5$ the law of dispersion behaves proportionally to $k^{3/2}$. Also, the ratio of densities of layers creates a significant influence on the balance of gravitational and capillary effects: an increase in the density ratio enhances dispersion which is a consequence of the influence of the surface tension forces.

2. Analysis of dependences of phase and group velocities on the wavenumber gives for parameters of $h_2=1$, $h_1=2$, $\rho=0.9$, $T=0.1$ at small wave numbers ($k < 2$), that is, the phase velocity is greater than the group velocity for large wavelengths which is characteristic for the gravitational waves. In the interval $0 < k < 2$, the phase velocity decreases to a certain minimum where it becomes equal to the group velocity. After that, growth begins and the group velocity begins to outpace the phase velocity. Thus, an anomalous dispersion of the inner wave packets appears.

3. The flux of energy of gravitational internal waves for the values of $h_2=1$, $T=0.1$, $\rho=0.5$, $k=0.5$ increases to some maximum value of $\Pi=0.06$ when the thickness of the lower layer increases to the value of ($0 < h_1 < 2$) and then gradually decreases to a certain limit. For the capillary waves at parameters $h_2=1$, $T=0.1$, $\rho=0.5$, $k=5$, the flux of energy of the internal waves is almost independent of the thickness of the lower layer and is approximately equal to $\Pi=0.0125$. For gravitational waves ($h_2=1$, $h_1=2$, $\rho=0.9$, $T=0.1$), the flux of energy is almost independent ($\Pi \approx 0.025$) of the wavenumber (in the interval of $0 < k < 1$). On the contrary, for the capillary waves, the flux of energy increases sharply with an increase in the wavenumber ($k > 1$). Also, the total flux of internal waves increases with a decrease in the ratio of layer densities which was calculated for the cases when $\rho \in \{0.5, 0.7, 0.9\}$. At small values of the coefficient of surface tension $T=0.05$, the flux of energy for gravitational waves decreases with an increase in k to a certain minimum $\Pi \approx 0.023$ and then increases quite rapidly when capillary forces begin to act. For large values of the coefficient of surface tension, the total flux ($T \in \{0.15, 0.20, 0.25\}$ at $\rho=0.9$) first increases slightly in the interval $0 < k < 1$, and then the growth rate increases.

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