INVARIANT DIFFERENTIAL GENERALIZATIONS IN PROBLEMS OF THE ELASTICITY THEORY AS APPLIED TO POLAR COORDINATES

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The method of argument functions has become famous for solving problems of continuum mechanics. The solution of problems of the elasticity theory in polar coordinates was the further development of this method. The same approaches are applied to solving problems of the theory of plasticity, the theory of elasticity, and the theory of dynamic processes. If regularities of the solution are determined correctly, then they should be continued in other fields including the problems of the theory of elasticity in polar coordinates.

The proposed approach features finding not the solution itself but the conditions for its existence. These conditions may include differential or integral relations which make it possible to close the solution in a general form. This becomes possible when additional functions are introduced into consideration or the argument functions of coordinates of the deformation zone. Basic dependences that satisfy the boundary or edge conditions as well as the functions that simplify the solution of the problem in general should be the carriers of the proposed argument functions. For various reasons, two basic dependences were used in the solution: trigonometric and exponential. Their arguments are two unknown argument functions.

In the process of transformations, a mathematical connection was established between them in a form of the Cauchy-Riemann relations which had a stable tendency to be repeated in problems of the continuum mechanics. From these positions, the flat problem was solved in the most detailed way, tested, and compared with the studies of other authors.

By reducing the solution to a particular result, a way to classical solutions was found which confirms its reliability. The result obtained is useful and important since it becomes possible to solve an extensive class of axisymmetric applied problems using the method of argument functions of a complex variable

Keywords: generalized approaches, argument functions, polar coordinates, Cauchy-Riemann relations, Laplace equations

1. Introduction

The emergence of new technical developments leads to an increase in the variety of applied problems of continuum mechanics. Physical and mathematical models, boundary, and edge conditions become more complicated. It becomes necessary to find an optimal result from the standpoint of their generalization, identify differential and integral relations that determine conditions of existence of closing solutions.

The method of argument functions was developed and improved in [1–6] making it possible to use practically the same approaches in solving problems of the continuum mechanics including the theories of plasticity, elasticity, and dynamic processes. Cauchy-Riemann differential relations are the generalizing factor for the argument functions. If there is any regularity in this, then it should manifest itself in the future as well, for example, when solving equations in different reference systems, including the polar coordinate system. Such approaches were defined [7, 8] and have found their further development in present-day publications.

The proven method for solving problems of continuum mechanics needs to be expanded for its use. This becomes relevant since several sections of the continuum mechanics are touched upon. When applying generalizing approaches to solving the problems shown in the method, it can be seen that the obtained regularities make it possible to formulate and solve new problems of the continuum mechanics including the solution of problems of the theory of elasticity in polar coordinates.

2. Literature review and problem statement

The study in the theory of plasticity was one of the first publications in which generalizing solutions using the method of argument functions were presented [1]. The method was further developed by considering applied issues of metal forming in the study on the theory of plasticity [2]. The development was somewhat reshaped as a new approach in the theory of plasticity in relation to the applied production issues and then proposed in [3]. The approaches of the method of argument functions were used in [4] as applied to dynamic problems of the theory of elasticity. Subsequently, the method was developed in the theory of elasticity [5, 6].

Along with the studies [7, 8], generalized approaches were formulated in a series of publications [9–11]. Classical

solutions of problems of the theory of elasticity were considered. The monograph [9] has presented the method of complex-variable function which can be applied after some refinements in conjunction with the method of argument functions. It all depends on whether the argument function can be represented as an analytical variable. Extensive use of tensor analysis serves as a generalization in [10]. The problem of this work was solved in a scalar form since the capabilities of the method in other coordinates were not completely clear. Solutions using the stress function method which differs significantly from the method of argument functions were presented in [11]. The results obtained in [11] do not allow one to estimate solutions using argument functions.

As is known, the main problem of the theory of elasticity is determining the stress-strain state of a solid. A possible linkage of solving the problems of the theory of elasticity to practical use was presented in [12]. In present-day publications, elements of generalization are reflected in a form of structural solutions of the problem [13] and integral relations for the assessment of kinematic perturbations [14]. The use of closing parameters [15] for the general form of the gradient solution can be to some extent analogs of the argument functions. However, the proposed generalizations do not enable the application of the results obtained in recent studies to the definition of closing differential relations.

To a limited extent, the studies in [16–18] can be an option of overcoming the above difficulties. Transformations and additional functions associated with basic dependences were considered. In the problem considered in [16], the Hankel's transform was applied to the basic differential Cauchy-Navier equilibrium equation to reduce the problem to an ordinary differential equation. In the case of the argument functions, transitions are used as well, however, the ones between partial differential equations. The Cauchy relations were used in [17], however, parameters of these relations are not the closing solution of the problem. Differential relations in the problem [18] were applied but they are inapplicable to the generalizations using argument functions. General approaches were considered in [19] where conditions of coupling in the sample-punch interaction were taken into account. The analysis has shown that they (conditions) do not adequately reflect the possibility of finding a concrete solution to the problem taking into account the application of the argument functions.

It was shown in [20] that there are transient conditions for introduction into consideration of additionally separated variables (analogy of the argument functions) when reformatting one type of differential equations into another. The very idea of transition is productive, however, the appearance of additional solutions, in this case, does not mean the determination of conditions for the existence of solutions.

A possibility of predicting one of the basic functions was considered in the problem considered in [21]. The trigonometric function was implemented in the structural formulation of a practical problem. The solution did not consider the argument functions as a closing component of the overall result.

Cyclic loading was shown in the case of simple shear which finds a corresponding response of internal stresses [22]. As in [21], a basic trigonometric function was introduced into consideration. Its capabilities were shown under various loads. Possibilities of its combination with argument functions were not shown. Using the example of [23], changes in the loading nature across the thickness of a compact specimen were determined. The maximum zone was closer to the surface which indicated unevenness of the material stress state. Taking into account the heterogeneity was ensured by introducing into consideration the coordinate function or the argument functions in this case. The coordinate dependences that were used did not serve as closing solutions to the problem. Loading at the base of a discontinuity was studied in [24]. The general approach to solving the problem was determined by the environment state. Inhomogeneity of the stress state was characterized by coordinate functions in combination with periodic dependences determined by a trigonometric expression. The choice of the basic trigonometric function was an important circumstance of the proposed solution, although there was no mentioning of a closing solution. A change in external loading causes a reaction from the medium according to an exponential law [25]. This is comparable with using a fundamental substitution in the method of argument functions. However, the functional purpose of the proposed dependence was different in the studies which did not allow one of the argument functions to be applied in the solution.

Alternating stresses and deformations occurring during loading are the main causes of degradation of the product strength and durability [26]. In a combination of basic functions, operating stresses were characterized in the course of a part loading. There were no additional functions. A method of R-functions was proposed in [27] which, in terms of functionality, has something in common with the method of argument functions. However, further analysis has shown that their application did not lead to the establishment of certain relationships. They are involved in other schemes of finding solutions, for example, applying the variational principles. The solution of a spatial problem of the theory of plasticity was considered in [28] in which its own rather complex approaches and generalizations were formulated. The work can be assessed as a new development in continuum mechanics. The solution of a spatial problem of the elasticity theory using series (asymptotic method) was shown in [29]. Like the previous development, it does not define any analogies with the argument functions method but makes it possible to draw a conclusion about the development of generalizing characteristics of the problem solution. A dynamic problem was presented in [30]. A solution using the boundary element method was considered. Analysis of the last three works shows a variety of approaches to solving similar problems of continuum mechanics. At the same time, the last three publications, without reducing their originality and novelty, are a clear example of absence in the literature of defining generalizations in solving problems of the continuum mechanics.

As a result, it was shown that the studies lack the tendencies of using generalizing approaches in solving problems of the continuum mechanics and, in particular, problems of the theory of elasticity. A significant field of problems was covered united by some approaches to formulation and solution of theoretical and practical problems: use of similar basic functions, some additional dependences that can lead to obtaining the final result, choice of approaches in the implementation of predictive functions, etc.

However, there are problems, still unresolved, related to how not the solutions themselves should be determined but the conditions of their existence. Such generalized approaches make it possible to predict results for new applied problems, expand possibilities of solutions to satisfy a

variety of boundary and edge conditions in the problems of ever-changing production.

The use of the method of argument functions of a complex variable which has shown its capabilities in solving diverse problems of continuum mechanics is a way to overcome such obstacles. The general regularities that have been identified make it possible to pose and solve new problems of the theory of elasticity, for example, the stress state study in polar coordinates using the argument functions.

3. The aim and objectives of the study

The study objective was to develop new approaches to solving problems of the continuum mechanics, in particular problems of the theory of elasticity taking into account invariant generalizations as applied to polar coordinates.

To achieve the objective, the following tasks were set:

– show the possibilities of using the method of argument functions in solving problems of the theory of elasticity in polar coordinates;

– determine generalizing relations in a differential form to enable obtaining the conditions for the existence of closing solutions of problems of the theory of elasticity;

– solve in an analytical form a plane problem of the theory of elasticity in polar coordinates using the method of argument functions;

– test the result obtained by the example of applied problems and compare it with the studies of other authors.

4. The methods used

The method of functions of a complex variable was used. Also, the method of argument functions was used which makes it possible to close the problem solution by introducing additional dependences and obtained generalizing differential relations into consideration. In addition, the method of comparison of the obtained practical result with theoretical and experimental data of other authors was used.

5. The study results

A plane problem of the theory of elasticity in polar coordinates was considered. To solve it, the following system of equations was used.

Equilibrium equations of the following form:

$$\frac{\partial \sigma_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial \tau_{\rho\phi}}{\partial \phi} + \frac{\sigma_\rho - \sigma_\phi}{\rho},$$

$$\frac{\partial \tau_{\phi\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_\phi}{\partial \phi} + 2\frac{\tau_{\phi\rho}}{\rho} = 0; \tag{1}$$

the condition of deformation continuity,

$$\nabla^2 \left(\sigma_\rho + \sigma_\phi\right) = 0; \tag{2}$$

the boundary conditions,

$$\tau_n = -\frac{\sigma_\phi - \sigma_\rho}{2}\sin 2\phi + \tau_{\rho\phi}\cos 2\phi, \tag{3}$$

where τ_n is the boundary contact shear stress; σ_ϕ, σ_ρ are the normal tangential and radial stresses, respectively; $\tau_{\rho\phi}$ is the shear stress in the deformation zone; ϕ is the angle of inclination of the contact area.

Expression (3) is convenient for simplifications which will allow us to linearize the boundary conditions in the future, that is, simplify them. It should be emphasized that the system (1) to (3) is applicable to both plane-stressed and plane-deformed states.

5. 1. The method of argument functions of a complex variable

Boundary conditions are an important factor in solving problems. Their knowledge allows one to determine unknown functions. It was shown in [31] how boundary conditions are formed based on certain approaches (the collocation method). The formation of boundary conditions in the method of argument functions makes it possible to define one of the basic trigonometric functions. Basic provisions of the mechanics of deformed solid are the basis for such a definition.

Expression (3) can be simplified by using the trigonometric law of distribution of contact stresses. To this end, it is necessary to know the difference between normal stresses and shear stress. The problem becomes even simpler if the relations connecting normal and shear stresses are known. The intensity of shear stresses for a plane deformed state takes the form

$$T_i = \frac{1}{2}\sqrt{\left(\sigma_\rho - \sigma_\phi\right)^2 + 4\tau_{\rho\phi}^2}$$

or

$$\sigma_\rho - \sigma_\phi = 2T_i\sqrt{1 - \left(\frac{\tau_{\rho\phi}}{T_i}\right)^2}. \tag{4}$$

The attractiveness of expression (4) lies in the fact that it is possible to express in some way the difference of normal stresses which is an unknown quantity.

To get rid of nonlinearity, the following dependence is taken [6, 21, 22, 24]:

$$\tau_{\rho\phi} = T_i\sin A\Phi, \tag{5}$$

where $A\Phi$ is the unknown coordinate function, or the first argument function; A is a constant correction value. Substitute (5) into the last equality to get:

$$\tau_n = -T_i\sin\left(A\Phi - 2\phi\right). \tag{6}$$

The boundary conditions were greatly simplified. Expressions (5), (6) are decisive in obtaining solutions to problems of the theory of elasticity in an analytical form. In addition to simplifications, it becomes possible to use fundamental substitution for the intensity of shear stresses [7, 25] since differential equations (1), (2) are assumed to be linear, that is:

$$T_i = C_\sigma \exp\theta, \tag{7}$$

where θ is an unknown function of coordinates or the second argument function; C_σ is a constant characterizing dimension of the intensity of shear stresses. Taking into account (6), (7), we can write down:

$$\tau_{\rho\phi} = C_\sigma \exp\theta \sin A\Phi, \tag{8}$$

further

$$\tau_n = -C_\sigma \exp\theta \sin(A\Phi - 2\phi).$$

Basic functions [24, 25] have appeared in formula (8). They satisfy boundary conditions and facilitate mathematical transformations when solving differential equations (1), (2).

The problem is reduced to the integration of differential equations of equilibrium (1) taking into account (8). In this regard, the problem is formulated as follows: under what conditions the argument of the function $A\Phi$ and θ can close the solution of the plane problem of the theory of elasticity, that is the system of equations (1) to (3) taking into account (6), (8) which will be identically satisfied upon substitution.

Using the method of complex variable [9], the unknown argument functions can be represented for shear stresses (8) in the form:

$$\tau_{\rho\phi} = C_\sigma \frac{\exp(\theta + iA\Phi) - \exp(\theta - iA\Phi)}{2i}. \tag{9}$$

The argument functions are assumed to be continuous, differentiable functions.

The solution of the system of equilibrium equations.

To determine normal stresses σ_ρ, σ_ϕ from equations (1), it is necessary to know coordinate derivatives from expression (9).

Normal stresses are introduced into consideration:

$$\sigma_\rho' = \sigma_\rho - \sigma_0 - f(\phi),$$

$$\sigma_\phi' = \sigma_\phi - \sigma_0 - f(\rho), \tag{10}$$

where σ_ϕ', σ_0, $f(\phi)$ are the deviatoric component of normal stress σ_ρ, hydrostatic pressure, and integration function, respectively; σ_ϕ', $f(\rho)$ are the deviatoric component of normal stress σ_ϕ and integration function, respectively. Expressions (10) will be substituted into differential equations (1) as was done in [32]. Separate the variables in a general form to get:

$$\sigma_\rho = -\int\left[\frac{1}{\rho}\frac{\partial\tau_{\rho\phi}}{\partial\phi} + \frac{\sigma_\rho - \sigma_\phi}{\rho}\right]d\rho + \sigma_0 + f(\phi),$$

$$\sigma_\phi = -\int\left[\rho\frac{\partial\tau_{\rho\phi}}{\partial\phi} + 2\tau_{\rho\phi}\right]d\phi + \sigma_0 + f(\rho). \tag{11}$$

Substitute the stress difference from expression (4) taking into account the boundary conditions. The following is obtained taking into account (5), (7):

$$\sigma_\rho - \sigma_\phi = 2T_i\sqrt{1 - \left(\frac{\tau_{\rho\phi}}{T_i}\right)^2} = C_\sigma \exp\theta \cos A\Phi.$$

Select the plus sign in the right-hand member. Substitute derivatives and differences of normal stresses in (11). Assuming that there can be a differential connection in a form of the Cauchy-Riemann relations [6] between the constituents of the argument functions, it is obvious that for polar coordinates:

$$\rho\theta_\rho = \mp A\Phi_\phi, \theta_\phi = \pm\rho A\Phi_\rho. \tag{12}$$

Passing with the help of (12) to opposite variables with their signs, the following is obtained:

$$\sigma_\rho = -\left\{ \begin{array}{l} \pm\int\frac{C_\sigma}{2i}\left[\begin{array}{l}(A\Phi_\rho - i\theta_\rho)\exp(\theta + iA\Phi) - \\ -(A\Phi_\rho + i\theta_\rho)\exp(\theta - iA\Phi)\end{array}\right]d\rho + \\ +2\int\frac{C_\sigma}{2\rho}\left[\begin{array}{l}\exp(\theta + iA\Phi) + \\ +\exp(\theta - iA\Phi)\end{array}\right]d\rho \end{array} \right\} +$$
$$+\sigma_0 + f(\phi), \tag{13}$$

$$\sigma_\phi = -\left\{ \begin{array}{l} \mp\int\frac{C_\sigma}{2i}\left[\begin{array}{l}(A\Phi_\phi - i\theta_\phi)\exp(\theta + iA\Phi) - \\ -(A\Phi_\phi + i\theta_\phi)\exp(\theta - iA\Phi)\end{array}\right]d\phi + \\ +2\int\frac{C_\sigma}{2i}\left[\begin{array}{l}\exp(\theta + iA\Phi) - \\ -\exp(\theta - iA\Phi)\end{array}\right]d\phi \end{array} \right\} +$$
$$+\sigma_0 + f(\rho), \tag{14}$$

where θ_ϕ, $A\Phi_\phi$, θ_ρ, $A\Phi_\rho$ are the partial derivatives of the argument functions with respect to the coordinates ϕ and ρ.

Introduce into consideration an imaginary unit i for (13), (14). The following is obtained after transformations when passing to real functions:

$$\sigma_\rho = \pm C_\sigma \exp\theta \cos A\Phi - I_1 + \sigma_0 + f(\phi), \tag{15}$$

$$\sigma_\phi = \mp C_\sigma \exp\theta \cos A\Phi - I_2 + \sigma_0 + f(\rho), \tag{16}$$

where

$$I_1 = 2C_\sigma\int\frac{1}{\rho}\exp\theta\cos A\Phi d\rho, \quad I_2 = 2C_\sigma\int\exp\theta\sin A\Phi d\phi.$$

The analysis shows that there is a mathematical relationship between the values of I_1 and I_2 when constraints on the argument functions of Cauchy-Riemann with upper signs are satisfied:

$$\frac{\partial I_1}{\partial\rho} = \frac{\partial I_2}{\partial\rho}, \quad \frac{\partial I_1}{\partial\phi} = \frac{\partial I_2}{\partial\phi},$$

under a condition that

$$\rho\theta_\rho = -A\Phi_\phi, \theta_\phi = \rho A\Phi_\rho.$$

Eventually,

$$\frac{\partial I_1}{\partial\rho} = \frac{\partial I_2}{\partial\rho} = 2C_\sigma\frac{1}{\rho}\exp\theta\cos A\Phi,$$

$$\frac{\partial I_1}{\partial\phi} = \frac{\partial I_2}{\partial\phi} = 2C_\sigma\exp\theta\sin A\Phi.$$

In this case, variables I_1 and I_2 can differ only in the integration constant which can be taken equal to zero, or:

$$I_1 = I_2 = I. \tag{17}$$

One should make sure whether the equality between the values of I_1, I_2, and the derivatives will hold if signs in the

Cauchy-Riemann relations change. Let us consider this issue in more detail:

$$\rho\theta_\rho = A\Phi_\phi, \theta_\phi = -\rho A\Phi_\rho.$$

Writing down similar partial derivatives and substituting the modified Cauchy-Riemann relations, we get the following

$$\frac{\partial I_1}{\partial \rho} = -\frac{\partial I_2}{\partial \rho}, \quad \frac{\partial I_1}{\partial \phi} = -\frac{\partial I_2}{\partial \phi},$$

or after integration

$$I_1 = -I_2 = I. \tag{18}$$

It can be shown that if relations (12) are fulfilled, the argument functions satisfy the Laplace equations. The following is obtained after transformations:

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0. \tag{19}$$

Argument functions are harmonic functions.

Taking into account (17) to (18), the following dependences are solutions of the system of equations (1) to (3):

$$\sigma_\rho = \pm C_\sigma \exp\theta\cos A\Phi \mp I + \sigma_0 + f(\phi),$$

$$\sigma_\phi = \mp C_\sigma \exp\theta\cos A\Phi \mp I + \sigma_0 + f(\rho), \tag{20}$$

$$\tau_{\rho\phi} = C_\sigma \exp\theta\sin A\Phi,$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi, \theta_\phi = \pm\rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0, \tag{21}$$

$$I_1 = 2C_\sigma \int\frac{1}{\rho}\exp\theta\cos A\Phi d\rho,$$

$$I_2 = 2C_\sigma \int\exp\theta\sin A\Phi d\phi,$$

$$\frac{\partial I_1}{\partial \phi} = \pm\frac{\partial I_2}{\partial \phi}, \quad \frac{\partial I_1}{\partial \rho} = \pm\frac{\partial I_2}{\partial \rho}.$$

Solution with a shift of the trigonometric function.

Solution (20) can be strengthened if we consider a more complex problem of the form:

$$\tau_{\rho\phi} = \left(C_{\sigma1}'\sin A\Phi + C_{\sigma1}''\cos A\Phi\right)\exp\theta. \tag{22}$$

In this case, it is necessary to check solutions (22) for compatibility with the boundary conditions (3), (4) assuming that:

$$C_{\sigma1}' = C_\sigma\cos A\Phi_0, \quad C_{\sigma1}'' = \pm C_\sigma\sin A\Phi_0,$$

taking into account (5), (6) and the above expressions, substitute in (22) to obtain:

$$\tau_{\rho\phi} = T_i\left(\sin A\Phi\cos A\Phi_0 \pm \cos A\Phi\sin A\Phi_0\right) =$$

$$= T_i\sin\left(A\Phi \pm A\Phi_0\right). \tag{23}$$

Taking into account (23), boundary conditions (4) must be satisfied. Then expression (23) can be used when integrating the equilibrium equations (1).

Substitute (23) into system (1) to obtain:

$$\sigma_\rho = \pm C_\sigma \exp\theta\cos\left(A\Phi \pm A\Phi_0\right)\mp I + \sigma_0 + f(\phi),$$

$$\sigma_\phi = \mp C_\sigma \exp\theta\cos\left(A\Phi \pm A\Phi_0\right)\mp I + \sigma_0 + f(\rho), \tag{24}$$

$$\tau_{\rho\phi} = C_\sigma \exp\theta\sin\left(A\Phi \pm A\Phi_0\right).$$

$$I_1 = 2C_\sigma \int\frac{1}{\rho}\exp\theta\cos\left(A\Phi \pm A\Phi_0\right)d\rho,$$

$$I_2 = 2C_\sigma \int\exp\theta\sin\left(A\Phi \pm A\Phi_0\right)d\phi,$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi, \theta_\phi = \pm\rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

Expression (24) is a more general solution with respect to (20). Taking the value $A\Phi_0=0$ in (24), dependences (20) are obtained.

There are opposite signs corresponding to signs (12) in front of square brackets of the basic expressions. These are different solutions that can be taken into account by the general approach. Let us consider a refined version of solving the system of equations (1) to (3).

Some clarifications in solving the problem.

Refinements are related to the change in signs in the Cauchy-Riemann differential relations. Let us consider what happens in the solution with a sign change in the Cauchy-Riemann relations (12). In this case, one more component may enter the solution (we will show it). We have the case (12):

$$\rho\theta_\rho = -\mp A\Phi_\phi, \theta_\phi = \pm\rho A\Phi_\rho,$$

hence, the initial data are:

$$\rho\theta_\rho' = -A\Phi_\phi, \theta_\phi' = \rho A\Phi_\rho,$$

$$\rho\theta_\rho'' = A\Phi_\phi, \theta_\phi'' = -\rho A\Phi_\rho.$$

Separate the variables and integrate to find θ' and θ'':

$$\theta' = -\frac{1}{2}\left(\int\frac{1}{\rho}A\Phi_\phi d\rho - \int\rho A\Phi_\rho d\phi\right),$$

$$\theta'' = \frac{1}{2}\left(\int\frac{1}{\rho}A\Phi_\phi d\rho - \int\rho A\Phi_\rho d\phi\right).$$

Introduce the notation:

$$\theta = \frac{1}{2}\left(\int\frac{1}{\rho}A\Phi_\phi d\rho - \int\rho A\Phi_\rho d\phi\right),$$

then

$$\theta^{'} = -\theta, \quad \theta^{''} = \theta.$$

Different signs in the Cauchy-Riemann relations give different signs in exponents of the exponential functions, that is

$$\rho\theta_{\rho}^{'} = -A\Phi_{\phi},$$

$$\theta_{\phi}^{'} = \rho A\Phi_{\rho}, \quad \rightarrow \theta^{'} = -\theta,$$

$$\rho\theta_{\rho}^{''} = A\Phi_{\phi},$$

$$\theta_{\phi}^{''} = -\rho A\Phi_{\rho}, \quad \rightarrow \theta^{''} = \theta.$$

Consequently, the change of signs in the Cauchy-Riemann relations in expressions (20), (24) leads to a change of signs not only in front of the basic functions $C\sigma\exp\theta\cos A\Phi$ but also in the signs of exponents. Taking into account the latter, we can write the following for (20):

$$\sigma_{\rho} = \pm C_{\sigma}\exp(\pm\theta)\cos A\Phi \mp I + \sigma_0 + f(\phi),$$

$$\sigma_{\phi} = \mp C_{\sigma}\exp(\pm\theta)\cos A\Phi \mp I + \sigma_0 + f(\rho), \qquad (25)$$

$$\tau_{\rho\phi} = C_{\sigma}\exp(\pm\theta)\sin A\Phi,$$

at

$$\rho\theta_{\rho} = \mp A\Phi_{\phi}, \theta_{\phi} = \pm\rho A\Phi_{\rho}, \quad \rho^2\theta_{\rho\rho} + \rho\theta_{\rho} + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_{\rho} + A\Phi_{\phi\phi} = 0.$$

Taking into account the shift of the trigonometric function, for (24):

$$\sigma_{\rho} = \pm C_{\sigma}\exp(\pm\theta)\cos(A\Phi \pm A\Phi_0) \mp I + \sigma_0 + f(\phi),$$

$$\sigma_{\phi} = \mp C_{\sigma}\exp(\pm\theta)\cos(A\Phi \pm A\Phi_0) \mp I + \sigma_0 + f(\rho), \qquad (26)$$

$$\tau_{\rho\phi} = C_{\sigma}\exp(\pm\theta)\sin(A\Phi \pm A\Phi_0),$$

at

$$\rho\theta_{\rho} = \mp A\Phi_{\phi}, \theta_{\phi} = \pm\rho A\Phi_{\rho}, \quad \rho^2\theta_{\rho\rho} + \rho\theta_{\rho} + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_{\rho} + A\Phi_{\phi\phi} = 0.$$

Let us consider a solution with two exponents having argument functions with opposite signs.

In accordance with the proposed approach, determine:

$$\tau_{\rho\phi} = (T_i^{'} + T_i^{''})\sin A\Phi =$$
$$= [C_{\sigma1}\exp(\theta) + C_{\sigma2}\exp(-\theta)]\sin A\Phi. \qquad (27)$$

Brackets in (27) can be represented through hyperbolic functions:

$$C_{\sigma1}\exp\theta + C_{\sigma2}\exp(-\theta) =$$
$$= (C_{\sigma1} + C_{\sigma2})\cosh\theta + (C_{\sigma1} - C_{\sigma2})\sinh\theta =$$
$$= C_{\sigma1}^{'}\cosh\theta + C_{\sigma2}^{''}\sinh\theta.$$

Express (27) through the function of a complex variable and obtain the following:

$$\tau_{\rho\phi} = -C_{\sigma1}\frac{\exp[\theta + iA\Phi] - \exp[\theta - iA\Phi]}{2i} +$$
$$+ C_{\sigma2}\frac{\exp[-\theta + iA\Phi] - \exp[-\theta - iA\Phi]}{2i}. \qquad (28)$$

Having the shear stress in the new formulation (28), we can proceed from the equilibrium equations (1) to finding normal stresses σ_{ρ}, σ_{ϕ}. To this end, it is necessary to determine coordinate derivatives from expression (28) and substitute the difference of normal tangential stresses into the equilibrium equations. Taking into account the latter, we have:

$$\sigma_{\rho} = -\left\{\begin{array}{l}\int\frac{C_{\sigma1}}{2i}\frac{1}{\rho}\begin{bmatrix}(\theta_{\phi} + iA\Phi_{\phi})\times \\ \times\exp(\theta + iA\Phi) - \\ -(\theta_{\phi} - iA\Phi_{\phi})\times \\ \times\exp(\theta - iA\Phi)\end{bmatrix}d\rho + \\ +\int\frac{1}{\rho}\frac{C_{\sigma2}}{2i}\begin{bmatrix}(-\theta_{\phi} + iA\Phi_{\phi})\times \\ \times\exp(-\theta + i\cdot A\Phi) - \\ -(-\theta_{\phi} - iA\Phi_{\phi})\times \\ \times\exp(-\theta - i\cdot A\Phi)\end{bmatrix}d\rho + \\ +2\int\left\langle\begin{array}{l}\frac{C_{\sigma1}}{2\rho}\begin{bmatrix}\exp(\theta + i\cdot A\Phi) + \\ +\exp(\theta - i\cdot A\Phi)\end{bmatrix} - \\ -\frac{C_{\sigma2}}{2\rho}\begin{bmatrix}\exp(-\theta + i\cdot A\Phi) + \\ +\exp(-\theta - i\cdot A\Phi)\end{bmatrix}\end{array}\right\rangle d\rho\end{array}\right\} +$$
$$+\sigma_0 + f(\phi). \qquad (29)$$

$$\sigma_{\phi} = -\left\{\begin{array}{l}\int\frac{C_{\sigma1}}{2i}\rho\begin{bmatrix}(\theta_{\rho} + iA\Phi_{\rho})\times \\ \times\exp(\theta + iA\Phi) - \\ -(\theta_{\rho} - iA\Phi_{\rho})\times \\ \times\exp(\theta - iA\Phi)\end{bmatrix}d\phi + \\ +\int\frac{C_{\sigma2}}{2i}\rho\begin{bmatrix}(-\theta_{\rho} + iA\Phi_{\rho})\times \\ \times\exp(-\theta + i\cdot A\Phi) - \\ -(-\theta_{\rho} - iA\Phi_{\rho})\times \\ \times\exp(-\theta - i\cdot A\Phi)\end{bmatrix}d\phi + \\ +2\int\left\langle\begin{array}{l}\frac{C_{\sigma1}}{2i}\begin{bmatrix}\exp(\theta + i\cdot A\Phi) - \\ -\exp(\theta - i\cdot A\Phi)\end{bmatrix} + \\ +\frac{C_{\sigma2}}{2i}\begin{bmatrix}\exp(-\theta + i\cdot A\Phi) - \\ -\exp(-\theta - i\cdot A\Phi)\end{bmatrix}\end{array}\right\rangle d\phi\end{array}\right\} +$$
$$+\sigma_0 + f(\rho). \qquad (30)$$

The resulting integral expressions must be transformed, that is integrated. It is necessary to go to one variable of integration. Two options are possible. When considering the options, use the Cauchy-Riemann relation in the form:

$$\rho\theta_{\rho} = -A\Phi_{\phi}, \theta_{\phi} = \rho A\Phi_{\rho}.$$

Perform a change of the variable for normal stresses (29), (30), transformation, reformatting by the imaginary unit

integrating with the transition to real functions to obtain the following:

$$\sigma_\rho = \pm \begin{bmatrix} C_{\sigma 1}\exp(\theta)- \\ -C_{\sigma 2}\exp(-\theta) \end{bmatrix} \cos A\Phi \mp I + \sigma_0 + f(\phi),$$

$$\sigma_\phi = \mp \begin{bmatrix} C_{\sigma 1}\exp(\theta)- \\ -C_{\sigma 2}\exp(-\theta) \end{bmatrix} \cos A\Phi \mp I + \sigma_0 + f(\rho),$$

$$\tau_{\rho\phi} = \begin{bmatrix} C_{\sigma 1}\exp(\theta)+C_{\sigma 2}\exp(-\theta) \end{bmatrix}\sin A\Phi, \qquad (31)$$

$$I_1 = 2\int \frac{1}{\rho}\begin{bmatrix} C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta) \end{bmatrix}\cos A\Phi d\rho,$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi, \theta_\phi = \pm\rho A\Phi_\rho, \quad \rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}=0,$$

$$\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}=0,$$

at the same time

$$\frac{\partial(I_1)}{\partial\rho} = \pm\frac{\partial(I_2)}{\partial\rho},$$

$$\frac{\partial(I_1)}{\partial\phi} = \pm\frac{\partial(I_2)}{\partial\phi}.$$

Similar approaches in determining normal shear stresses at a shift of the trigonometric function:

$$\sigma_\rho = \pm \begin{bmatrix} C_{\sigma 1}\exp(\theta)- \\ -C_{\sigma 2}\exp(-\theta) \end{bmatrix} \cos(A\Phi\pm A\Phi_0) \mp I +$$
$$+\sigma_0 + f(\phi),$$

$$\sigma_\phi = \mp \begin{bmatrix} C_{\sigma 1}\exp(\theta)- \\ -C_{\sigma 2}\exp(-\theta) \end{bmatrix} \cos(A\Phi\pm A\Phi_0) \mp I +$$
$$+\sigma_0 + f(\rho),$$

$$\tau_{\rho\phi} = \begin{bmatrix} C_{\sigma 1}\exp(\theta)+C_{\sigma 2}\exp(-\theta) \end{bmatrix}\sin(A\Phi\pm A\Phi_0), \qquad (32)$$

$$I_1 = 2\int \frac{1}{\rho}\begin{bmatrix} C_{\sigma 1}\exp(\theta)- \\ -C_{\sigma 2}\exp(-\theta) \end{bmatrix}\cos(A\Phi\pm A\Phi_0)d\rho,$$

$$I_2 = 2\int \begin{bmatrix} C_{\sigma 1}\exp(\theta)+ \\ +C_{\sigma 2}\exp(-\theta) \end{bmatrix}\sin(A\Phi\pm A\Phi_0)A\Phi d\phi,$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi, \theta_\phi = \pm\rho A\Phi_\rho, \quad \rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}=0,$$

$$\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}=0.$$

$$\frac{\partial(I_1)}{\partial\rho} = \pm\frac{\partial(I_2)}{\partial\rho}, \quad \frac{\partial(I_1)}{\partial\phi} = \pm\frac{\partial(I_2)}{\partial\phi}.$$

Signs in (31), (32) in front of square brackets mean that the derivation was performed at different signs in the Cauchy-Riemann relations (12).

5. 2. Invariant differential generalizations in the problem

When formulating the problem, one should take into account certain approaches to further implementation [33]. The Lamé strain potential method for an analytical solution is extended to plane gradient elasticity of a simple type. The proposed method was applied to express certain components of generalization of the scalar functions making it possible to use it in solving the continuity equation. At the same time, this approach does not ensure the identification of those generalizations that define the method of argument functions. An acceptable feature consists in that this scheme clearly demonstrates the ability to express unknown quantities through generalizing dependences during formulation and solution of the problem.

It can be seen from (22), (24) to (26) that to complete the problem, it is necessary to know the value of hydrostatic pressure σ_0. To this end, let us use the Laplace equation (2). After some transformations, the Laplace equation (2) takes the form:

$$\nabla^2(\sigma_\rho+\sigma_\phi)=0, \sigma_\rho+\sigma_\phi=2\sigma_0 \to n\sigma_0,$$

$$\nabla^2(n\sigma_0) = \frac{\partial^2(n\sigma_0)}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial(n\sigma_0)}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2(n\sigma_0)}{\partial\phi^2}=0. \qquad (33)$$

By analogy with [33], let us express σ_0 through the generalized component included in formulas (20), (24) to (26) for normal stresses, and I. This will make it possible to get rid of the integral values of I in the above expressions in the future. The following dependences are the determining format:

$$R = C_\sigma \exp\theta\cos A\Phi =$$
$$= C_\sigma \frac{\exp(\theta+iA\Phi)+\exp(\theta-iA\Phi)}{2}. \qquad (34)$$

$$I_1 = 2C_\sigma\int\frac{1}{\rho}\exp\theta\cos A\Phi d\rho =$$
$$= 2C_\sigma\int\frac{\exp(\theta+iA\Phi)+\exp(\theta-iA\Phi)}{2\rho}d\rho. \qquad (35)$$

$$I_2 = 2C_\sigma\int\exp\theta\sin A\Phi d\phi =$$
$$= 2C_\sigma\int\frac{\exp(\theta+iA\Phi)-\exp(\theta-iA\Phi)}{2i}d\phi. \qquad (36)$$

Upon analyzing the previously obtained result, a decision is made in the form:

$$\sigma_0 = R\pm I_1 =$$
$$= C_\sigma\exp\theta\cos A\Phi \pm \int\frac{1}{\rho}2C_\sigma\exp\theta\cdot\cos A\Phi d\rho. \qquad (37)$$

or

$$\sigma_0 = R\pm I_2 =$$
$$= C_\sigma\exp\theta\cos A\Phi \pm \int 2C_\sigma\exp\theta\sin A\Phi d\phi. \qquad (38)$$

The main thing is that (37), (38) satisfy equation (33). It is necessary to find what conditions the argument functions must meet in order that expressions (37), (38) satisfy differential equation (33). Substitute (37), (38) in (33):

$$\nabla^2(\sigma_0) = \rho^2\frac{\partial^2(R\pm I)}{\partial\rho^2}+\rho\frac{\partial(R\pm I)}{\partial\rho}+\frac{\partial^2(R\pm I)}{\partial\phi^2}=0.$$

The equations will be identically satisfied if:

$$\rho^2 \frac{\partial^2(R)}{\partial\rho^2} + \rho\frac{\partial(R)}{\partial\rho} + \frac{\partial^2(R)}{\partial\phi^2} = 0, \quad \rho^2 \frac{\partial^2(I_1)}{\partial\rho^2} + \rho\frac{\partial(I_1)}{\partial\rho} + \frac{\partial^2(I_1)}{\partial\phi^2} = 0,$$

$$\rho^2 \frac{\partial^2(I_2)}{\partial\rho^2} + \rho\frac{\partial(I_2)}{\partial\rho} + \frac{\partial^2(I_2)}{\partial\phi^2} = 0. \tag{39}$$

Let us consider sequentially solution of equations (39) which taken together determine the general solution of the equation (33).

Let us use the method of argument functions and find what conditions they must meet in order that expressions (39) be solutions of the continuity equation (33).

Determine derivatives with respect to ρ and ϕ, substitute into the Laplace equation (39) with further rearrangements, decomposition of the difference squares and then the square difference to obtain the following:

$$\rho^2 R_{\rho\rho} + \rho R_\rho + R_{\phi\phi} =$$
$$= \frac{C_\sigma}{2}\left\{ \begin{bmatrix} \left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)+ \\ +i\left(\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}\right)+ \\ +\left(\rho\theta_\rho+A\Phi_\phi\right)\left(\rho\theta_\rho-A\Phi_\phi\right)+ \\ +\left(\theta_\phi-\rho A\Phi_\rho\right)\left(\theta_\phi+\rho A\Phi_\rho\right)+ \\ +2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right) \end{bmatrix} \exp(\theta+iA\Phi)+ \\ + \begin{bmatrix} \left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)- \\ -i\left(\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}\right)+ \\ +\left(\rho\theta_\rho+A\Phi_\phi\right)\left(\rho\theta_\rho-A\Phi_\phi\right)+ \\ +\left(\theta_\phi-\rho A\Phi_\rho\right)\left(\theta_\phi+\rho A\Phi_\rho\right)- \\ -2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right) \end{bmatrix} \exp(\theta-iA\Phi) \right\}. \tag{40}$$

After transformations, a difference of squares was obtained in equation (40) which introduces undesirable nonlinearity. If we take the expansion brackets equal to zero:

$$\rho\theta_\rho + A\Phi_\phi = 0, \quad \rho\theta_\rho - A\Phi_\phi = 0,$$

$$\theta_\phi - \rho A\Phi_\rho = 0, \quad \theta_\phi + \rho A\Phi_\rho = 0, \tag{41}$$

then simplifications are possible after substitution of (40) into (41):

$$\rho^2 R_{\rho\rho} + \rho R_\rho + R_{\phi\phi} =$$
$$= \frac{C_\sigma}{2}\left\{ \begin{bmatrix} \left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)+ \\ +i\begin{pmatrix}\rho^2 A\Phi_{\rho\rho}+ \\ +\rho A\Phi_\rho+A\Phi_{\phi\phi}\end{pmatrix}+ \\ +2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right) \end{bmatrix} \exp(\theta+iA\Phi)+ \\ + \begin{bmatrix} \left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)- \\ -i\begin{pmatrix}\rho^2 A\Phi_{\rho\rho}+ \\ +\rho A\Phi_\rho+A\Phi_{\phi\phi}\end{pmatrix}- \\ -2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right) \end{bmatrix} \exp(\theta-iA\Phi) \right\}. \tag{42}$$

In a case of (41), a mathematical connection appears between derivatives of the argument functions in a form of the Cauchy-Riemann relations in polar coordinates of the form:

$$\rho\theta_\rho = \mp A\Phi_\phi, \quad \theta_\phi = \pm\rho A\Phi_\rho. \tag{43}$$

The same relation (12) was used when integrating the equilibrium equations (1). If this relation was used in (12) in a form of an assumption, it was determined as a result of a correct derivation in the case of (43). Of interest is the fact that different differential equations (1) and (39) feature the same approaches when finding the main solution. Relation (43) will be used more than once in what follows. Let us consider the brackets in equation (42) with taking into account the Cauchy-Riemann relation:

$$2i\left(\rho\,\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi\right) = 2i\left(-\theta_\phi A\Phi_\phi + \theta_\phi A\Phi_\phi\right) = 0.$$

Equation (42) will get simplified even more and take the form:

$$\rho^2 R_{\rho\rho} + \rho R_\rho + R_{\phi\phi} =$$
$$= \frac{C_\sigma}{2}\left\{ \begin{bmatrix} \left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)+ \\ +i\begin{pmatrix}\rho^2 A\Phi_{\rho\rho}+ \\ +\rho A\Phi_\rho+A\Phi_{\phi\phi}\end{pmatrix} \end{bmatrix} \exp(\theta+iA\Phi)+ \\ + \begin{bmatrix} \left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)- \\ -i\begin{pmatrix}\rho^2 A\Phi_{\rho\rho}+ \\ +\rho A\Phi_\rho+A\Phi_{\phi\phi}\end{pmatrix} \end{bmatrix} \exp(\theta-iA\Phi) \right\}. \tag{44}$$

The exponential operators in equation (44) are virtually the same, except for the signs. They are represented by the same differential expressions. Using the Cauchy-Riemann relations (43), one can show that the equations in (44) taken in parentheses are also equal to zero. This has already been shown in (19), hence:

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

When substituting the last expressions in (44), an identity is obtained. It should be emphasized that the Laplace equations for the argument functions are identically equal to zero for different combinations of signs in the Cauchy-Riemann relations. The presented sequence of derivation shows that the Laplace equation (33) was mostly identically satisfied and its solution R takes the form:

$$R = \mp C_\sigma \frac{\exp(\theta+iA\Phi)+\exp(\theta-iA\Phi)}{2} =$$
$$= \mp C_\sigma \exp(\theta)\cos A\Phi, \tag{45}$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi,$$

$$\theta_\phi = \pm\rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

To obtain the final solution to the continuity equation (33), it is necessary to show that functions I_1 and I_2 satisfy the Laplace equations (33) as well since they are part of the final result of (37) to (39). Substitute (35), (36) into (39) taking into account:

$$\frac{\partial I_1}{\partial \phi} = \frac{\partial I_2}{\partial \phi}, \quad \rho\theta_\rho = -A\Phi_\phi,$$

$$\theta_\phi = \rho A\Phi_\rho,$$

$$\frac{\partial I_1}{\partial \phi} = -\frac{\partial I_2}{\partial \phi}, \quad \rho\theta_\rho = A\Phi_\phi$$

$$\theta_\phi = -\rho A\Phi_\rho,$$

and obtain the following for the function I_1 after transformations:

$$\rho^2 I_{1\rho\rho} + \rho I_{1\rho} + I_{1\phi\phi} =$$
$$= \rho 2 C_\sigma \frac{\left[(\theta_\rho + iA\Phi_\rho)\exp(\theta + iA\Phi) + (\theta_\rho - iA\Phi_\rho)\exp(\theta - iA\Phi)\right]}{2} +$$
$$+ 2 C_\sigma \frac{\left[(\theta_\phi + iA\Phi_\phi)\exp(\theta + iA\Phi) - (\theta_\phi - iA\Phi_\phi)\exp(\theta - iA\Phi)\right]}{2i}. \quad (46)$$

Correspondingly:

$$\rho^2 I_{1\rho\rho} + \rho I_{1\rho} + I_{1\phi\phi} =$$
$$= \rho 2 C_\sigma \frac{\left[(\theta_\rho + iA\Phi_\rho)\exp(\theta + iA\Phi) + (\theta_\rho - iA\Phi_\rho)\exp(\theta - iA\Phi)\right]}{2} -$$
$$- 2 C_\sigma \frac{\left[(\theta_\phi + iA\Phi_\phi)\exp(\theta + iA\Phi) - (\theta_\phi - iA\Phi_\phi)\exp(\theta - iA\Phi)\right]}{2i}. \quad (47)$$

By passing to one variable in (46), (47), and using corresponding Cauchy-Riemann relations, we make sure that they are identically satisfied. As a result, a general solution was obtained taking into account (37) in the form:

$$\sigma_0 = \mp nC_\sigma \frac{\exp(\theta + iA\Phi) + \exp(\theta - iA\Phi)}{2} \pm$$
$$\pm 2 C_\sigma \int \frac{\exp(\theta + iA\Phi) + \exp(\theta - iA\Phi)}{2\rho} d\rho =$$
$$= \mp nC_\sigma \exp\theta\cos A\Phi \pm I_1 =$$
$$= \mp nC_\sigma \exp\theta\cos A\Phi \pm 2C_\sigma \int \frac{1}{\rho}\exp\theta\cos A\Phi d\rho, \quad (48)$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi,$$

$$\theta_\phi = \pm\rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

Let us consider differential equation (39) and dependence (36) for I_2.

By determining the coordinate derivatives (36), substituting into the Laplace equation (39), the following is obtained after transformations:

$$\rho^2 I_{2\rho\rho} + \rho I_{2\rho} + I_{2\phi\phi} =$$

$$= 2C_\sigma \frac{1}{2i} \int \left\{ \begin{array}{l} \left[\begin{array}{l} (\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) + \\ + i\left(\begin{array}{l} \rho^2 A\Phi_{\rho\rho} + \\ + \rho A\Phi_\rho + A\Phi_{\phi\phi} \end{array} \right) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + \\ + (\theta_\phi^2 - \rho^2 A^2\Phi_\rho^2) + \\ + 2i\left(\begin{array}{l} \rho^2\theta_\rho A\Phi_\rho + \\ + \theta_\phi A\Phi_\phi \end{array} \right) \end{array} \right] \exp(\theta + iA\Phi) - \\ - \left[\begin{array}{l} (\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) - \\ - i\left(\begin{array}{l} \rho^2 A\Phi_{\rho\rho} + \\ + \rho A\Phi_\rho + A\Phi_{\phi\phi} \end{array} \right) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + \\ + (\theta_\phi^2 - \rho^2 A^2\Phi_\rho^2) - \\ - 2i\left(\begin{array}{l} \rho^2\theta_\rho A\Phi_\rho + \\ + \theta_\phi A\Phi_\phi \end{array} \right) \end{array} \right] \exp(\theta - iA\Phi) \end{array} \right\} d\phi. \quad (49)$$

All derivatives were found under the same integral which makes it possible to simplify the derivation. The integrand (49) is actually coinciding with equation (40) which allows it to be reduced to identity. In this case, there is no need to consider a solution twice because of a sign change in Cauchy-Riemann relations. As a result, the expression for the average stress σ_0 can be written in the form:

$$\sigma_0 = \mp nC_\sigma \frac{\exp(\theta + iA\Phi) + \exp(\theta - iA\Phi)}{2} \pm$$
$$\pm 2 C_\sigma \int \frac{\exp(\theta + iA\Phi) - \exp(\theta - iA\Phi)}{2i} d\rho =$$
$$= \mp nC_\sigma \exp\theta\cos A\Phi \pm I_2 =$$
$$= \mp nC_\sigma \exp\theta\cos A\Phi \pm 2C_\sigma \int \exp\theta\sin A\Phi d\phi, \quad (50)$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi,$$

$$\theta_\phi = \pm\rho A\Phi_\rho.$$

5. 3. Solutions of the plane problem of the theory of elasticity in polar coordinates

Generalizations of expressions (48), (50) are possible if we use relations (21), (22). Then:

$$\sigma_0 = \mp nC_\sigma \exp\theta\cos A\Phi \pm I. \quad (51)$$

Substitute the generalized value of hydrostatic pressure (51) into (20) and (24)–(26) to obtain:

$$\sigma_\rho = \left[\pm C_\sigma \exp(\pm\theta)\cos A\Phi\right]+$$
$$+\left[\mp nC_\sigma \exp(\pm\theta)\cos A\Phi\right]+f(\phi),$$

$$\sigma_\phi = \left[\mp C_\sigma \exp(\pm\theta)\cos A\Phi\right]+$$
$$+\left[\mp nC_\sigma \exp(\pm\theta)\cos A\Phi\right]+f(\rho),$$

$$\tau_{\rho\phi} = C_\sigma \exp(\pm\theta)\sin A\Phi. \qquad (52)$$

For the shift:

$$\sigma_\rho = \left[\pm C_\sigma \exp(\pm\theta)\cos(A\Phi\pm A\Phi_0)\right]+$$
$$+\left[\mp nC_\sigma \exp(\pm\theta)\cos(A\Phi\mp A\Phi_0)\right]+f(\phi),$$

$$\sigma_\phi = \left[\mp C_\sigma \exp(\pm\theta)\cos(A\Phi\pm A\Phi_0)\right]+$$
$$+\left[\mp nC_\sigma \exp(\pm\theta)\cos(A\Phi\mp A\Phi_0)\right]+f(\rho),$$

$$\tau_{\rho\phi} = C_\sigma \exp(\pm\theta)\sin(A\Phi\pm A\Phi_0), \qquad (53)$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi,$$

$$\theta_\phi = \pm\rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}=0,$$

$$\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}=0.$$

Working expressions (52), (53) are simpler than (20), (24) to (26) since the need to calculate values of I has disappeared. The last formulas contain the defining expressions $C_\sigma\exp(\pm\theta)\cos A\Phi$ which satisfy the equilibrium equations and the equation of deformation continuity. This feature makes it possible to expand their applicability by shifting the sign and the result by the value of the mean stress as is the case in the Mohr's circles.

For a refined solution of (31), (32), determination of the mean stress σ_0 should be considered separately.

We have refined solutions of (31), (32) with wider possibilities in comparison with (52), (53):

$$\sigma_\rho = \pm\left[C_{\sigma1}\exp(\theta)-C_{\sigma2}\exp(-\theta)\right]\times$$
$$\times\cos(A\Phi\pm A\Phi_0)\mp I+\sigma_0+f(\phi),$$

$$\sigma_\phi = \mp\left[C_{\sigma1}\exp(\theta)-C_{\sigma2}\exp(-\theta)\right]\times$$
$$\times\cos(A\Phi\pm A\Phi_0)\mp I+\sigma_0+f(\rho),$$

$$\tau_{\rho\phi} = \left[C_{\sigma1}\exp(\theta)+C_{\sigma2}\exp(-\theta)\right]\times$$
$$\times\sin(A\Phi\pm A\Phi_0).$$

To complete the problem solution, it is necessary to know average stress σ_0 and the integral values of I which can be determined from the Laplace equation (33). It has been shown that

$$I_1 = 2\int\frac{1}{\rho}\left[\begin{matrix}C_{\sigma1}\exp(\theta)-\\-C_{\sigma2}\exp(-\theta)\end{matrix}\right]\cos(A\Phi\pm A\Phi_0)d\rho,$$

$$I_2 = 2\int\left[\begin{matrix}C_{\sigma1}\exp(\theta)+\\+C_{\sigma2}\exp(-\theta)\end{matrix}\right]\sin(A\Phi\pm A\Phi_0)A\Phi d\phi,$$

$$\frac{\partial I_1}{\partial\rho}=\mp\frac{\partial I_2}{\partial\rho},$$

$$\frac{\partial I_1}{\partial\phi}=\mp\frac{\partial I_2}{\partial\phi}.$$

The solution should be related to the defining functions of equation (32),

$$R = \left[C_{\sigma1}\exp(\theta)-C_{\sigma2}\exp(-\theta)\right]\times$$
$$\times\cos(A\Phi\pm A\Phi_0),$$

and integral expressions of I_1, I_2.

For a solution, equation (33) must be identically satisfied. In this case, the sign in front of the indicated variables is not essential. The solution is sought in the form as for (34) to (36)

$$\sigma_0 = R\pm I_1 = R\pm I$$

or

$$\sigma_0 = R\pm I_2 = R\pm I. \qquad (54)$$

The problem is formulated as follows: what conditions should be met by the argument functions in order that the coordinate functions (54) satisfy the differential equation (33).

Represent σ_0, R, I_1, I_2 through the function of a complex variable:

$$R =$$
$$=\pm\left\{\begin{matrix}C_{\sigma1}\dfrac{\exp[\theta+i(A\Phi\mp A\Phi_0)]+\exp[\theta-i(A\Phi\mp A\Phi_0)]}{2}-\\-C_{\sigma2}\dfrac{\exp[-\theta+i(A\Phi\mp A\Phi_0)]+\exp[-\theta-i(A\Phi\mp A\Phi_0)]}{2}\end{matrix}\right\},$$

$$I_1 = \int\left\langle\begin{matrix}2C_{\sigma1}\dfrac{1}{2\rho}\left\{\begin{matrix}\exp[\theta+i(A\Phi\mp A\Phi_0)]+\\+\exp[\theta-i(A\Phi\mp A\Phi_0)]\end{matrix}\right\}-\\-2C_{\sigma2}\dfrac{1}{2\rho}\left\{\begin{matrix}\exp[-\theta+i(A\Phi\mp A\Phi_0)]+\\+\exp[-\theta-i(A\Phi\mp A\Phi_0)]\end{matrix}\right\}\end{matrix}\right\rangle d\rho, \quad (55)$$

$$I_2 = \int\left\langle\begin{matrix}2C_{\sigma1}\dfrac{1}{2i}\left\{\begin{matrix}\exp[\theta+i(A\Phi\pm A\Phi_0)]-\\-\exp[\theta-i(A\Phi\pm A\Phi_0)]\end{matrix}\right\}+\\+2C_{\sigma2}\dfrac{1}{2i}\left\{\begin{matrix}\exp[-\theta+i(A\Phi\pm A\Phi_0)]-\\-\exp[-\theta-i(A\Phi\pm A\Phi_0)]\end{matrix}\right\}\end{matrix}\right\rangle d\phi.$$

Substitution of dependences (55) into (39) and the transformation have resulted in obtaining of working equations of the following form:

$$\rho^2 \frac{\partial^2(R)}{\partial \rho^2} + \rho \frac{\partial(R)}{\partial \rho} + \frac{\partial^2(R)}{\partial \phi^2} =$$

$$= \frac{C_{\sigma 1}}{2} \left\{ \begin{matrix} \exp\left[\theta + i(A\Phi \mp A\Phi_0)\right] \times \\ \times \left\langle \begin{matrix} (\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) + i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + (\theta_\phi^2 - \rho^2 A^2\Phi_\rho^2) + \\ + 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\rangle + \exp\left[\theta - i(A\Phi \mp A\Phi_0)\right] \left\langle \begin{matrix} (\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) - \\ - i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + (\theta_\phi^2 - \rho^2 A^2\Phi_\rho^2) - \\ - 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\rangle \end{matrix} \right\} -$$

$$- \frac{C_{\sigma 2}}{2} \left\{ \begin{matrix} \exp\left[-\theta + i(A\Phi \mp A\Phi_0)\right] \left\langle \begin{matrix} -(\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) + i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + (\theta_\phi^2 - \rho^2 A^2\Phi_\rho^2) - 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\rangle + \\ + \exp\left[-\theta - i(A\Phi \mp A\Phi_0)\right] \left\langle \begin{matrix} -(\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) - i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + (\theta_\phi^2 - \rho^2 A^2\Phi_\rho^2) + 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\rangle \end{matrix} \right\}. \tag{56}$$

Further, for the function I_1 taking into account signs of the Cauchy-Riemann relations.

Option 1. $\rho\theta_\rho = -A\Phi_\phi$ $\theta_\phi = \rho A\Phi_\phi$

$$\rho^2 I_{1\rho\rho} + \rho I_{1\rho} + I_{1\phi\phi} =$$

$$= 2C_{\sigma 1} \left\{ \begin{matrix} \left[\frac{\rho}{2}(\theta_\rho + iA\Phi_\rho) + \frac{1}{2i}(\theta_\phi + iA\Phi_\phi)\right] \times \\ \times \exp\left[\theta + i(A\Phi \pm A\Phi_0)\right] + \\ + \left[\frac{\rho}{2}(\theta_\rho - iA\Phi_\rho) - \frac{1}{2i}(\theta_\phi - iA\Phi_\phi)\right] \times \\ \times \exp\theta - i(A\Phi \pm A\Phi_0) \end{matrix} \right\} -$$

$$- 2C_{\sigma 2} \left\{ \begin{matrix} \left[\frac{\rho}{2}(-\theta_\rho + iA\Phi_\rho) - \frac{1}{2i}(-\theta_\phi + iA\Phi_\phi)\right] \times \\ \times \exp\left[-\theta + i(A\Phi \pm A\Phi_0)\right] + \\ + \left[\frac{\rho}{2}(-\theta_\rho - iA\Phi_\rho) + \frac{1}{2i}(\theta_\phi - iA\Phi_\phi)\right] \times \\ \times \exp\left[-\theta - i(A\Phi \pm A\Phi_0)\right] \end{matrix} \right\}. \tag{57}$$

Option 2. $\rho\theta_\rho = A\Phi_\phi,$ $\theta_\phi = -\rho A\Phi_\phi,$

$$\rho^2 I_{1\rho\rho} + \rho I_{1\rho} + I_{1\phi\phi} =$$

$$= 2C_{\sigma 1} \left\{ \begin{matrix} \left[\frac{\rho}{2}(\theta_\rho + iA\Phi_\rho) - \frac{1}{2i}(\theta_\phi + iA\Phi_\phi)\right] \times \\ \times \exp\left[\theta + i(A\Phi \pm A\Phi_0)\right] + \\ + \left[\frac{\rho}{2}(\theta_\rho - iA\Phi_\rho) + \frac{1}{2i}(\theta_\phi - iA\Phi_\phi)\right] \times \\ \times \exp\left[\theta - i(A\Phi \pm A\Phi_0)\right] \end{matrix} \right\} -$$

$$- 2C_{\sigma 2} \left\{ \begin{matrix} \left[\frac{\rho}{2}(-\theta_\rho + iA\Phi_\rho) + \frac{1}{2i}(-\theta_\phi + iA\Phi_\phi)\right] \times \\ \times \exp\left[-\theta + i(A\Phi \pm A\Phi_0)\right] - \\ - \left[\frac{\rho}{2}(-\theta_\rho - iA\Phi_\rho) - \frac{1}{2i}(\theta_\phi - iA\Phi_\phi)\right] \times \\ \times \exp\left[-\theta - i(A\Phi \pm A\Phi_0)\right] \end{matrix} \right\}. \tag{58}$$

Two solutions of (57), (58) correspond to two Cauchy-Riemann options.

Further for the function of I_2:

$$\rho^2 I_{2\rho\rho} + \rho I_{2\rho} + I_{2\phi\phi} =$$

$$= \rho^2 \int \left\langle \begin{matrix} 2C_{\sigma 1} \frac{1}{2i} \left\{ \begin{matrix} (\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) + \\ + i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + \\ + (\theta_\phi^2 - \rho^2 A^2\Phi_\phi^2) + \\ + 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\} \times \\ \times \exp\left[\theta + i(A\Phi \pm A\Phi_0)\right] - \\ -2C_{\sigma 1} \frac{1}{2i} \left\{ \begin{matrix} (\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) - \\ - i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + \\ + (\theta_\phi^2 - \rho^2 A^2\Phi_\phi^2) - \\ - 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\} \times \\ \times \exp\left[\theta - i(A\Phi \pm A\Phi_0)\right] + \\ +2C_{\sigma 2} \frac{1}{2i} \left\{ \begin{matrix} -(\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) + \\ + i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + \\ + (\theta_\phi^2 - \rho^2 A^2\Phi_\phi^2) - \\ - 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\} \times \\ \times \exp\left[-\theta + i(A\Phi \pm A\Phi_0)\right] - \\ -2C_{\sigma 2} \frac{1}{2i} \left\{ \begin{matrix} -(\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi}) - \\ - i(\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi}) + \\ + (\rho^2\theta_\rho^2 - A^2\Phi_\phi^2) + \\ + (\theta_\phi^2 - \rho^2 A^2\Phi_\phi^2) + \\ + 2i(\rho^2\theta_\rho A\Phi_\rho + \theta_\phi A\Phi_\phi) \end{matrix} \right\} \times \\ \times \exp\left[-\theta - i(A\Phi \pm A\Phi_0)\right] \end{matrix} \right\rangle d\phi. \tag{59}$$

Returning to (56), analysis shows that near the 4 exponents of the equation:

$$\exp\left[\theta+i\left(A\Phi\mp A\Phi_0\right)\right],$$

$$\exp\left[\theta-i\left(A\Phi\mp A\Phi_0\right)\right],$$

$$\exp\left[-\theta+i\left(A\Phi\mp A\Phi_0\right)\right],$$

$$\exp\left[-\theta-i\left(A\Phi\mp A\Phi_0\right)\right],$$

there are 4 operators that contain parentheses with the same differential expressions. As for (40), generalizing differential relations of the following form were found:

$$\rho\theta_\rho=\mp A\Phi_\phi,\ \theta_\phi=\pm\rho A\Phi_\rho,$$

which ultimately vanish identically like all parentheses and then operators. The equation of deformation continuity (56) turns into identity. In addition, the following differential dependences were established:

$$\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}=0,$$

$$\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}=0,$$

allowing us to unambiguously find the coordinate functions for arguments of the basic dependences. Hence, it follows that

$$R=\left[C_{\sigma1}\exp(\theta)-C_{\sigma2}\exp(-\theta)\right]\times$$
$$\times\cos\left(A\Phi\pm A\Phi_0\right), \tag{60}$$

is the basic part of solving the problem of deformation continuity if the Cauchy-Riemann relation is satisfied.

In the two options, (57), (58), the Laplace equations were identically satisfied if the corresponding Cauchy-Riemann conditions were met. Thus, the expression (55) is a solution to the Laplace equation (57), (58), however, with different signs, that is:

$$I_1=\int\left\langle\begin{array}{l}2C_{\sigma1}\dfrac{1}{2\rho}\left\{\begin{array}{l}\exp\left[\theta+i\left(A\Phi\mp A\Phi_0\right)\right]+\\+\exp\left[\theta-i\left(A\Phi\mp A\Phi_0\right)\right]\end{array}\right\}-\\-2C_{\sigma2}\dfrac{1}{2\rho}\left\{\begin{array}{l}\exp\left[-\theta+i\left(A\Phi\mp A\Phi_0\right)\right]+\\+\exp\left[-\theta-i\left(A\Phi\mp A\Phi_0\right)\right]\end{array}\right\}\end{array}\right\rangle d\rho. \tag{61}$$

$$I_1=-\int\left\langle\begin{array}{l}2C_{\sigma1}\dfrac{1}{2\rho}\left\{\begin{array}{l}\exp\left[\theta+i\left(A\Phi\mp A\Phi_0\right)\right]+\\+\exp\left[\theta-i\left(A\Phi\mp A\Phi_0\right)\right]\end{array}\right\}-\\-2C_{\sigma2}\dfrac{1}{2\rho}\left\{\begin{array}{l}\exp\left[-\theta+i\left(A\Phi\mp A\Phi_0\right)\right]+\\+\exp\left[-\theta-i\left(A\Phi\mp A\Phi_0\right)\right]\end{array}\right\}\end{array}\right\rangle d\rho. \tag{62}$$

It was shown that the solution of the continuity equation must contain two components (61) and (62) differing in signs. The same components take place in finding normal stresses (37), (38) by integrating the equilibrium equations.

Let us consider the Laplace equation (39) for the third function (55). Substitute derivatives into the Laplace equation, determine the general integral, and rearrange to obtain the following:

$$\rho^2 I_{2\rho\rho}+\rho I_{2\rho}+I_{2\phi\phi}=$$

$$=\rho^2\int\left\langle\begin{array}{l}2C_{\sigma1}\dfrac{1}{2i}\left\{\begin{array}{l}\left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)+\\+i\left(\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}\right)+\\+\left(\rho^2\theta_\rho^2-A^2\Phi_\phi^2\right)+\\+\left(\theta_\phi^2-\rho^2 A^2\Phi_\phi^2\right)+\\+2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right)\end{array}\right\}\times\\ \times\exp\left[\theta+i\left(A\Phi\pm A\Phi_0\right)\right]-\\-2C_{\sigma1}\dfrac{1}{2i}\left\{\begin{array}{l}\left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)-\\-i\left(\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}\right)+\\+\left(\rho^2\theta_\rho^2-A^2\Phi_\phi^2\right)+\\+\left(\theta_\phi^2-\rho^2 A^2\Phi_\phi^2\right)-\\-2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right)\end{array}\right\}\times\\ \times\exp\left[\theta-i\left(A\Phi\pm A\Phi_0\right)\right]+\\+2C_{\sigma2}\dfrac{1}{2i}\left\{\begin{array}{l}-\left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)+\\+i\left(\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}\right)+\\+\left(\rho^2\theta_\rho^2-A^2\Phi_\phi^2\right)+\\+\left(\theta_\phi^2-\rho^2 A^2\Phi_\phi^2\right)-\\-2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right)\end{array}\right\}\times\\ \times\exp\left[-\theta+i\left(A\Phi\pm A\Phi_0\right)\right]-\\-2C_{\sigma2}\dfrac{1}{2i}\left\{\begin{array}{l}-\left(\rho^2\theta_{\rho\rho}+\rho\theta_\rho+\theta_{\phi\phi}\right)-\\-i\left(\rho^2 A\Phi_{\rho\rho}+\rho A\Phi_\rho+A\Phi_{\phi\phi}\right)+\\+\left(\rho^2\theta_\rho^2-A^2\Phi_\phi^2\right)+\\+\left(\theta_\phi^2-\rho^2 A^2\Phi_\phi^2\right)+\\+2i\left(\rho^2\theta_\rho A\Phi_\rho+\theta_\phi A\Phi_\phi\right)\end{array}\right\}\times\\ \times\exp\left[-\theta-i\left(A\Phi\pm A\Phi_0\right)\right]\end{array}\right\rangle d\phi. \tag{63}$$

The equation (63) variables are largely the same as the equation (56) variables. In addition, all operators at the exponents have the same dependences as for (56) including the differences of squares. The latter are characterized by the Cauchy-Riemann differential relations $\rho\theta_\rho=\mp A\Phi_\phi$, $\theta_\phi=\pm\rho A\Phi_\rho$.

The analysis shows that the use of differential relations (12) nullifies the parentheses of all operators, therefore, the integrand (63) turns into an identity, and the expression:

$$I_2=\int\left\langle\begin{array}{l}2C_{\sigma1}\dfrac{1}{2i}\left\{\begin{array}{l}\exp\left[\theta+i\left(A\Phi\pm A\Phi_0\right)\right]-\\-\exp\left[\theta-i\left(A\Phi\pm A\Phi_0\right)\right]\end{array}\right\}+\\+2C_{\sigma2}\dfrac{1}{2i}\left\{\begin{array}{l}\exp\left[-\theta+i\left(A\Phi\pm A\Phi_0\right)\right]-\\-\exp\left[-\theta-i\left(A\Phi\pm A\Phi_0\right)\right]\end{array}\right\}\end{array}\right\rangle d\phi, \tag{64}$$

is a solution of the continuity equation (63) if condition of (12), (43) is met. It should be added that the equation will be satisfied identically for both combinations of signs in relations (12), (43).

Based on the obtained solutions of the continuity equation (60) to (62), (64), we can finally write down expressions for determining the hydrostatic pressure σ_0:

$$\sigma_0 = \mp \begin{bmatrix} C_{\sigma 1}\exp(\pm\theta)- \\ -C_{\sigma 2}\exp(\mp\theta) \end{bmatrix}\cos(A\Phi \pm A\Phi_0)\pm I_1 =$$

$$= \mp \begin{bmatrix} C_{\sigma 1}\exp(\pm\theta)- \\ -C_{\sigma 2}\exp(\mp\theta) \end{bmatrix}\cos(A\Phi \pm A\Phi_0)\pm I_2,$$

or

$$\sigma_0 = \mp \begin{bmatrix} C_{\sigma 1}\exp(\pm\theta)- \\ -C_{\sigma 2}\exp(\mp\theta) \end{bmatrix}\cos(A\Phi \pm A\Phi_0)\pm I, \tag{65}$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi,$$

$$\theta_\phi = \pm\rho A\Phi_\phi,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

Refined solutions of the system of equilibrium equations (31), (32) make it possible to use the result of (65). Substitute to obtain the following after simplification:

$$\sigma_\rho = \pm[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos A\Phi \mp$$
$$\mp n[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos A\Phi + f(\phi),$$

$$\sigma_\phi = \mp[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos A\Phi \mp$$
$$\mp n[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos A\Phi + f(\rho), \tag{66}$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi,$$

$$\theta_\phi = \pm\rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

For the shift trigonometric function, the following is obtained after simplifications:

$$\sigma_\rho = \pm[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos(A\Phi \pm A\Phi_0)\mp$$
$$\mp n[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos(A\Phi \pm A\Phi_0)+f(\phi),$$

$$\sigma_\phi = \mp[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos(A\Phi \pm A\Phi_0)\mp$$
$$\mp n[C_{\sigma 1}\exp(\theta)-C_{\sigma 2}\exp(-\theta)]\cos(A\Phi \pm A\Phi_0)+f(\rho), \tag{67}$$

at

$$\rho\theta_\rho = \mp A\Phi_\phi,$$

$$\theta_\phi = \pm\rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

Expressions (66), (67) supplement formulas (52), (53). The main regularities associated with the problem solution are observed at each solution stage. First of all, this is the commonality from the position from which we have managed to find the result. In many simplifying transformations, the Cauchy-Riemann differential relations participated which closed solution at the final stage. It is desirable to compare the obtained result of (52), (53), (66), (67) with the results of studies of other authors.

5. 4. Testing and comparison of the study results with the studies of other authors

The method of argument functions proposed in this work was checked in the process of comparing with the study results of other authors for problems of the continuum mechanics in the theories of plasticity [1–3], elasticity in Cartesian coordinates [5, 6], and the theory of dynamic problems [4]. To achieve reliability of the result obtained, it is advisable to carry out such a comparison in this work as well, only with respect to the polar coordinates.

Work [34] has presented solution of the problem in polar coordinates using the stress function, in the following form:

$$f_\sigma(\rho,\phi) = \frac{D}{\rho}\sin\phi. \tag{68}$$

By comparing the result of (68) with the third formula (25) obtained in this work, we have the following:

$$f_\sigma(\rho,\phi) = C_\sigma\exp\theta\sin A\Phi, \tag{69}$$

at

$$\rho\theta_\rho = -A\Phi_\phi,$$

$$\theta_\phi = \rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

By solving the Laplace equation, the simplest result can be obtained, that is:

$$A\Phi = AA_1\phi. \tag{70}$$

By substituting into the Cauchy-Riemann relations and integrating, the second argument functions can be found:

$$\rho\theta_\rho = -AA_1,$$

$$\theta_\rho = -AA_1\frac{1}{\rho},$$

$$\rightarrow \theta = -AA_1\ln\rho + C_1.$$

Upon choosing the boundary conditions, the following is obtained:

$$\theta = AA_1 \ln \frac{D}{\rho}. \tag{71}$$

Expressions (70), (71) must be checked for compatibility with the Laplace equation. In the latter case:

$$\theta_\rho = -AA_1 \frac{1}{\rho}, \quad \theta_{\rho\rho} = AA_1 \frac{1}{\rho^2},$$

or

$$\rho^2 AA_1 \frac{1}{\rho^2} + \rho\left(-AA_1 \frac{1}{\rho}\right) + 0 \equiv 0.$$

An identity was obtained which shows that the result obtained by means of the method of argument functions is acceptable. Further, substituting (70), (71) into (69), write down the following:

$$f_\sigma(\rho,\phi) = C_\sigma \exp\left(AA_1 \ln \frac{D}{\rho}\right) \sin AA_1 \phi =$$
$$= C_\sigma \left(\frac{D}{\rho}\right)^{AA_1} \sin AA_1 \phi. \tag{72}$$

When simplified

$$C_\sigma = AA_1 = 1 \rightarrow f(\rho,\phi) = \left(\frac{D}{\rho}\right)\sin\phi,$$

the result of (68) was obtained, as required. In this case, expression (72) is considered a special case of a solution of (25).

As mentioned above, there is a need to obtain different solutions for one of the argument functions by solving the Laplace equation. When solving the Laplace equations, we have a series of coordinate dependences:

$$A\Phi_1 = \phi,$$

$$A\Phi_2 = \ln\rho,$$

$$A\Phi_3 = \phi\ln\rho,$$

$$A\Phi_4 = AA_6\phi. \tag{73}$$

The first argument functions (73) satisfy the Laplace equations:

To determine the second argument function, we use the Cauchy-Riemann relations: $\rho\theta_\rho = -A\Phi_\phi$, $\theta_\phi = \rho A\Phi_\rho$, when

$$A\Phi_{1\rho} = 0, \quad A\Phi_{1\phi} = 1,$$

$$A\Phi_{2\rho} = \frac{1}{\rho},$$

$$A\Phi_{2\phi} = 0, \quad A\Phi_{3\rho} = \phi\frac{1}{\rho}, \quad A\Phi_{1\phi} = \ln\rho,$$

$$A\Phi_{4\rho} = 0, \quad A\Phi_{4\phi} = AA_6.$$

After substitution in Cauchy-Riemann and integration, we have:

$$\theta_1 = -\ln\rho,$$

$$\theta_2 = \phi,$$

$$\theta_3 = -\frac{\ln^2\rho}{2} + \frac{\phi^2}{2},$$

$$\theta_4 = -AA_6 \ln\rho. \tag{74}$$

Functions (74) are verified by the Laplace equations:

$$\rho^2 (-\ln\rho)_{\rho\rho} + \rho(-\ln\rho)_\rho + (-\ln\rho)_{\phi\phi} = 0,$$

$$\rho^2 (\phi)_{\rho\rho} + \rho(\phi)_\rho + (\phi)_{\phi\phi} = 0,$$

$$\rho^2 \left(-\frac{\ln^2\rho}{2} + \frac{\phi^2}{2}\right)_{\rho\rho} +$$
$$+ \rho\left(-\frac{\ln^2\rho}{2} + \frac{\phi^2}{2}\right)_\rho + \left(-\frac{\ln^2\rho}{2} + \frac{\phi^2}{2}\right)_{\phi\phi} = 0,$$

$$\rho^2 (-AA_6 \ln\rho)_{\rho\rho} +$$
$$+ \rho(-AA_6 \ln\rho)_\rho + (-AA_6 \ln\rho)_{\phi\phi} = 0.$$

Thus, all argument functions (73), (74) satisfy the conditions of existence of a solution to the system of equations (1) to (3) and close it in this formulation. It is seen that the field of analytical solutions of applied problems can be extended in the cases convenient for boundary conditions.

It is of interest to compare the obtained result with theoretical solutions by a number of other authors. For example, the solution of the problem from the theory of elasticity (action of a concentrated force on the wedge tip) is known [11, 35, 36]. Let us consider the option (15), (16), or (25) after simplifications at $f(\phi) = f(\rho) = 0$:

$$\sigma_\rho = C_\sigma \exp\theta\cos A\Phi - I + \sigma_0,$$

$$\sigma_\phi = -C_\sigma \exp\theta\cos A\Phi - I + \sigma_0,$$

$$\tau_{\rho\phi} = C_\sigma \exp\theta\sin A\Phi,$$

where

$$\rho\theta_\rho = -A\Phi_\phi, \theta_\phi = \rho A\Phi_\rho,$$

$$\rho^2\theta_{\rho\rho} + \rho\theta_\rho + \theta_{\phi\phi} = 0,$$

$$\rho^2 A\Phi_{\rho\rho} + \rho A\Phi_\rho + A\Phi_{\phi\phi} = 0.$$

Using the boundary conditions, determine the coefficient n, the constant C_σ, and the value of σ_0.

Boundary conditions:

at $\phi = \alpha$, $\rho = \rho_1$, $A\Phi = A\Phi_1$,

$\theta = \theta_1$, $\sigma_\phi = 0$, $\sigma_\rho - \sigma_\phi = -2k_1$.

Then

$$\sigma_\rho - \sigma_\phi = 2C_\sigma \exp\theta_1 \cos A\Phi_1 = -2k_1,$$

$$0 = -C_\sigma \exp\theta\cos A\Phi - I + \sigma_0,$$

$$\sigma_0 = C_\sigma \exp\theta\cos A\Phi + I,$$

$$\sigma_\rho = 2C_\sigma \exp\theta\cos A\Phi, \tag{75}$$

taking into account the boundary conditions and expression (75), it follows that

$$C_\sigma = -\frac{k_1}{\exp\theta_1 \cos A\Phi_1}, \quad n = 1,$$

$$\sigma_\rho = -\frac{2k_1}{\cos A\Phi_1}\exp(\theta - \theta_1)\cos A\Phi,$$

$$\tau_{\rho\phi} = -\frac{k_1}{\cos A\Phi_1}\exp(\theta - \theta_1)\sin A\Phi. \tag{76}$$

Make use of expression (73), (74) for the first option. Then:

$$A\Phi_1 = \alpha, \quad \theta_1 = -\ln\rho_1.$$

Using the substitution of (71), (72) where instead of $D \to \rho_1$, at $AA_1 = 1$, $\theta = -\ln\rho$:

$$\theta - \theta_1 = \theta_1 - \theta = \ln\frac{\rho_1}{\rho}.$$

Substituting the last formulas in (76), the following is obtained:

$$\sigma_\rho = -\frac{2k_1}{\cos\alpha}\exp\left(\ln\frac{\rho_1}{\rho}\right)\cos\phi = -\frac{2k_1}{\cos\alpha}\cdot\frac{\rho_1}{\rho}\cdot\cos\phi,$$

$$\sigma_\phi = 0,$$

$$\tau_{\rho\phi} = -\frac{k_1}{\cos\alpha}\exp\left(\ln\frac{\rho_1}{\rho}\right)\sin\phi = -\frac{k_1}{\cos\alpha}\frac{\rho_1}{\rho}\sin\phi.$$

at $\rho \to \infty$, σ_ρ and $\tau_{\rho\phi} \to 0$.

Let us determine the value of k_1. To this end, write down the equilibrium equation for the upper cut-off part of the wedge, as it was done in [11, 35, 36] and in works by other authors:

$$\int_{-\alpha}^{\alpha} -\sigma_\rho \rho\, d\phi \cos\phi = \frac{2k_1\rho_1}{\cos\alpha}\cdot\int_{-\alpha}^{\alpha}\cos^2\phi\, d\phi = P.$$

After integration, the following can be written:

$$k_1 = \frac{P\cos\alpha}{2\rho_1\left(\alpha + \frac{1}{2}\sin 2\alpha\right)} \to \sigma_\rho =$$

$$= -2\frac{P\cos\alpha}{2\rho_1\left(\alpha + \frac{1}{2}\sin 2\alpha\right)\cos\alpha}\frac{\rho_1}{\rho}\cos\phi$$

or

$$\sigma_\rho = -\frac{P}{\left(\alpha + \frac{1}{2}\sin 2\alpha\right)}\frac{1}{\rho}\cos\phi,$$

$$\tau_{\rho\phi} = -\frac{P}{2\left(\alpha + \frac{1}{2}\sin 2\alpha\right)}\frac{1}{\rho}\sin\phi. \tag{77}$$

Expression (77) for normal stresses coincides with formula (3) in paragraph 30 of the study [35] and in the works

by other authors [11, 36]. This example is remarkable in that the simplified solution using the method of argument functions of the complex variable coincided with the classical solution of this problem. It was not shown in (77) that the tangential stresses $\tau_{\rho\phi}$ are equal to zero. In the presented option, when formulating the problem, tangential stresses of opposite signs must be present on the lateral surfaces of the wedge.

Let us consider a more general case which also has something in common with the work [11]. To compare the results, we shall use formulas (52), (53) at $f(\phi) = f(\rho) = A\Phi_0 = 0$:

$$\sigma_\rho = \left[C_{\sigma1}\exp(\theta) - C_{\sigma2}\exp(-\theta)\right]\cos(A\Phi) \mp I + \sigma_0,$$

$$\sigma_\phi = -\left[C_{\sigma1}\exp(\theta) - C_{\sigma2}\exp(-\theta)\right]\cos(A\Phi) \mp I + \sigma_0,$$

$$\tau_{\rho\phi} = \left[C_{\sigma1}\exp(\theta) + C_{\sigma2}\exp(-\theta)\right]\sin(A\Phi).$$

Let us consider the same problem with the action of a concentrated force on the wedge tip. Take a solution option in the form:

$$A\Phi = \phi,$$

$$\theta = -\ln\rho.$$

Boundary conditions: at $\phi = \alpha$, $\rho = \rho_1$,

$$A\Phi = A\Phi_1 = \alpha,$$

$$\theta = \theta_1, \quad \sigma_\phi = 0,$$

$$\tau_{\rho\phi} = 0, \quad \rho = \rho_1,$$

$$\sigma_\rho - \sigma_\phi = -2k_1,$$

then:

$$\sigma_\rho - \sigma_\phi = 2\left[C_{\sigma1}\exp(\theta_1) - C_{\sigma2}\exp(-\theta_1)\right]\cos A\Phi_1 = -2k_1,$$

$$\sigma_\rho = -\left[C_{\sigma1}\exp(\theta) - C_{\sigma2}\exp(-\theta)\right]\cos A\Phi \mp I + \sigma_0 = 0,$$

$$\tau_{\rho\phi} = \left[C_{\sigma1}\exp(\theta_1) + C_{\sigma2}\exp(-\theta_1)\right]\sin(A\Phi_1) = 0.$$

Use the boundary conditions to obtain:

$$\sigma_0 = \left[C_{\sigma1}\exp(\theta) - C_{\sigma2}\exp(-\theta)\right]\cos A\Phi \pm I,$$

$$C_{\sigma2} = -C_{\sigma1}\exp^2(\theta_1),$$

$$C_{\sigma1} = -\frac{k_1}{2\exp\theta_1\cos A\Phi_1},$$

$$\sigma_\rho = 2\left[C_{\sigma1}\exp(\theta) - C_{\sigma2}\exp(-\theta)\right]\cos(A\Phi),$$

or

$$\sigma_\rho = -2\frac{k_1}{\exp\theta_1\cos A\Phi_1}\times$$

$$\times\left[\exp(\theta) + \exp^2\theta_1\exp(-\theta)\right]\cos\phi =$$

$$= -2\frac{k_1}{\cos A\Phi_1}\exp(\theta - \theta_1)\times$$

$$\times\left[1 + \exp^2\theta_1\exp^2(-\theta)\right]\cos\phi, \tag{78}$$

$$\tau_{\rho\phi} = -\frac{k_1}{2\exp\theta_1\cos A\Phi_1}\times$$

$$\times\left[\exp(\theta)-\exp^2\theta_1\exp(-\theta)\right]\sin\phi =$$

$$= -\frac{k_1}{2\exp\theta_1\cos A\Phi_1}\exp(\theta-\theta_1)\times$$

$$\times\left[1-\exp^2\theta_1\exp^2(-\theta)\right]\sin\phi. \tag{79}$$

Thus, at $\phi=\alpha$, $\tau_{\rho\phi}=0$, $\sigma_\phi=0$, $\sigma_\rho=-2k_1$. This indicates the fulfillment of the boundary conditions. The following was obtained previously:

$$\theta-\theta_1=\theta_1-\theta=\ln\frac{\rho_1}{\rho}.$$

Substitute the last formula in (78) to (79) to obtain:

$$\sigma_\rho = -2\frac{k_1}{\cos A\Phi_1}\left(\frac{\rho_1}{\rho}\right)\left[1+\left(\frac{\rho_1}{\rho}\right)^2\right]\cos\phi, \tag{80}$$

$$\tau_{\rho\phi} = -\frac{k_1}{2\cos A\Phi_1}\left(\frac{\rho_1}{\rho}\right)\left[1-\left(\frac{\rho_1}{\rho}\right)^2\right]\sin\phi, \tag{81}$$

at $\rho\to\infty$, σ_ρ and $\tau_{\rho\phi}\to0$.

Let us determine the value of k_1. To do this, write an equilibrium equation for the upper cut-off part of the wedge, as it was done in [35] and in works by other authors:

$$\int_{-\alpha}^{\alpha}-\sigma_\rho\rho d\phi\cos\phi=\frac{2k_1\rho_1}{\cos\alpha}\cdot m(\rho)\int_{-\alpha}^{\alpha}\cos^2\phi d\phi=P,$$

where

$$m(\rho)=\left[1+\exp^2\left(\ln\frac{\rho_1}{\rho}\right)\right].$$

It follows that:

$$k_1=\frac{P\cos\alpha}{2\rho_1\left(\alpha+\frac{1}{2}\sin2\alpha\right)m(\rho)}. \tag{82}$$

Substitute (82) in (80), (81) and obtain:

$$\sigma_\rho = -\frac{P}{\left(\alpha+\frac{1}{2}\sin2\alpha\right)}\left(\frac{1}{\rho}\right)\cos\phi,$$

$$\tau_{\rho\phi} = -\frac{P}{4\cdot\left(\alpha+\frac{1}{2}\sin2\alpha\right)}\left(\frac{1}{\rho}\right)\frac{m'(\rho)}{m(\rho)}\sin\phi, \tag{83}$$

where

$$m'(\rho)=\left[1-\left(\frac{\rho_1}{\rho}\right)^2\right].$$

Solution of (83) does not differ from expression (77) obtained earlier and is confirmed by classical solutions. Tangential stress has slightly changed in comparison with (77). It became possible to satisfy the boundary condition for the shear stress due to the variable $m'(\rho)$.

Analysis of the obtained solutions of varying complexity, when compared with the studies by other authors, shows that (20) and (24), (52), (53), (66), (67) have a generalized character. Solutions in a particular case coincide with the results of similar developments by other authors

The conditions for the existence of solutions to various problems are invariant, both in the theory of elasticity, the theory of plasticity, and the theory of dynamic processes. The Cauchy-Riemann relations are widely used in transformations, in solutions themselves which simplifies the final result and the process of its finding. The revealed generalizations have made it possible to obtain solutions to the problem of the theory of elasticity in polar coordinates.

6. Discussion of the study results

The proposed solution features identification of differential conditions of its existence using the argument functions, that is, the Cauchy-Riemann relations and Laplace equations for polar coordinates.

The obtained study results can be explained by:
– using the method of argument functions of a complex variable;
– obtaining of invariant differential generalizations in a form of the Cauchy-Riemann relations including the solution for polar coordinates (20), (24), (52), (53), (66), (67);
– obtaining of generalizations of disparate elements of solutions in literature sources which has made it possible to identify this problem as unsolved;
– the study results were compared with the classical solution of some problems of the theory of elasticity [8–11, 18] and with solutions made by present-day authors [13, 14, 27]. The possibilities of using the proposed method in continuum mechanics were shown which, in addition to the theory of elasticity, includes the theory of plasticity and the theory of dynamic processes. The analysis shows that the proposed mathematical apparatus can be used in the theory of plastic metalworking, geomechanics, the interaction of elastic bodies, non-stationary problems associated with the transfer of interaction in a form of a wave process.

Limitations include boundaries of applicability of solutions. These approaches do not apply to solutions of the biharmonic equation using the argument functions in polar coordinates. However, this will expand the capabilities of the method. This will ensure the emergence of additional opportunities, both for solutions and implementation of boundary conditions in the continuum mechanics.

The disadvantages of the study include cumbersomeness and volume of the derivation. This is primarily explained by the lack of accumulated material on this issue.

When solving problems of the continuum mechanics, defining generalizations were revealed in the method of argument functions. However, this is not enough for using it in new problems. There is a need to extend the method and perhaps not only in problems of continuum mechanics.

7. Conclusions

1. Generalizing approaches to solving problems of the theory of elasticity using argument functions of a complex variable in polar coordinates were developed. The fundamental difference from the known solutions consists in

that they are performed in Cartesian coordinates which contain less complex differential equations. Expansion of capabilities of the method argument functions in the theory of elasticity due to its use for solving problems of polar coordinates is a qualitative indicator of the study results.

2. Generalizing Cauchy-Riemann relations and Laplace equations in polar coordinates were determined in a differential form. Identification of invariant differential relations of diverse problems of the continuum mechanics including problems of the theory of elasticity is the defining indicator of the study results.

3. Using the method of argument functions, a plane problem of the theory of elasticity was solved in polar coordinates. The fundamental difference is in the use of the argument functions of complex variables in solving the problem of the elasticity theory. The application of the method to solving more complex problems of the theory of elasticity and prediction of results is a qualitative indicator of the study results.

4. The obtained results were tested and compared with the studies of other authors. The results were compared with the results of solving applied problems of the theory of elasticity. A more general solution by the method of argument functions was simplified and reduced to a special case which was compared with the solutions obtained by the method of the stress function. At the same time, arguments of trigonometric and exponential functions used by the authors were satisfied by the Cauchy-Riemann relations which were determined in this study.

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References

1. Chigurinski, V. (1999). The study of stressed and deformed metal state under condition of no uniform plastic medium flow. Metalurgija, 38 (1), 31–37.

2. Chygyryns'kyy, V. V. (2004). Analysis of the state of stress of a medium under conditions of inhomogeneous plastic flow. Metalurgija, 43 (2), 87–93.

3. Chigirinsky, V. V., Legotkin, G. I., Slepynin, A. G., Kozlov, V. I., Dragobetsky, V. V. (2015). Mechanisms of plastic deformation in case of production of thin-walled rolled stock of the special purpose. Metallurgical and Mining Industry, Dnipropetrovsk, 11, 222–230.

4. Chigirinsky, V., Putnoki, A. (2017). Development of a dynamic model of transients in mechanical systems using argument-functions. Eastern-European Journal of Enterprise Technologies, 3 (7 (87)), 11–22. doi: https://doi.org/10.15587/1729-4061.2017.101282

5. Chygyryns'kyy, V. V., Shevchenko, V. G., Mamuzic, I., Belikov, S. B. (2010). A new solution of the harmonic-functions in the theory of elasticity. Materials and technology, 44 (4), 219–222.

6. Chigirinsky, V., Naumenko, O. (2019). Studying the stressed state of elastic medium using the argument functions of a complex variable. Eastern-European Journal of Enterprise Technologies, 5 (7 (101)), 27–35. doi: https://doi.org/10.15587/1729-4061.2019.177514

7. Tihonov, A. N., Samarskiy, A. A. (1977). Uravneniya matematicheskoy fiziki. Moscow: Nauka, 735.

8. Sedov, L. I. (2004). Mehanika sploshnoy sredy. Sankt-Peterburg: Lan', 560.

9. Mushelishvili, N. I. (1966). Nekotorye osnovnye zadachi matematicheskoy teorii uprugosti. Moscow: Nauka, 709.

10. Lur'e, A. I. (1979). Teoriya uprugosti. Moscow: Nauka, 939.

11. Timoshenko, S. P., Gud'er, Dzh. (1979). Teoriya uprugosti. Moscow: «Nauka», 560.

12. Belmas, I. V., Kolosov, D. L., Onyshchenko, S. V. (2018). Stress-strain state of rubber-cable tractive element of tubular shape. Naukovyi Visnyk Natsionalnoho Hirnychoho Universytetu, 2, 60–69. doi: https://doi.org/10.29202/nvngu/2018-2/5

13. Pozharskiy, D. A. (2017). Kontaktnaya zadacha dlya ortotropnogo poluprostranstva. Mehanika tverdogo tela, 3, 100–108.

14. Georgievskiy, D. V., Tlyustangelov, G. S. (2017). Eksponentsial'nye otsenki vozmushcheniy zhestkoplasticheskogo rastekaniya-stoka kol'tsa. Izvestiya Rossiyskoy akademii nauk. Mehanika tverdogo tela, 4, 135–144.

15. Vasil'ev, V. V., Lurie, S. A. (2017). New Solution of Axisymmetric Contact Problem of Elasticity. Mechanics of Solids, 52 (5), 479–487. doi: https://doi.org/10.3103/s0025654417050028

16. Ike, C. C. (2019). Hankel transformation method for solving the Westergaard problem for point, line and distributed loads on elastic half-space. Latin American Journal of Solids and Structures, 16 (1). doi: https://doi.org/10.1590/1679-78255313

17. Georgievskii, D. V. (2014). Compatibility equations in systems based on generalized Cauchy kinematic relations. Mechanics of Solids, 49 (1), 99–103. doi: https://doi.org/10.3103/s0025654414010117

18. Sneddon, I. N., Berri, D. S.; Grigolyuka, E. I. (Ed.) (1961). Klassicheskaya teoriya uprugosti. Moscow: Gos. izd-vo fiz.-mat. lit-ry, 219.

19. Maciolka, P., Jedrzejewski, J. (2017). Evaluation of different approaches to 3d numerical model development of the contact zone between elementary asperities and flat surface. Journal of Machine Engineering, 4 (17), 40–53. doi: https://doi.org/10.5604/01.3001.0010.7004

20. Stampouloglou, I. H., Theotokoglou, E. E. (2009). Additional Separated-Variable Solutions of the Biharmonic Equation in Polar Coordinates. Journal of Applied Mechanics, 77 (2). doi: https://doi.org/10.1115/1.3197157

21. Qian, H., Li, H., Song, G., Guo, W. (2013). A Constitutive Model for Superelastic Shape Memory Alloys Considering the Influence of Strain Rate. Mathematical Problems in Engineering, 2013, 1–8. doi: https://doi.org/10.1155/2013/248671

22. El-Naaman, S. A., Nielsen, K. L., Niordson, C. F. (2019). An investigation of back stress formulations under cyclic loading. Mechanics of Materials, 130, 76–87. doi: https://doi.org/10.1016/j.mechmat.2019.01.005

23. Lopez-Crespo, P., Camas, D., Antunes, F. V., Yates, J. R. (2018). A study of the evolution of crack tip plasticity along a crack front. Theoretical and Applied Fracture Mechanics, 98, 59–66. doi: https://doi.org/10.1016/j.tafmec.2018.09.012

24. Li, J., Zhang, Z., Li, C. (2017). Elastic-plastic stress-strain calculation at notch root under monotonic, uniaxial and multiaxial loadings. Theoretical and Applied Fracture Mechanics, 92, 33–46. doi: https://doi.org/10.1016/j.tafmec.2017.05.005

25. Pathak, H. (2017). Three-dimensional quasi-static fatigue crack growth analysis in functionally graded materials (FGMs) using coupled FE-XEFG approach. Theoretical and Applied Fracture Mechanics, 92, 59–75. doi: https://doi.org/10.1016/j.tafmec.2017.05.010

26. Correia, J. A. F. O., Huffman, P. J., De Jesus, A. M. P., Cicero, S., Fern ndez-Canteli, A., Berto, F., Glinka, G. (2017). Unified two-stage fatigue methodology based on a probabilistic damage model applied to structural details. Theoretical and Applied Fracture Mechanics, 92, 252–265. doi: https://doi.org/10.1016/j.tafmec.2017.09.004

27. Sinekop, N. S., Lobanova, L. S., Parhomenko, L. A. (2015). Metod R-funktsiy v dinamicheskih zadachah teorii uprugosti. Kharkiv: KhGUPT, 95.

28. Radeev, Yu. N. (2007). Prostranstvennaya zadacha matematicheskoy teorii plastichnosti. Samara: Iz-vo Samarskiy universitet, 464.

29. Mehtiev, M. F. (2008). Asimptoticheskiy analiz nekotoryh prostranstvennyh zadach teorii uprugosti dlya polyh tel. Baku: NAN Azerbaydzhana, 320.

30. Krupoderov, A. V., Shcherbakov, S. S. (2013). Reshenie nekotoryh dinamicheskih zadach teorii uprugosti metodom granichnyh elementov. Teoreticheskaya i prikladnaya mehanika, 28, 294–300.

31. Hussein, N. S. (2014). Solution of a Problem Linear Plane Elasticity with Mixed Boundary Conditions by the Method of Boundary Integrals. Mathematical Problems in Engineering, 2014, 1–11. doi: https://doi.org/10.1155/2014/323178

32. Malinin, N. N. (1975). Prikladnaya teoriya plastichnosti i polzuchesti. Moscow: Mashinostroenie, 400.

33. Papargyri-Beskou, S., Tsinopoulos, S. (2014). Lamé's strain potential method for plane gradient elasticity problems. Archive of Applied Mechanics, 85 (9-10), 1399–1419. doi: https://doi.org/10.1007/s00419-014-0964-5

34. Zhemochkin, B. N. (1948). Teoriya uprugosti. Moscow: Stroyvoenmorizdat, 240.

35. Nikiforov, S. N. (1955). Teoriya uprugosti i plastichnosti. Moscow: GILSI, 284.

36. Bezuhov, N. I. (1968). Osnovy teorii uprugosti, plastichnosti i polzuchesti. Moscow: Vysshaya shkola, 512.