

*This paper proposes a theoretically substantiated and universal new method to calculate the three-dimensional stressed-strained state of the statically loaded multi-link orthotropic shell of arbitrary thickness, made of heterogeneous material (a composite). The numerical-analytical RVR method used in this work is based on the Reissner principle, Vekua method, the R-function theory, as well as the algorithm of two-way assessment of the accuracy of approximate solutions to variational problems. In contrast to the classical principles by Lagrange and Castigliano, the application of the mixed variational Reissner principle yields an increase in the accuracy of solving boundary-value problems due to the independent variation of the displacement vector and the stress tensor. Vekua method makes it possible, as a result of expanding the desired functions into a Fourier series based on Legendre polynomials, to replace a solution to the three-dimensional problem with a regular sequence of solutions to the two-dimensional problems in the process of refining the models of shells. The R-function theory that takes into consideration, at the analytical level, the geometric information on boundary-value problems for multi-relationship regions is necessary to build the structures of solutions that accurately meet different boundary conditions. When studying spatial boundary-value problems, the constructed algorithm for a two-way integrated assessment of the accuracy of approximate solutions makes it possible to automate the search for such a number of approximations at which the process of solutions' convergence becomes persistent. For an orthotropic spherical shell made from the material of non-uniform thickness and weakened by the pole holes, the RVR-method capabilities are shown on the numerical examples of solving the relevant boundary-value problems. The results of the reported research have been discussed, as well as the features typical of the new method, which could be effectively applied when designing responsible shell-type elements of structures in the different sectors of modern industry*

**Keywords:** *orthotropic shell of inhomogeneous thickness with holes, Reissner principle, R-function theory*

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# CALCULATION OF THE SPHERICAL ELEMENTS OF NON-UNIFORM THICKNESS FOR STRUCTURES WITH HOLES BASED ON THE VARIATIONAL RVR-METHOD

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## 1. Introduction

The widespread use of composite materials whose structure is heterogeneous in the various sectors of modern industry is one of the promising areas for improving existing structures, as well as for designing new ones. The intensive increase in the application of composites is due to a significant decrease in the mass of responsible elements

in articles, as well as an increase in their performance and reliability.

Engineering practice also often involves such structural elements as the elastic shells, weakened by holes (cutouts), whose strength and rigidity may affect the performance and reliability of a structure in general. In this regard, we note that the mechanics of a deformable solid still consider relevant those issues that relate to solving the boundary-value problems

for multi-connected anisotropic shells in a three-dimensional statement. Solving such problems is typically associated with significant mathematical and computational complexity, which must be overcome when performing specific calculations. Therefore, it is a relevant and practically significant scientific task to devise effective and reliable methods for calculating the strength and rigidity of anisotropic shells of inhomogeneous thickness with holes of arbitrary shapes and sizes.

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## 2. Literature review and problem statement

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Studying the spatial boundary-value problems involving heterogeneous shells weakened by holes is immeasurably more difficult than similar problems for homogeneous shells. This is primarily due to that the solving equations include variable coefficients, which depend on the coordinates of the study area. At the same time, significant progress in resolving the issue of calculating the stressed-strained state of structures made from heterogeneous materials (composites) is impossible without the use of reliable methods involving the basic ratios from a three-dimensional theory of elasticity. Thus, study [1] analyzes the interlayer normal stress in curved layered components considering the need for mandatory use of the three-dimensional model. The three-dimensional stressed state of a multi-layered composite rod under the influence of torque was investigated in [2], whose authors applied a layered displacement method while their numerical results are compared with the analytical solutions based on the Kantorovich method. The three-dimensional analysis of stresses in plate structures under conditions of instability is reported in work [3]. The behavior of nonlinear stability of rigid multi-layered composite sloping shells is estimated in [4] by using a new semi-analytic method.

It is known that the construction of direct approximate algorithms to solve the boundary-value problems in a shell theory has increasingly involved different variational methods [5]. Study [6] reported a Jacobi-Ritz approach for the dynamic analysis of composite sloping shells with arbitrary boundary conditions. The functions of each segment displacement for shells are represented by the Jacobi polynomial function while the Ritz method, used for the solution procedure, is based on the energy approach. The new results of the free oscillations of layered shells are reported; the accuracy and reliability of the methodology are confirmed by comparing it with the scientific literature. Paper [7] uses the Ritz method to investigate the free oscillations of combined spherical and cylindrical shells of a heterogeneous thickness. The energy method and the theory of the deformation of the first order displacement were used to derive a semi-analytic solution. At the same time, to prove the validity of the reported results, they are compared with the results obtained by using a finite-element method (FEM) and experiments.

Study [8] suggests a generalized finite-difference method (GFDM) to analyze the stresses of three-dimensional composite elastic materials. In the authors' calculations, the composite material is split into several sub-regions, along the boundaries of which the conditions of compatibility of movements and the balance of efforts are imposed. It is noted that such an approach leads to a system with sparse matrices containing up to 500,000 unknowns; the reported numerical example is solved using the developed GFDM code. A new element of the continual shell is proposed in [9] for

the three-dimensional simulation of multi-layered shell-type structures of arbitrary geometry when using FEM.

The boundary-value problems on shells whose structure is heterogeneous for thickness are tackled in [10, 11]. In particular, the Vekua method [12] was applied in [10] to derive equations of the equilibrium of the heterogeneous isotropic spherical shell of constant thickness; [11] investigated the strained state of a sloping transversal-isotropic spherical shell with a circular opening. It should be noted that the presence of holes in the designed anisotropic article leads to the need to take into consideration the significant impact of stress concentrators on the carrying capacity of the examined shell-type structure [13]. In particular, the results of work [14] show that that the exact three-dimensional characteristic of tense states is important for assessing the structural integrity and predicting the fatigue growth of end-to-end cracks emanating from the hole.

In the mechanics of a deformable solid, of relevance are the issues related to solving boundary-value problems for anisotropic shells with holes of arbitrary shapes and sizes in a three-dimensional statement [15]. One of the promising possibilities of the new numerical-analytical method (termed an RVR-method), proposed in work [15], is its application to calculate heterogeneous multi-link shells for the case when the dependences of elastic characteristics of the material are known. The RVR method, which was theoretically substantiated in [16, 17], is based on the Reissner principle [18–20], a Vekua method [12], the R-function theory [21, 22], as well as a two-way algorithm for assessing the accuracy of approximate solutions to variational problems. The application of the mixed variational Reissner principle, in contrast to the classical principles by Lagrange and Castigliano, leads to an increase in the accuracy of solving boundary-value problems due to the independent variation of the displacement vector and the stress tensor. The Vekua method makes it possible to replace a solution to the spatial boundary-value problem from the linear theory of elasticity [23] with a regular sequence of solutions to the two-dimensional tasks in the software-implemented algorithm for refining the models of shell deformation. The R-function theory mathematical apparatus is necessary to create the analytical structures of solutions that precisely meet different variants of the boundary conditions set on the boundary surfaces of the shells. It is important to note that the effectiveness of the proposed RVR method for deriving reliable results was tested in resolving a large number of technical tasks studied by other authors. In particular, paper [13] shows a satisfactory match between the RVR-based numerical values of the stress concentration factor and the experimental data reported in the scientific literature for an isotropic cylinder with an elliptical hole.

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## 3. The aim and objectives of the study

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The aim of this study is to calculate the stressed-strained state of a statically loaded orthotropic spherical shell of inhomogeneous thickness with holes based on the variational RVR-method.

To accomplish the aim, the following tasks have been set:

- to build the structures of solutions that precisely meet all the boundary conditions of the examined area of a heterogeneous shell with holes;
- by using the RVR-method, derive numerical results to assess the effect of the heterogeneity of the material of a spherical thick-walled shell on its stressed-strained state.

**4. Materials and methods to study the orthotropic spherical shells of inhomogeneous thickness with holes**

**4.1. Examples of the dependences of a shell's elastic characteristics when considering a one-dimensional variant of the material's heterogeneity**

Consider an elastic shell whose Poisson coefficients  $\nu_{ij}$  are constant; the moduli of elasticity  $E_i$  and shear  $G_{ij}$  (at  $i \neq j=1,2,3$ ) are the arbitrary functions of the coordinate  $\zeta$  along the thickness of the shell (at  $|\zeta| \leq 1$ ):

$$E_i = E_i^0 f_i(n, \delta, \zeta); \quad G_{ij} = G_{ij}^0 f_{ij}(n, \delta, \zeta), \quad (1)$$

where the constant quantities  $E_i^0, G_{ij}^0$  are the values of elastic characteristics of a homogeneous shell;  $n, \delta$  are the parameters of heterogeneity of the material.

Note that functions (1) belong to the most common one-dimensional variant of heterogeneity reported in the scientific literature [10, 11]. When applying an RVR method [15], considering the dependence of elastic shell characteristics on two or three coordinates could only create time-cost-related difficulties in the numerical implementation of the set boundary-value problem. For the material of a particular heterogeneous shell, the functions  $f_i(n, \delta, \zeta)$  and  $f_{ij}(n, \delta, \zeta)$  in ratios (1) take a different form. Consider the following analytical dependences  $E_i$  (similar to  $G_{ij}$ ):

$$\left. \begin{aligned} (a): \quad E_i &= E_i^0 \left[ 1 + \delta(1 + \zeta)^n \right]; \\ (b): \quad E_i &= E_i^0 e^{\delta(1+\zeta)^n}; \\ (c): \quad E_i &= E_i^0 \frac{e^{1+\zeta}}{1 + \delta(1 + \zeta)^n}; \\ (d): \quad E_i &= E_i^0 \left\{ 1 + \ln \left[ 1 + \delta(1 + \zeta)^n \right] \right\}. \end{aligned} \right\} \quad (2)$$

Here, the distribution of the  $E_i$  value along the thickness of the shell corresponds to the power (2a), exponential (2b), exponential-power (2c), and logarithmic (2d) laws of change.

**4.2. Deriving an analytical expression of the Reissner variational equation for orthotropic spherical shells**

To improve the accuracy of solving boundary-value problems, it is advisable to independently determine parameters of the stressed and strained state, which could be implemented by applying the variational Reissner principle with the functional  $I_R$  [18–20]. This functional has the property that the  $u_i$  components of the displacement vector  $\mathbf{u}$  and the  $\sigma_{ij}$  components of the stress tensor  $\sigma$  could be set arbitrarily disregarding the implementation of boundary conditions, equilibrium equations, and continuity conditions. In this case, due to the independent approximation of  $\mathbf{u}$  and  $\sigma$ , the variational Reissner equation  $\delta I_R = 0$  yields a system of differential equations of the first order relative to the desired quantities. In turn, the equations of classical variational statements have a higher order, require laborious mathematical operations, and significantly complicate the structures of solutions that accurately meet the boundary conditions of the problem.

The numerical implementation of the Reissner variational statement was significantly hindered by difficulties in assessing the accuracy of solutions caused by the lack of an extremum at the stationary point of the Reissner functional  $I_R$ . This issue was resolved by the theorem proven in studies [15, 16]: «The sequences of Ritz method coincide with the exact solution to a boundary-value problem stated on the

basis of the Reissner principle if the structures of the solutions accurately meet all boundary conditions». Therefore, the RVR-method employs R-functions [21, 22] to account, at the analytical level, for the geometric information of the examined boundary-value problems for multi-linked regions and to build the structures of solutions that precisely meet different variants of the boundary conditions.

We shall use the RVR-method to calculate the stressed-strained state of an orthotropic spherical shell with two coaxial circular holes of the same radius  $r_0$ , loaded by the constant internal pressure of intensity  $q_0$  [24]. We introduce a system of curvilinear coordinates  $\vartheta, \varphi$ , and  $z$  in the middle surface  $\Omega_s$  of the radius  $R$  of the shell with a structure  $h$  heterogeneous in thickness (Fig. 1). Here,  $\vartheta$  is the angle between the normal  $\mathbf{n}$  to  $\Omega_s$  and the rotation axis;  $\varphi$  is the angle between the fixed meridian plane and the meridian plane passing through the point in question  $\Omega_s$ ;  $z$  is the distance along the normal  $\mathbf{n}$  ( $-h/2 \leq z \leq h/2$ ). At the same time, the elastic-equivalent directions of the orthotropy of the material coincide with the lines of the main curvature of the spherical shell (its meridians and parallels).

Let  $\vartheta_0 = \pi/2$  be the angle that determines the plane against which the pole holes are symmetrical. We place at the intersection between the middle surface  $\Omega_s$  and the plane  $\vartheta_0 = \pi/2$  the origin of the dimensionless coordinate  $\vartheta = (\vartheta - \vartheta_0) / (\vartheta_0 - \vartheta_1)$ . The  $\vartheta$  value interval is determined by the expression  $|\vartheta| \leq 1$  when the  $\vartheta$  angle changes from  $\vartheta = \vartheta_1$  to  $\vartheta = \vartheta_2$  (respectively, the equations of the lateral surfaces of the two circular holes). Due to the symmetrical arrangement of holes relative to the  $\vartheta = \vartheta_0$  plane and the axisymmetric loading of the shell, its calculation comes down to studying the meridian section  $\Omega$ . In this case, the analytical form of the Reissner variational equation (a stationarity condition of the functional  $I_R$ ) in the coordinate system  $\{\vartheta, \varphi, z\}$  (Fig. 1) takes the following form at  $u_2 = 0, \sigma_{12} = 0$ , and  $\sigma_{23} = 0$  (at  $\chi = 1 + z/R$ ):

$$\iint_{\Omega} \left\{ \begin{aligned} & - \left[ \frac{1}{R\chi} \left[ \frac{\partial \sigma_{11}}{\partial \theta} + (\sigma_{11} - \sigma_{22}) \times \right. \right. \\ & \left. \left. \times \text{ctg} \theta + \sigma_{13} \right] + \frac{1}{\chi^2} \frac{\partial \chi^2 \sigma_{13}}{\partial z} \right] \delta u_1 - \\ & - \left[ \frac{1}{R\chi} \left( \frac{\partial \sigma_{13}}{\partial \theta} + \sigma_{13} \text{ctg} \theta - \right. \right. \\ & \left. \left. - \sigma_{11} - \sigma_{22} \right) + \frac{1}{\chi^2} \frac{\partial \chi^2 \sigma_{33}}{\partial z} \right] \delta u_3 + \\ & + \left[ \frac{1}{R\chi} \left( \frac{\partial u_1}{\partial \theta} + u_3 \right) - \frac{1}{E_1} \sigma_{11} + \right. \\ & \left. + \frac{\nu_{12}}{E_2} \sigma_{22} + \frac{\nu_{13}}{E_3} \sigma_{33} \right] \delta \sigma_{11} + \\ & + \left[ \frac{1}{R\chi} (u_1 \text{ctg} \theta + u_3) - \frac{1}{E_2} \sigma_{22} + \right. \\ & \left. + \frac{\nu_{12}}{E_2} \sigma_{11} + \frac{\nu_{23}}{E_3} \sigma_{33} \right] \delta \sigma_{22} + \\ & + \left[ \frac{\partial u_3}{\partial z} - \frac{1}{E_3} \sigma_{33} + \frac{\nu_{13}}{E_3} \sigma_{11} + \frac{\nu_{23}}{E_3} \sigma_{22} \right] \delta \sigma_{33} + \\ & + \left[ \frac{1}{R\chi} \left( \frac{\partial u_3}{\partial \theta} - u_1 \right) + \frac{\partial u_1}{\partial z} - \frac{1}{G_{13}} \sigma_{13} \right] \delta \sigma_{13} \end{aligned} \right\} \times \quad (3)$$

$$\times 2\pi R^2 \chi^2 \sin \theta d\theta dz = 0,$$

where  $\nu_{12}, \nu_{13}, \nu_{23}$  are the Poisson coefficients;  $E_1, E_2, E_3$  are the Young moduli in the main directions of the shell's orthotropy;  $G_{13}$  is the shear module.

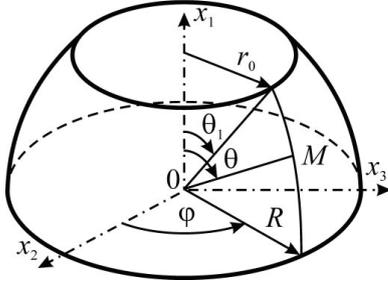


Fig. 1. The estimated elastic area of a spherical shell

### 5. The results of studying the stated boundary-value problems

#### 5.1. Building the structures of solutions to the examined problems

The entire surface of the spherical shell under study consists of the side surfaces  $\Gamma_s$  ( $\vartheta = \vartheta_s$ ,  $s=1,2$ ) of two pole circular openings and an internal  $\Gamma^-$  ( $z = -h/2$ ) and an external  $\Gamma^+$  ( $z = h/2$ ) surfaces, whose equations are determined by the functions  $\Omega_s$  and  $\Omega_+$ ,  $\Omega_-$  (at  $\zeta = 2z/h$ ,  $|\zeta| \leq 1$ ):

$$\omega_1 = \frac{\theta^2}{\theta_1^2} - 1; \quad \omega_2 = 1 - \frac{\theta^2}{\theta_2^2}; \quad \omega_- = 1 + \zeta; \quad \omega_+ = 1 - \zeta. \quad (4)$$

Introduce a dimensionless coordinate  $\vartheta = [(\vartheta_2 - \vartheta_1)\vartheta + \vartheta_1 + \vartheta_2]/2$  at  $|\vartheta| \leq 1$ . Then, it is possible to replace the functions  $\Omega_1$  and  $\Omega_2$  that determine the equations of the surfaces  $\Gamma_s$  ( $s=1, 2$ ), in the case of similar holes ( $\vartheta_2 = \pi - \vartheta_1$ ), when displacing the coordinate origin to section  $\vartheta = \pi/2$ , with one function  $\Omega_{12} = 1 - \vartheta^2$ . The structures of solutions for the independently variable, in the Reissner functional  $I_R$ , desired components of the displacements  $u_i$  and the stresses  $\sigma_{ij}$  would be represented, for the set axisymmetric problem, by the following finite series:

$$\left. \begin{aligned} u_1 &= \sum_{k=0}^{l_1-1} \sum_{p=0}^{m_1} a_1^{n_1} S_p(\vartheta) P_k(\zeta); \\ u_3 &= \sum_{k=0}^{l_3-1} \sum_{p=0}^{m_3} a_3^{n_3} C_p(\vartheta) P_k(\zeta); \\ \chi \sigma_{11} &= \omega_{12} \sum_{k=0}^{l_{11}-1} \sum_{p=0}^{m_{11}} a_{11}^{n_{11}} C_p(\vartheta) P_k(\zeta); \\ \chi \sigma_{22} &= \sum_{k=0}^{l_{22}-1} \sum_{p=0}^{m_{22}} a_{22}^{n_{22}} C_p(\vartheta) P_k(\zeta); \\ \chi \sigma_{13} &= \sigma_{13}^* + \omega_\zeta \omega_{12} \sum_{k=0}^{l_{13}-1} \sum_{p=0}^{m_{13}} a_{13}^{n_{13}} S_p(\vartheta) P_k(\zeta); \\ \chi^2 \sigma_{33} &= \sigma_{33}^* + \omega_\zeta \sum_{k=0}^{l_{33}-1} \sum_{p=0}^{m_{33}} a_{33}^{n_{33}} C_p(\vartheta) P_k(\zeta), \end{aligned} \right\} \quad (5)$$

where  $a_i^{n_i}$ ,  $a_{ij}^{n_{ij}}$  are the desired constants;  $C_p(\vartheta)$  and  $S_p(\vartheta)$  are the even and odd approximating functions of the coordinate  $\vartheta$ ;  $P_k(\zeta)$  are the Legendre polynomials of the coordinate  $\zeta$ ;  $\Omega_\zeta = \Omega_+ \Omega_-$ ;  $n_i = k(m_i + 1) + p + 1$ ;  $n_i \rightarrow n_{ij}$ .

Consider a common case in technology when the holes are closed with lids that transmit to the shell only the action of the cutting force  $Q_0^*$  whose magnitude could be determined from a lid equilibrium condition:

$$2\pi r_0 Q_0^* + q_0 \pi r_0^2 = 0. \quad (6)$$

The function  $\sigma_{13}^*$  in structure (5), due to the asymmetry of the transverse tangent stress value  $\sigma_{13}$  relative to the  $\vartheta$  coordinate, is set in the following form:

$$\sigma_{13}^* = -\frac{3Q_0^*}{2h} (1 - \zeta^2) \vartheta = \frac{3q_0 r_0}{4h} (1 - \zeta^2) \vartheta. \quad (7)$$

Note that at the surface  $\Gamma_s$  of the hole ( $\vartheta = \pm 1$ ) the stress  $\sigma_{13} = \sigma_{13}^*/\chi$  is statically equivalent to an integrated characteristic – the force  $Q_0^*$ . The analytical expression of the function  $\sigma_{33}^*$  in (5) takes the following form:

$$\sigma_{33}^* = -\frac{1}{2} (1 - h/2R)^2 \omega_+ q_0. \quad (8)$$

The boundary conditions on the holes' surfaces  $\Gamma_s$  ( $s=1,2$ ), as well as at the face internal  $\Gamma^-$  ( $\zeta = -1$ ) and external  $\Gamma^+$  ( $\zeta = 1$ ) surfaces of the shell:

$$\left. \begin{aligned} \Gamma_s : \sigma_{11} &= 0; \quad \sigma_{13} = \sigma_{13}^*/\chi, \\ \Gamma^+ : \sigma_{13} &= 0; \quad \sigma_{33} = 0, \\ \Gamma^- : \sigma_{13} &= 0; \quad \sigma_{33} = -q_0. \end{aligned} \right\} \quad (9)$$

Similarly to works [13, 15], in structures (5), the numbers  $l_i$ ,  $l_{ij}$  ( $i=1,2$ ),  $l_{13}$ ,  $l_{33}$  of the approximations of the desired displacements and stresses for the shell thickness determine its shift model when setting a combination of quantities ( $l_1, l_3, l_{ij}, l_{13}, l_{33}$ ). In this case,  $l_1$  is the number of terms held in expanding, along the  $z$  coordinate, the meridian displacement  $u_i$ ;  $l_3$  – the normal displacement  $u_3$ ;  $l_{ij}$  – tangential stresses  $\sigma_{ij}$ ;  $l_{13}$  – the transverse tangential stresses  $\sigma_{13}$  and  $l_{33}$  – the transverse normal stress  $\sigma_{33}$ . In the [12] terminology, in case  $l_1 = l_{ij}$ , the value  $N = l_1 - 1$  is a parameter that characterizes the order of the  $N$ -th approximation in the considered theory of shells.

#### 5.2. Deriving numerical results to assess the effect of the heterogeneity of a spherical thick-walled shell's material on its stressed-strained state

After fitting a solution structure (5) to the Reissner variational equation (3) and following the numerical integration of double integrals, the examined boundary-value problem is reduced to solving a system of linear algebraic equations relative to the desired constants  $a_i^{n_i}$  and  $a_{ij}^{n_{ij}}$ . Using their derived values, we determine all the characteristics of the stressed-strained state of the estimated area of a shell of inhomogeneous thickness.

Consider the orthotropic spherical belt with the following parameters:

$$R = 0.6 \text{ m}; \quad \theta_1 = \pi/4; \quad \theta_2 = 3\pi/4; \quad q_0 = 1 \text{ MPa}. \quad (10)$$

In this case, the Poisson coefficients are constant ( $\nu_{ij} = 0.3$ ), and the moduli of elasticity  $E_i$  and shear  $G_{13}$  are the functions of the coordinate  $\zeta$  of form (2), where the constant quantities  $E_i^0$  and  $G_{13}^0$  accept the values of the elastic characteristics of a homogeneous shell:

$$E_1^0 = E_3^0 = 20 \text{ GPa}; \quad G_{13}^0 = 1 \text{ GPa}. \quad (11)$$

According to expressions (2), we shall focus on the following distribution options, used in the calculations (at the assigned  $n$  and  $\delta$  values), of the functions  $f_i(\zeta) = E_i/E_i^0$  (similar to  $f_{ij}(\zeta) = G_{ij}/G_{ij}^0$ ) along the thickness of the shell:

$$\left. \begin{aligned}
 \text{(a): } f_i(\zeta) &= 1 - 0.25(1 + \zeta); \\
 \text{(b): } f_i(\zeta) &= 1 - 0.0625(1 + \zeta)^3; \\
 \text{(c): } f_i(\zeta) &= e^{0.25(1 + \zeta)^2 \ln 2}; \\
 \text{(d): } f_i(\zeta) &= \frac{e^{1 + \zeta}}{1 + (e^2 - 0.5)(1 + \zeta)}; \\
 \text{(e): } f_i(\zeta) &= 1 + \ln[1 + 0.5(e - 1)(1 + \zeta)].
 \end{aligned} \right\} \quad (12)$$

A visual illustration of the  $f_i(\zeta)$  function distribution along the shell's thickness ( $|\zeta| \leq 1$ ) is shown in Fig. 2 for the case of a change (increase or decrease) in the elasticity module in half. For a convenient comparison and analysis of the results obtained, the  $\delta$  parameter accepts a value for each considered function  $f_i(\zeta)$  at which equality  $E_i|_{\zeta=1} = 2E_i^0$  (or  $E_i|_{\zeta=-1} = 0.5E_i^0$ ) holds. The letter designations  $s_1, s_3, e_2, es_1$ , and  $ln_1$  next to charts in Fig. 2 correspond to the linear (12a), cubic (12b), exponential (12c), exponential-power (12d), and logarithmic (12e) laws of change in the ratio  $E_i/E_i^0$ .

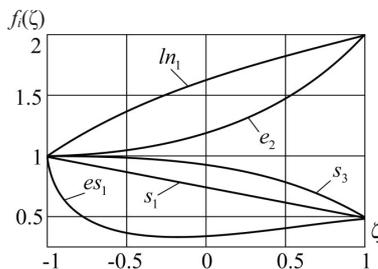


Fig. 2. Distribution charts of functions  $f_i(\zeta)$  (12) for the shell thickness

Introduce the reduced displacements  $\tilde{u}_i$  and stresses  $\tilde{\sigma}_{ij}$ , as well as the dimensionless coordinate  $r_\theta$ :

$$\tilde{u}_i = \frac{100 u_i}{h}; \quad \tilde{\sigma}_{ij} = \frac{\sigma_{ij}}{q_0}; \quad r_\theta = \frac{r_0 + (\theta - \theta_1)R}{r_0} = 1 + \frac{\theta - \theta_1}{\sin \theta_1}. \quad (13)$$

The quantities  $\tilde{u}_i$  and  $\tilde{\sigma}_{ij}$  accept, when calculating them at  $\zeta = -1, \zeta = 0$ , and  $\zeta = 1$ , the form of  $\tilde{u}_i^-, \tilde{u}_i^0$  or  $\tilde{u}_i^+$  (similarly,  $\tilde{\sigma}_{ij}^-, \tilde{\sigma}_{ij}^0$ , or  $\tilde{\sigma}_{ij}^+$ ). Designate  $p_0 = q_0 R / 2h$ , then  $\tilde{\sigma}_{11}(\tilde{\sigma}_\theta)$ , and  $\tilde{\sigma}_{22}(\tilde{\sigma}_\varphi)$  coincide with an accuracy to the multiplier  $2h/R$  with the following coefficients of the concentration of the meridian  $k_\theta$  and circumferential  $k_\varphi$  stresses:

$$\begin{aligned}
 k_\theta &= \frac{\sigma_\theta}{p_0} = \frac{\sigma_{11}}{q_0} \frac{2h}{R} = \tilde{\sigma}_{11} \frac{2h}{R}; \\
 k_\varphi &= \frac{\sigma_\varphi}{p_0} = \frac{\sigma_{22}}{q_0} \frac{2h}{R} = \tilde{\sigma}_{22} \frac{2h}{R}.
 \end{aligned} \quad (14)$$

At  $\zeta = -1, \zeta = 0$ , and  $\zeta = 1$ , quantities (14) take the form  $k_\theta^-, k_\theta^0$  and  $k_\theta^+$  (similarly,  $k_\varphi$ ). Following the numerical implementation of the problem, Fig. 3, 4 show, for an orthotropic ( $E_2 = 0.5E_1$ ) thick-walled ( $h/R = 0.2$ ) shell, along the  $r_\theta$  coordinate, the displacement charts of  $\tilde{u}_1^0, \tilde{u}_3^0$  (13) and coefficients  $k_\theta^-$  and  $k_\theta^+$  (14) for the case of using a shell's model of the fifth approximation at  $l_1 = l_{ii} = 5$  in (5).

Symbols 1, 2, 3, 4, and 5 next to the charts correspond to the dependences  $s_1$  (12a),  $s_3$  (12b),  $e_2$  (12c),  $es_1$  (12d), and  $ln_1$  (12e), built in Fig. 2; the bar line shows the chart for a homogeneous spherical shell.

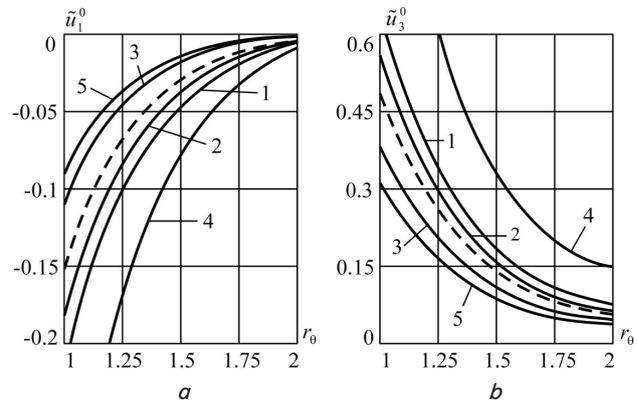


Fig. 3. Displacement distribution charts: a – for  $\tilde{u}_1^0$ ; b – for  $\tilde{u}_3^0$

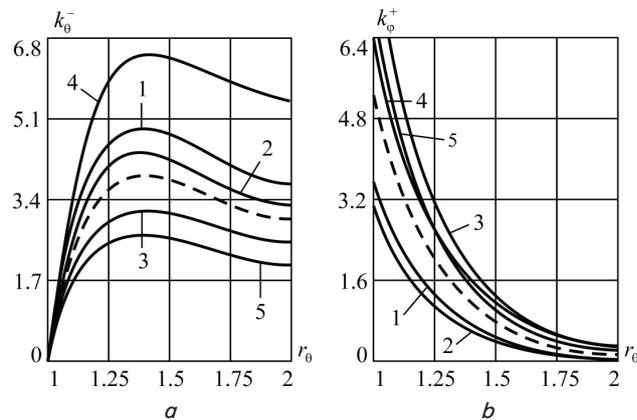


Fig. 4. Coefficient distribution charts: a – for  $k_\theta^-$ ; b – for  $k_\theta^+$

### 6. Discussion of results of calculating the stressed-strained state of a heterogeneous spherical shell with holes

By using the proposed RVR-method we derived numerical results of solving complex spatial problems for a heterogeneous spherical shell under different laws of continuous change in the elastic characteristics for thickness. The calculations were performed when considering the linear, cubic, exponential, exponentially-power, and logarithmic laws of change in the elasticity module. It follows from the shown graphic results (Fig. 3, 4) that the degree of heterogeneity of the elastic shell's material significantly affects its strength. In particular, it was found that the stressed-strained state of the examined shell significantly depends on how much the law of a continuous change in the elastic characteristics of the shell material for thickness would differ from their linear distribution.

It should be noted that the means of verifying the validity of the results reported in our study was a software-implemented algorithm of *a posteriori* integrated double assessment of the accuracy of approximate solutions to mixed variational problems [17]. Its employment in the variational RVR-method allows for the automated search, within structure (5), of such a number of approximations at which the process of solutions' convergence is robust while the ultimate results become reliable.

Note that this work reports the calculation of a heterogeneous spherical shell. The next stage to advance the possibilities of the proposed RVR-method implies the study of a shell of an arbitrary Gaussian curvature (in particular, numerical computation of cylindrical and torus-shaped shells).

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## 7. Conclusions

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1. Based on the non-extreme Reissner principle, we have formulated a variational statement of spatial boundary-value problems for a statically loaded heterogeneous orthotropic spherical shell weakened by the curvilinear holes of arbitrary shape and size. For the shell in question, the analytical expressions have been given for a Reissner variational equation and for the structures of solutions that accurately meet all the boundary conditions of the examined elastic area of a shell of arbitrary thickness.

2. By using the RVR-method, we have demonstrated the numerical calculations that are of interest to engineer-

ing practice when assessing the effect of the heterogeneity of the spherical shell's material on its stressed-strained state. The analysis of these results confirms the effective use of the RVR-method [13, 15] in solving complex spatial problems on the heterogeneous shells with holes, the concentration of stresses near which could significantly affect the carrying capacity of structures. The new numerical-analytical RVR-method for calculating non-thin homogeneous and continuously heterogeneous shells with holes could be used in the design of responsible shell-type elements of structures in the various sectors of modern industry.

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*This paper reports the synthesized two-mass antiphase resonance vibratory machine with a vibration exciter in the form of a passive auto-balancer. In the vibratory machine, platforms 1 and 2 are viscoelastically attached to the stationary bed and are tied together viscoelastically. A passive auto-balancer is mounted on platform 2.*

*It has been established that the vibratory machine has two resonant frequencies and two corresponding forms of platform oscillations. Such values for the supports' parameters have been analytically selected at which:*

*– there is an antiphase mode of motion at which platforms 1 and 2 oscillate in the opposite phase and the principal vector of forces acting on the bed (when disregarding the forces of gravity) is zero;*

*– the frequency of platform oscillations under an antiphase mode coincides with the second resonance frequency.*

*The antiphase mode occurs when the loads in an auto-balancer get stuck in the vicinity of the second resonance frequency, which is caused by the Sommerfeld effect.*

*The dynamic characteristics of a vibratory machine have been investigated by numerical methods. It has been established that in the case of small internal and external resistance forces:*

*– there are five theoretically possible modes of load jamming;*

*– the antiphase (second) form of platform oscillations is theoretically implemented under jamming modes 3 and 4;*

*– jamming mode 3 is locally asymptotically stable while jamming mode 4 is unstable;*

*– for the loads to get stuck in the vicinity of the second resonance frequency, the vibratory machine must be provided with the initial conditions close to jamming mode 3, or the rotor must be smoothly accelerated to the working frequency;*

*– the dynamic characteristics of the vibratory machine during operation can be controlled in a wide range by changing both the rotor speed and the number of loads in the auto-balancer.*

*The reported results are applicable for the design of resonant antiphase two-mass vibratory machines for general purposes*

*Keywords: inertial vibration exciter, resonant vibrations, antiphase vibratory machine, auto-balancer, two-mass vibratory machine, Sommerfeld effect*

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# SYNTHESIZING A RESONANCE ANTI- PHASE TWO-MASS VIBRATORY MACHINE WHOSE OPERATION IS BASED ON THE SOMMERFELD EFFECT

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## 1. Introduction

Among the vibratory machines for various applications, resonance vibratory machines appear promising [1]. In them, vibration exciters of lower mass generate vibrations at a greater

amplitude, thereby improving the energy efficiency, reliability, and durability of vibratory machine operation. The basic principles of designing such machines were considered in [1, 2].

The inertial vibration exciters of a lower mass excite more intense vibrations compared to electromagnetic vibration