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DEVELOPMENT OF A METHOD TO LINEARIZE THE QUADRATIC ASSIGNMENT PROBLEM

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The paper presents a new powerful technique to linearize the quadratic assignment problem. There are so many techniques available in the literature that are used to linearize the quadratic assignment problem. In all these linear formulations, both the number of variables and the linear constraints significantly increase. The quadratic assignment problem (QAP) is a well-known problem whereby a set of facilities are allocated to a set of locations in such a way that the cost is a function of the distance and flow between the facilities. In this problem, the costs are associated with a facility being placed at a certain location. The objective is to minimize the assignment of each facility to a location. There are three main categories of methods for solving the quadratic assignment problem. These categories are heuristics, bounding techniques and exact algorithms. Heuristics quickly give near-optimal solutions to the quadratic assignment problem. The five main types of heuristics are construction methods, limited enumeration methods, improvement methods, simulated annealing techniques and genetic algorithms. For every formulated QAP, a lower bound can be calculated. We have Gilmore-Lawler bounds, eigenvalue related bounds and bounds based on reformulations as bounding techniques. There are four main classes of methods for solving the quadratic assignment problem exactly, which are dynamic programming, cutting plane techniques, branch and bound procedures and hybrids of the last two. The QAP has application in computer backboard wiring, hospital layout, dartboard design, typewriter keyboard design, production process, scheduling, etc. The technique proposed in this paper has the strength that the number of linear constraints increases by only one after the linearization process

Keywords: quadratic assignment problem, Koopmans and Beckmann formulation, linear binary form

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1. Introduction

The quadratic assignment problem (QAP) is a well-known problem and this is a problem whereby a set of facilities are allocated to a set of locations in such a way that the cost is a function of the distance and flow between the facilities. In this problem, the costs are associated with a facility being placed at a certain location. The objective is to minimize the assignment of each facility to a location as given in [1, 2].

The QAP has application in wiring a computer backboard, in designing a hospital layout and in the dartboard design whereby in the game of darts points are scored by hitting specific marked areas of the board. The QAP is also used in the keyboard design of a typewriter, production process and scheduling.

2. Literature review and problem statement

In the paper [1], the QAP was linearized and the numbers of constraints and variables were kept to a minimal level. This linear formulation is based on the original nonlinear

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form given in [2]. Even though the best was done to keep the numbers of constraints to a minimal level, the QAP is still very difficult to solve. There is a need to reduce the number of extra or additional constraints that result from the linearization process. The lowest number of additional constraints one can think of is one (i. e. a single constraint). Taking a look at the available methods for the difficult QAP, we have the following approaches. There are three main categories of methods for solving the quadratic assignment problem. These categories are heuristics, bounding techniques and exact algorithms.

These are algorithms that quickly give near-optimal solutions to the quadratic assignment problem that are given in [3, 4]. In [3], extensive computational experiments for solving quadratic assignment problems using various options of a hybrid genetic algorithm were performed. The paper [4] addresses the problem of how to solve QAP under the Adleman-Lipton-sticker model. There are five main classes of heuristics for the quadratic assignment problem and these are:

1. Construction methods.
2. Limited enumeration methods.
3. Improvement methods.
4. Simulated annealing techniques.
5. Genetic algorithms.

The major challenge with these heuristics is that the solutions obtained are near-optimal and, in terms of costs, the difference between the optimal solution and the near-optimal one is big for large QAPs.

For a formulated quadratic assignment problem, a lower bound can be calculated. There are several types of bounds that can be calculated for a quadratic assignment problem as given in [5, 6]. The paper [5] relaxes the QAP as an LP relaxation defined in a lifted high dimensional variable space with order $O(n^4)$ variables and constraints. The paper [6] explores polyhedral methods for the quadratic assignment problem. These bounds are:

1) Gilmore-Lawler bounds;
2) eigenvalue related bounds;
3) bounds based on reformulations.

Lower bounds on their own are not very effective in solving QAPs but are important in some ways. Besides being used to approximate optimal solutions, they can be used within the context of heuristics or exact methods.

There are four main classes of methods for solving the quadratic assignment problem exactly as given in [7, 8]. The classes of methods for solving the QAP are well presented in [7]. The paper [8] gives the properties of the maximization form of QAP, which are important in solving this difficult problem. The exact methods are:

1) dynamic programming;
2) cutting plane techniques;
3) branch and bound procedures;
4) hybrids of the last two.

Research on these four methods has shown that the hybrids are the most successful of these methods for solving instances of the quadratic assignment problem. The QAP remains a very difficult problem and the hunt for an efficient consistent algorithm for this problem must continue.

The QAP has application in wiring, hospital layout, dartboard design, typewriter keyboard design, production process, scheduling, etc.:

a) steinberg wiring problem.

When wiring a computer backboard, there is a need to minimize the total amount or length of wire used. The main reason we need to minimize the amount of wire or length of wire is to minimize costs. In addition, minimizing the total length of the wiring will improve computing time. To achieve this, the wiring problem is formulated as a QAP and this problem is now known as Steinberg wiring problem;

b) hospital layout.

In designing a hospital layout, there are so many important factors that must be considered. These important factors include the patients, hospital staff, clinics, X-ray room, emergency room, drug store, etc. In designing the hospital layout, the objective is to minimize the total distance a patient in need of urgent care must travel before being treated. This problem is formulated as a QAP;

c) dartboard design.

A competitive sport in which two or more players barehandedly throw small sharp-pointed missiles at a round target or dartboard is called darts or dart-throwing. In darts, points are scored by hitting specific marked areas of the board. These areas follow a principle of points increasing towards the center of the board. The dartboard design problem can be formulated as a QAP;

d) typewriter keyboard design.

The use of smartphones and tablets is increasing significantly these days.

For one to enter data or text on these modern devices, virtual keyboards are now being used instead of the conventional hardware keyboards. The challenge is what is the best virtual keyboard layout for these devices? This problem is modeled as a quadratic assignment problem;

e) production.

In production processes, orders for a number of products must be scheduled on a number of similar production lines so as to minimize the sum of product-dependent changeover costs, production costs and time-constraint penalties. This is a production problem that can be modeled as a quadratic assignment problem;

f) scheduling.

Scheduling is very important in big hospitals, large universities, rail operations, large bus companies, airlines, etc. As an example, assignment of classes at a university can be scheduled in such a way that very few similar classes would be in the same time slot. In order to do this, the problem can be formulated as a quadratic assignment problem.

More on applications of the quadratic assignment problem can be found in [9].

There are four main variants of the quadratic assignment problem. These are the quadratic bottleneck assignment problem (QBAP), the biquadratic assignment problem (BQAP), the quadratic semi-assignment problem (QSAP), and the generalized quadratic assignment problem (GQAP).

Suppose we are given a set of n facilities and a set of n locations. Suppose it is also given that for each pair of locations, a distance is specified and for each pair of facilities a weight or flow is also specified. The quadratic bottleneck problem is the problem of assigning all facilities to different locations with the goal of minimizing the maximum of the distances multiplied by the corresponding flows.

A biquadratic assignment problem can be defined as a quadratic assignment problem with cost coefficients formed by the products of two four-dimensional arrays.

In the quadratic semi-assignment problem (QSAP), we are given again two coefficient matrices, which are the flow matrix and a distance matrix. In this case, there are n objects and m locations and are in such a way that $n>m$. The objective is to assign all objects to locations and at least one object to each location so as to minimize the overall distance covered by the flow of materials (or people) moving between different objects.

The GQAP involves the minimization of a total pairwise interaction cost among m equipment, tasks or other entities and placement of these entities into n possible destinations and is dependent upon existing resource capacities.

For more on quadratic assignment variants, readers are encouraged to see [10]. There are so many mathematical formulations for QAP. In this chapter, we use the linear form proposed in [1]. This linear form is an extension of the formulation introduced by [2]. In this formulation, we assume that new buildings are to be placed on a piece of land and n sites have been identified as sites for the buildings. We also assume that each building has a special function.

The number of variables and the number of constraints are used to measure complexity. As an example, the paper [11] gives a method whose complexity is of order:

$$O\left(L\sqrt{n}\left(n+m\right)\overline{\left\|M\right\|_F}\overline{\kappa}\in^{-2}\right).$$

In this complexity function, n is the number of variables in the cost function, m is the number of constraints, L is the bit length of the input data, $\overline{\left\|M\right\|_F}$ is an upper bound to the Frobenius norm of the linear systems of equations that appear and $\in^{-2}$ is the target precision. The more constraints and/or variables in the linear program, the more complex the model becomes. In other words, it makes sense to keep these two factors to the minimum when formulating and linearizing the quadratic assignment problem.

The formulated linear problem can be solved in many ways. The first way is to use branch and bound related algorithms [12–14]. In [12], reformulation and bounds are used to enhance the performance of the branch and bound algorithm for knapsack problem. In [13], a cut-and-branch algorithm for the quadratic knapsack problem is proposed, in which a cutting-plane phase is followed by a branch-and-bound phase. The basics of the branch and bound algorithm and simple illustrations are given in [14]. Dynamic programming (DP) can be used to solve some versions of knapsack models [15]. The main weakness of DP related approaches for knapsack models is that they become inefficient as the problem increases in size. Interior-point based algorithms, which are good for large problems, can also be used [16, 17]. In [16], it was shown that any linear binary problem can be converted into a convex quadratic problem. Interior point algorithms can solve convex quadratic problems in polynomial time. In [17], the traveling salesman problem is formulated as a convex quadratic problem. Other methods that combine heuristics and exact technics can also be used. More on these methods can be found in [18–21]. The paper [18] aims to find better algorithms for solving parameter reduction problems of soft sets and gives their potential applications. The paper [19] proposes a Petri net based mathematical programming approach to practical problems, in which we generate integer linear programming problems from Petri net models instead of the direct mathematical formulation. In [20], a rectangle blanket is used to speed up the computations in computer vision applications. In [21], a new formulation based on the definition of new binary variables has been proposed to convert practical problems in solar systems to binary linear programming (BLP). The proposed method finds the global optimum solution more efficiently than any other method available so.

3. The aim and objectives of the study

The aim of the study is to develop a method for the quadratic assignment problem. To achieve the set aim, the following tasks have been solved:

– to linearize the quadratic assignment formulation;

– to reduce the number of extra constraints to only one.

4. Materials and methods used to develop the method for the QAP

In this case, a theoretical approach was used to develop the proposed method for the quadratic assignment problem. This was done by first linearizing the nonlinear quadratic assignment problem. After that, there was a need to reduce the number of extra linear constraints to only one.

Let a_{ij} be the walking distance between sites i and j; b_{kl} be the number of people per week who circulate between buildings k and l.

Then the Koopmans-Beckmann formulation of the QAP is given as (1):

Maximize:

$$Z=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}a_{ij}b_{kl}x_{ik}x_{jl}+\sum_{i=1}^{n}\sum_{k=1}^{n}c_{ik}x_{ik}.$$

Such that:

$$\sum_{i=1}^{n}x_{ij}=1,\quad 1\le j\le n,$$

$$\sum_{j=1}^{n}x_{ij}=1,\quad 1\le i\le n,$$

$$x_{ij}\in\left\{0,1\right\},\quad 1\le i\le n,\quad 1\le j\le n. \tag{1}$$

In this formulation, there are n^2 variables and $2n$ constraints [2].

The current technique can linearize the Koopmans-Beckmann model to the form given in (2).

Maximize:

$$z=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}a_{ij}b_{kl}y_{ijkl},$$

Such that:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}y_{ijkl}=n^2, \tag{2}$$

$$x_{ik}+x_{jl}\ge 2y_{ijkl},\quad \forall i,j,k,l,$$

$$y_{ij}\ge x_{ik}+x_{jl}-1.$$

Solving this linearized QAP model becomes very difficult as n increases in size. This linearized model has (n^4+n^2) variables and $O(n^4)$ constraints. This is very difficult to manage as n becomes large.

The Koopmans-Beckmann formulation is a special case of a quadratic binary problem.

Let a general case of the quadratic binary problem be represented in (3).

Minimize:

$$Z=\sum_{i=1}^{n}\sum_{j=1}^{n}c_{ij}^{0}x_ix_j+\sum_{k}^{n}c_k^1x_k.$$

Such that:

$$a_{11}x_1+a_{12}x_2+...+a_{1n}x_n\le b_1,$$

$$a_{21}x_1+a_{22}x_2+...+a_{2n}x_n\le b_2,$$

$$a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n\le b_m, \qquad (3)$$

where a_{ij}, b_i, c_{ij}^0 and c_k^1 are constants, $1\le i\le m$, $1\le j\le n$,

$$x_i,x_j,x_k\in\{0,1\},\ \ 1\le i\le n,\ \ 1\le j\le n,\ \ 1\le k\le n.$$

In this formulation, all the variables are binary but there are a very large number of variables and constraints.

If $i=j$ then $(x_i)^2=(x_j)^2$. For binary integer variables, we have the following.

$$x_i(x_i-1)=0,$$

$$x_i^2-x_i=0,$$

$$x_i=x_i^2. \qquad (4)$$

Thus, x_i^2 can be replaced by x_i in the objective function. Similarly, x_j^2 can also be replaced by x_j in the objective function. Note that this substitution on its own does not change the number of variables in the problem.

If i is not equal to j then in the worst case there are $\frac{n(n-1)}{2}$ combinations of such variables in the objective function.

Proof 1. Suppose that:

– the case of two variables x_1 and x_2: then in the worst case, we can have the product x_1x_2 as the only possible combination of variables;

– the case of three variables x_1, x_2 and x_3: then in the worst case, we can have products x_1x_2, x_1x_3 or x_2x_3 as the possible combinations of variables. Thus, these three variables give 3 possible combinations;

– the case of n variables x_1, x_2, x_3, ..., x_{n-1}, and x_n: then in the worst case, we can have x_1x_2, x_1x_3, ..., x_1x_n, x_2x_3, x_2x_4, ..., x_{n-1}, x_n as the possible combinations. This results in:

$$(n-1)+(n-2)+...+1=\sum_{1}^{n-1}t=\frac{n(n-1)}{2}$$

possible combinations. The total number of combinations increases as the number of variables n increases.

The variable combinations of the form x_ix_j where i is not equal to j must be removed in order to make the objective function linear. This is done by using the following substitution.

Variable substitution. Let:

$$x_ix_j=\lambda_r^2. \qquad (5)$$

Since x_i and x_j are binary then the only possible products of these two are 0 and 1. This implies that λ_r^2 is also a binary variable and $r=1,2,...,\frac{n(n-1)}{n}$.

In other words, this can be summarized as given in (6–8).

$$x_i+x_j=2\lambda_r^1+\lambda_r^2, \qquad (6)$$

$$\lambda_r^1+\lambda_r^2\le 1, \qquad (7)$$

$$\lambda_r^1,\lambda_r^2\in\{0,1\}\ \text{and}\ r=1,2,...,\frac{n(n-1)}{n}.. \qquad (8)$$

Proof.

To prove this, we show that the solution space for $\Omega(x_i,x_j)=\{0, 1\}$ is also the solution space for $\Omega(\lambda_r)$. In addition, every point in $\Omega(x_i, x_j)$ has a corresponding point in $\Omega(\lambda_r)$ and that $x_ix_j=\lambda_r^2$ for all corresponding points.

Solution space for x_i,x_j, i. e. $\Omega(x_i, x_j)$:

– if $x_i=0$ and $x_j=0$, then $x_ix_j=0$;

– if $x_i=1$ and $x_j=0$, then $x_ix_j=0$;

– if $x_i=0$ and $x_j=1$, then $x_ix_j=0$;

– if $x_i=1$ and $x_j=1$, then $x_ix_j=1$;

– if $\lambda_r^1=1$ and

$$\lambda_2^2=0\Rightarrow x_i+x_j=2\Rightarrow x_i=x_j=1\Rightarrow x_ix_j=1.$$

There are two cases to consider:

– case one:

$$\lambda_r^1=0\ \text{and}\ \lambda_r^2=1\Rightarrow x_i+x_j=1,\Rightarrow x_i=1$$

and $x_j=0\Rightarrow x_ix_j=0$.

– case two:

$$\lambda_r^1=0\ \text{and}\ \lambda_r^2=1\Rightarrow x_i+x_j=1,\Rightarrow x_i=0$$

and $x_j=1\Rightarrow x_ix_j=0$.

If $\lambda_r^1=0$ and

$$\lambda_r^2=0\Rightarrow x_i+x_j=0\Rightarrow x_i=x_j=0\Rightarrow x_ix_j=0.$$

$$\therefore\Omega(\lambda_r)=\{0,1\}.$$

Corresponding points. Point in $\Omega(x_ix_j)$. Corresponding point in $\Omega(\lambda_r)$.

$$x_i=0\ \text{and}\ x_j=0,\ \ \lambda_r^1=0\ \text{and}\ \lambda_r^2=0,$$

$$x_i=1\ \text{and}\ x_j=0,\ \ \lambda_r^1=0\ \text{and}\ \lambda_r^2=1,$$

$$x_i=0\ \text{and}\ x_j=1,\ \ \lambda_r^1=0\ \text{and}\ \lambda_r^2=1,$$

$$x_i=1\ \text{and}\ x_j=1,\ \ \lambda_r^1=1\ \text{and}\ \lambda_r^2=0.$$

In other words, both the nonlinear and linear forms are made up of binary variables only.

This linearization process has serious weaknesses. Two additional variables are added for every product of variables x_i, x_j where i is not equal to j appearing in the objective function. For every any quadratic binary problem, there are $\frac{n(n-1)}{2}$ such products, and this was shown in Section 2. The number of new variables is given in (9).

$$2\times\frac{n}{2}(n-1)=n(n-1)\ \text{new variables}. \qquad (9)$$

This gives a total of $n(n-1)$ new variables $+n$ original variables, which is equal to n^2 variables. Also, two additional constraints are added for every product of variables x_i, x_j where i is not equal to j appearing in the objective function. The total number of new constraints is given in (10).

$$2\times\frac{n}{2}(n-1)=n(n-1)\text{ constraints.} \tag{10}$$

The total number of constraints $(\bar{m})$ is given by (11). This comes from, $\bar{m}=m$ original constraints$+n(n–1)$ original constraints, which is simplified to (11).

$$\left(n^2+m-n\right)\text{ variables.} \tag{11}$$

This gives the total number of variables, which becomes large as the number of variables increases.

The linearized model becomes as given in (12).

Minimize:

$$Z=\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0\lambda_r^1+\sum_i^n\bar{c}_k^1x_k.$$

Such that:

$$\begin{aligned}&a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n\le b_1,\\&a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n\le b_2,\\&\ldots\\&a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n\le b_m,\\&x_i+x_j=2\lambda_r^1+\lambda_r^2,\quad\forall i\ne j,\\&\lambda_r^1+\lambda_r^2\le1,\quad\forall i\ne j,\\&x_i,x_j,x_k\in\{0,1\},\quad1\le i\le n,\quad1\le j\le n,\quad1\le k\le n,\\&\lambda_r^1,\lambda_r^2\in\{0,1\}\text{ and }r=1,2,\ldots,\frac{n(n-1)}{n}.\end{aligned} \tag{12}$$

The number of additional constraints in this linearized model is large and there is a need to reduce it to the smallest manageable level possible.

4. 1. Minimization of the number of additional constraints in the linear model

Solving a linear model with $n(n–1)$ additional constraints and $(n^2+m–n)$ additional variables is a mammoth task for very large QAPs. There is a need to minimize the number of additional constraints.

The following two additional constraints can be combined into one:

$$x_i+x_j=2\lambda_r^1+\lambda_r^2.$$

$$\lambda_r^1+\lambda_r^2\le1.$$

The additional constraint in (6) can be expressed as given in (13) and (14).

$$x_i+x_j=2\lambda_r^1+2\lambda_r^2-\lambda_r^2. \tag{13}$$

$$x_i+x_j+\lambda_r^2=2\lambda_r^1+2\lambda_r^2. \tag{14}$$

$$x_i+x_j+\lambda_r^2=2\left(\lambda_r^1+\lambda_r^2\right). \tag{15}$$

From (7) $\lambda_r^1+\lambda_r^2\le1$, thus (15) is reduced to (17).

$$x_i+x_j+\lambda_r^2\le2(1). \tag{16}$$

$$x_i+x_j+\lambda_r^2\le2. \tag{17}$$

This reduces the number of additional constraints and variables to $\frac{n(n-1)}{2}$.

The minimized additional variables $\frac{n(n-1)}{2}$ is still a mammoth task for large QAPs. The number of additional constraints and variables increases by $\frac{500(500-1)}{2}=124.750.$ There is still a need to further minimize the number of additional constraints. This can be done by combining all the additional constraints into one.

$$\begin{aligned}&x_1+x_2+\lambda_1^2\le2,\quad x_1+x_3+\lambda_2^2\le2,\ \ldots\\&x_{\ell-1}+x_\ell+\lambda_\ell^2\le2,\end{aligned} \tag{18}$$

where $\ell=\frac{n(n-1)}{2}.$

Combining ℓ additional constraints, we have (19).

$$x_1+x_2+\lambda_1^2+x_1+x_3+\lambda_2^2+\ldots+x_{n-1}^2+x_n+\lambda_\ell^2\le2+2+\ldots+2, \tag{19}$$

$$(n-1)(x_1+x_2+\ldots+x_n)+\left(\lambda_1^2+\lambda_2^2+\ldots+\lambda_\ell^2\right)\le2\ell. \tag{20}$$

Then the linearized model becomes as given in (21).

Minimize:

$$Z=\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0\lambda_r^1+\sum_i^n\bar{c}_k^1x_k.$$

Such that:

$$\begin{aligned}&a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n\le b_1,\\&a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n\le b_2,\\&\ldots\\&a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n\le b_m,\\&(n-1)(x_1+x_2+\ldots+x_n)+\left(\lambda_1^2+\lambda_2^2+\ldots+\lambda_\ell^2\right)\le2\ell.\end{aligned} \tag{21}$$

Note that the number of additional constraints increases by only one as given in (19).

Since $\lambda_r^1+\lambda_r^2\le1$, then (20) is simplified to (22), which further is simplified to (23).

Minimize:

$$Z=\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0\left(1-\lambda_r^2\right)+\sum_i^n\bar{c}_k^1x_k,$$

Such that:

$$\begin{aligned}&a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n\le b_1,\\&a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n\le b_2,\\&\ldots\\&a_{m1}x_1+a_{m2}x_2+\ldots+a_{mn}x_n\le b_m,\\&(n-1)(x_1+x_2+\ldots+x_n)+\left(\lambda_1^2+\lambda_2^2+\ldots+\lambda_\ell^2\right)\le2\ell.\end{aligned} \tag{22}$$

Minimize:

$$Z=\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0-\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0\lambda_r^2+\sum_i^n\bar{c}_k^1x_k.$$

Such that:

$$\begin{aligned}&a_{11}x_1+a_{12}x_2+...+a_{1n}x_n\le b_1,\\&a_{21}x_1+a_{22}x_2+...+a_{2n}x_n\le b_2,\\&...\\&a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n\le b_m,\\&(n-1)(x_1+x_2+...+x_n)+(\lambda_1^2+\lambda_2^2+...+\lambda_\ell^2)\le 2\ell.\end{aligned}\tag{23}$$

Since $\left[\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0\right]$ is constant, then (23) is reduced to (24).

Minimize:

$$Z=\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0-\sum_{r=1}^{\frac{n}{2}(n-1)}\bar{c}_r^0\lambda_r^2+\sum_i^n\bar{c}_k^1x_k.$$

Such that:

$$\begin{aligned}&a_{11}x_1+a_{12}x_2+...+a_{1n}x_n\le b_1,\\&a_{21}x_1+a_{22}x_2+...+a_{2n}x_n\le b_2,\\&...\\&a_{m1}x_1+a_{m2}x_2+...+a_{mn}x_n\le b_m,\\&(n-1)(x_1+x_2+...+x_n)+(\lambda_1^2+\lambda_2^2+...+\lambda_\ell^2)\le 2\ell.\end{aligned}\tag{24}$$

This is another way of linearizing the quadratic assignment problem whereby the number of linear constraints increases by only one constraint.

5. Results of the development of the method for QAP

5. 1. Linearize the quadratic assignment problem

In the numerical illustration, the given problem in (23) is linearized by introducing 6 new variables and 6 new constraints as given in (26).

5. 2. Reduce the number of extra constraints to only one and solve

The linearized problem given in (26) has 6 extra constraints. These 6 extra constraints are reduced to only one as given in (27).

Numerical illustration.

Solve the quadratic assignment problem given in (25) using the proposed approach.

Minimize:

$$\begin{aligned}z=&36x_1+32x_2+33x_3+30x_4+16x_1^2+25x_1x_2+\\&+26x_1x_3+28x_1x_4+28x_2x_3+32x_2x_4+27x_3^2+31x_3x_4.\end{aligned}$$

Such that:

$$\begin{aligned}&24x_1+32x_2+30x_3+31x_4\ge 63,\\&34x_1+28x_2+32x_3+37x_4\ge 68,\\&31x_1+35x_2+28x_3+33x_4\ge 65,\end{aligned}\tag{25}$$

where $x_j\in\{0,1\}, j=1,2,3,4$.

There are three stages in solving this problem. These stages are linearizing the problem, reducing the additional constraints in the linear model and then solving it.

Making the model linear.

Let $x_j=x_j^2$ and $x_ix_j=-\lambda_r^2$ then the linear model in the numerical illustration becomes as given in (26).

Minimize:

$$\begin{aligned}z=&36x_1+32x_2+33x_3+30x_4+\\&+(16x_1^2=16x_1)+(-25\lambda_1^2)+(-26\lambda_2^2)+(-28\lambda_3^2)+\\&+(-28\lambda_4^2)+(-32\lambda_5^2)+(27x_3^2=27x_3)+(-31\lambda_6^2).\end{aligned}$$

Such that:

$$\begin{aligned}&24x_1+32x_2+30x_3+31x_4\ge 63,\\&34x_1+28x_2+32x_3+37x_4\ge 68,\\&31x_1+35x_2+28x_3+33x_4\ge 65.\end{aligned}$$

Extra linear constraints:

$$\begin{aligned}&x_1+x_2+\lambda_1^2\le 2,\\&x_1+x_3+\lambda_2^2\le 2,\\&x_1+x_4+\lambda_2^2\le 2,\\&x_2+x_3+\lambda_2^2\le 2,\\&x_2+x_4+\lambda_2^2\le 2,\\&x_3+x_4+\lambda_2^2\le 2,\end{aligned}\tag{26}$$

where $x_i\in\{0,1\}$, $i=1,2,3,4$ and $\lambda_j^2\in\{0,1\}$, $j=1,2,3,4,5,6$.

The problem is now linear and our next stage is to reduce the number of extra constraints.

If the six extra constraints are combined into one, then this is reduced to (27).

Minimize:

$$\begin{aligned}z=&52x_1+32x_2+60x_3+30x_4-25\lambda_1^2-26\lambda_2^2-\\&-28\lambda_3^2-28\lambda_4^2-32\lambda_5^2-31\lambda_6^2,\end{aligned}$$

Such that:

$$\begin{aligned}&24x_1+32x_2+30x_3+31x_4\ge 63,\\&34x_1+28x_2+32x_3+37x_4\ge 68,\\&31x_1+35x_2+28x_3+33x_4\ge 65,\\&3x_1+3x_2+3x_3+3x_4+\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2+\lambda_6^2\le 12,\end{aligned}\tag{27}$$

where $x_i\in\{0,1\}$, $i=1,2,3,4$; $\lambda_j^2\in\{0,1\}$, $j=1,2,3,4,5,6$.

The number of extra constraints in the linear model has been reduced to one and we now need to solve a linear model.

Solving a linear integer model.

Solving a linear model, the optimal solution is obtained as given in (28).

$$x_1 = x_2 = x_4 = \lambda_3^2 = \lambda_5^2 = \lambda_6^2 = 1,$$
$$x_3 = \lambda_1^2 = \lambda_2^2 = \lambda_4^2 = 0. \quad (28)$$

The problem is now solved to give an optimal solution shown in (28).

6. Discussion of numerical illustration

The given problem is in (25) and to solve this problem it must be linearized first. In order to linearize it, products of two different variables in the objective function are considered. In this case, products of different variables are x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4 and x_3x_4. There are six such products and it implies there must be 6 new variables in the linear formulation. In other words, the number of new variables depends on the products of different variables. The number of new constraints also depends on the number of given variables. In this case, there are four variables x_1, x_2, x_3 and x_4. The number of new constraints is the same as the number of all possible combinations ($^4C_2$=6) of these four variables. Using the noted patterns, any QAP can be easily linearized.

The second stage in the proposed method is to reduce the number of extra constraints in the linearized form of the QAP. The linearized form is given in (26) and there are 6 extra constraints. In order to reduce the 6 extra constraints, they are combined into one constraint by mere adding as given in (27). This is an excellent feature that is inherent in the proposed method. It is the only method whereby the number of extra constraints increases by only one. We are not aware of any other method that can do that. The only limitation of the proposed algorithm is that there are no computational results to compare with other methods at the moment.

To linearize the quadratic assignment problem given in Section 5. 1, we need a total of 6 extra constraints. Thus, the new linear problem has a total of 4 original variables, plus 3 original constraints, 6 new constraints and 6 new variables. The 6 new or extra constraints can be reduced to only 1 constraint as given in (25). This can be solved more efficiently than the form given in (24) to obtain the optimal solution given in (26). The strength of the proposed approach is that the number of constraints increases by one no matter what size.

7. Conclusions

1. The quadratic assignment formulation was linearized. A numerical illustration is presented in which 6 new constraints and 6 new variables are added to linearize the given quadratic assignment problem.

2. In every linearizing process, using the proposed approach, the number of new constraints is reduced to only one. This is illustrated in the numerical example whereby the new 6 extra constraints are reduced to only one.

The complexity of most models including (QAP) is measured in terms of the number of variables (n) and the number of constraints (m). Reducing any of these two parameters or both of them can significantly reduce the complexity of the model. In this paper, we proposed a technique to fix the increase in the number of extra constraints to only one. We are not aware of any method that can do that.

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