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UNIVERSAL METHOD FOR SOLVING OPTIMIZATION PROBLEMS UNDER THE CONDITIONS OF UNCERTAINTY IN THE INITIAL DATA

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This paper proposes a method to solve a mathematical programming problem under the conditions of uncertainty in the original data.

The structural basis of the proposed method for solving optimization problems under the conditions of uncertainty is the function of criterion value distribution, which depends on the type of uncertainty and the values of the problem's uncertain variables. In the case where independent variables are random values, this function then is the conventional theoretical-probabilistic density of the distribution of the random criterion value; if the variables are fuzzy numbers, it is then a membership function of the fuzzy criterion value.

The proposed method, for the case where uncertainty is described in the terms of a fuzzy set theory, is implemented using the following two-step procedure. In the first stage, using the membership functions of the fuzzy values of criterion parameters, the values for these parameters are set to be equal to the modal, which are fitted in the analytical expression for the objective function. The resulting deterministic problem is solved. The second stage implies solving the problem by minimizing the comprehensive criterion, which is built as follows. By using an analytical expression for the objective function, as well as the membership function of the problem's fuzzy parameters, applying the rules for operations over fuzzy numbers, one finds a membership function of the criterion's fuzzy value. Next, one calculates a measure of the compactness of the resulting membership function of the fuzzy value of the problem's objective function whose numerical value defines the first component of the integrated criterion. The second component is the rate of deviation of the desired solution to the problem from the previously received modal one.

Absolutely similarly designed is the computational procedure for the case where uncertainty is described in the terms of a probability theory. Thus, the proposed method for solving optimization problems is universal in relation to the nature of the uncertainty in the original data. An important advantage of the proposed method is the ability to use it when solving any problem of mathematical programming under the conditions of fuzzily assigned original data, regardless of its nature, structure, and type

Keywords: mathematical programming problem, uncertainty in the original data, universal solution method

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1. Introduction

A characteristic feature of the development of the modern theory of operations research is the complexity of mathematical models of real systems and processes, associated with the desire to improve their accuracy and efficiency. The serious difficulties that arise along the way are due to the incomplete and inaccurate initial data on the status of the objects under study and the environment in which they operate [1–5]. The issues to adequately account for uncertainty manifest themselves in solving many real-world tasks, including the rational allocation of resources, inventory management, routing and transportation management tasks, management of technical systems and technological processes, the structural and parametric optimization of complex systems, scheduling, and many others. The diversity of the arsenal of modern mathematics is a natural consequence and, above all, is determined by objective differences in the mathematical models for relevant practical tasks. The basic mathematical apparatus, originally used in solving these and, especially, optimization problems, is the probability theory. In this case, the initial approach was that the original problems, stated under the conditions of uncertainty, were reduced to the standard mathematical programming problems by replacing all random problem parameters with their averages [6–10]. The fundamental drawback of such optimization on average is the difficult-to-predict uncertainty of the result. This flaw is particularly demonstrative in situations where the variances of random problem parameters are large. In this regard, a different approach was proposed, which used the probability that the efficiency of the system would be at least above the set threshold as an alternative criterion for optimizing the system. Therefore, to solve the problem, one needs to derive a function of the distribution of the random value of the corresponding indicator. It is clear that the actual possibilities of implementing this approach in most practical tasks are limited by the difficulties of building appropriate mathematical models. The situation is further complicated by the increasingly clear understanding of the inadequacy of standard theoretical-probabilistic models to describe systems and how they function given a small sample of initial data. At the same time, the axiomatic limitations of the possibility of using probability theory technologies are violated. The impossibility of correctly calculating the unknown distribution densities of random parameters of the formed models leads to results whose implementation quality level is not predictable. Further development of the theory is associated with the emergence of an effective and advanced alternative to the probability theory under these conditions – the theory of fuzzy sets. The methods from this theory successfully solve the problems of fuzzy logic, the construction of fuzzy derivation systems, the task of analyzing fuzzy relationships and mapping. This theory is successfully developing in the following areas. Strict rules for fuzzy and bifuzzy numbers of the (L–R)-type have been devised [15]. A method for determining the density of distribution for fuzzy numbers is proposed, which makes it possible to calculate the moments of these values. Characteristic numbers for fuzzy numbers, used in the tasks of finding their compositions, have been introduced. The limiting theorem for the sum of a large number of loosely connected fuzzy numbers has been proven, leading to its Gaussian distribution [16].

The expansion of the arsenal of analytical and computational methods from the fuzzy set theory significantly increases the field of its practical use, bringing it closer to theoretical-probabilistic. It should be noted that the most significant and structurally important result of the continuing improvement of the fuzzy set theory is the emerging possibility of unifying the arsenal of approaches, techniques, and methods for dealing with practical tasks, regardless of the nature and type of uncertainty in the optimization problems under the conditions of uncertainty. It follows that when one solves problems with different types of uncertainty while maintaining the basic idea of building a method, its formal content and computational pattern may differ only in inconsequential details related to differences in setting the original data. However, that does not eliminate the task of developing common methods for solving optimization problems under the conditions of uncertainty. Still, the success of forming an adequate criterion and obtaining the actual solution to the optimization problem depends on the level of complexity of the relevant mathematical model. Thus, it is a relevant task to devise a common, uncertainty-free method for solving mathematical programming problems. In the future, for certainty, the analysis of known results based on the methods for solving optimization problems under conditions of uncertainty, as well as the description of the proposed approaches to solving them will be performed in relation to the theory of fuzzy sets.

2. Literature review and problem statement

The overall mathematical model of an optimization problem with fuzzily defined parameters is conventionally stated [13, 14] as follows: it is required to find a set of variables $X=(x_1,x_2,...,x_n)$, which maximizes the objective function:

$$f(X;\, a_1,a_2,...,a_q) \tag{1}$$

and satisfies the following constraints:

$$G_i(X;\, b_{i1},b_{i2},...,b_{ip}) \le 0,\quad i=1,2,...,m, \tag{2}$$

where the parameters a_k, $k=1,2,...,q$, and b_{il}, $i=1,2,...,m$, $l=1,2,...,p$, are the fuzzy numbers with the assigned membership functions:

$$\mu_k(a_k),\quad k=1,2,...,q,\quad \nu_{il}(b_{il}),$$

$$i=1,2,...,m,\quad l=1,2,...,p. \tag{3}$$

The general approach to solving a fuzzy problem of mathematical programming [13, 17–19] is to transform original problem (1), (2) with fuzzy parameters into the deterministic problem of mathematical programming. The wording of the problem received takes the following form: it is required to find the sets $X=(x_1,x_2,...,x_n)$, $A=(a_1,a_2,...,a_q)$, $B=(b_{ij})$, which maximize (1) that satisfy constraints (2) and the following additional limitations:

$$\mu_k(a_k) \ge \alpha,\quad k=1,2,...,q, \tag{4}$$

$$\nu_{il}(b_{il}) \ge \alpha,\quad i=1,2,...,m,\quad l=1,2,...,p. \tag{5}$$

Here α is the value of the membership function of the problem's parameters that is chosen from any natural consideration.

The set X^*, obtained from solving the problem (1) to (5), belongs to a combination of maximizing alternatives with a power not less than α. With the same power, the value $f(X^*,A)$ belongs to the fuzzy assessment of this alternative X^* [1]. The drawbacks of this approach are obvious:

– the increased dimensionality and computational complexity of the resulting problem compared to the original one;

– the non-predictability of the result of converting the levels of uncertainty of the original data into the uncertainty of the solution;

– a problem-solving procedure involves choosing the same power of the membership to all the fuzzily assigned parameters of the problem with the same α value; in this case, there are no recommendations for this value.

Another common approach is related to the concept of the expected fuzzy value proposed in [20]. This method provides a determined function of the problem's variables that are varied and thus converts the original fuzzy problem to a standard mathematical programming problem. The shortcomings of optimization on average have already been discussed.

A completely different idea leads to a method based on the operation of building a membership function of the fuzzy value of the problem's objective function. Using this feature also makes it possible to transition from the original fuzzy problem to a clear optimization problem. Applying this method is especially effective when only the parameters for the problem's objective function are fuzzy. In this case, the procedure for constructing the membership function of the fuzzy value of the problem's objective function is implemented as follows from [21]. Using the membership functions of the fuzzy parameters for the objective function calculates their values for which the level of membership is in any way set (for example, equal to α). These fuzzy parameter values naturally determine the value of the membership function of the objective function at the same level of α. Then, by changing the value of α, the corresponding values of the membership function of the objective function are calculated, the set of which is then approximated by a suitable curve. The resulting numerical sets are used in a standard way to calculate the analytical description of the membership function of the objective function, which ensures that a clear solution is found.

Knowing the membership function of the problem's objective function can be used to assess the level of preference for one fuzzy function value over another. Taking into consideration this circumstance, the next solution to the problem implements a procedure to find a sequence of solutions, in which the next solution is preferable to the previous one. At the same time, any method of zero-order can be used. For example, the branch and boundary method was used in [21] to solve the problem. It is clear that the effectiveness of the methods based on this variant of the original problem depends significantly on the level of complexity of the analytical description of the objective function. Therefore, they can actually be used only in problems of low dimensionality.

Let us finally consider another known approach [18]. Let the membership function $\mu\left(f\left(X,A\right)\right)$ of the fuzzy value of the problem's objective function is built using (1), (3) according to the rules given in [16, 17] (or by any other technique). Next, select some specific fixed value $\alpha<1$ for the $\mu\left(f\left(X,A\right)\right)$ membership level and solve the following equation:

$$\mu\left(f\left(X,A\right)\right)=\mu\left(y\right)=\alpha. \quad (6)$$

Since the membership functions are upward convex, the resulting equation has two roots:

$$y_{1,2}=\left(\mu_1^{-1}\left(\alpha\right),\mu_2^{-1}\left(\alpha\right)\right).$$

Choose the smaller one from these roots, L_1, and state the problem to find an X^* set that maximizes $L_1(X)$ and satisfies constraints (2). It is clear that, in this case, the body of uncertainty that corresponds to the membership function of the fuzzy value of the problem's objective function is moved as much as possible to the right, in the region of the large values of the objective function. There are obvious drawbacks to this approach. First, the resulting solution to the X^* problem depends on which of the resulting roots L_1 or L_2 in equation (6) is used to solve the maximization problem. At the same time, it is clear that the solutions are different. Second, the question of the choice of value for α, on which the desired result undoubtedly depends, remains open.

The noted shortcomings of known methods of solving the problems of mathematical programming under the conditions of uncertainty in the original data allow us to state the problem of searching for an alternative technology. At the same time, our brief analysis of available publications related to solving problems of mathematical programming reveals that there are no attempts to devise any common approach to solving such problems, which is not rigidly tied to their nature and structure, as well as to the type of uncertainty.

3. The aim and objectives of the study

The aim of this work is to devise a universal method for solving the problems of mathematical programming under the conditions of uncertainty.

To accomplish the aim, the following tasks have been set:

– to develop the concept of building a universal method for solving optimization problems;

– to construct computational schemes that implement a universal procedure for solving optimization problems.

4. Method for solving the problems of mathematical programming under the conditions of uncertainty

4. 1. The formal description of the concept of building a universal method for solving the problems of mathematical programming under the conditions of uncertainty

Let a mathematical programming problem be set that matches the formalism $K\left(A,Z\right)$, describing the objective function depending on the set of parameters A and the set of varied variables Z. In addition, the formalism $G\left(B,Z\right)$, is assigned that describes the region of permissible solutions, determined by a set of parameters B. The composition of formalisms $K\left(A,Z\right)$ and $G\left(B,Z\right)$ sets a certain formalism $M\left\{K\left(A,Z\right),G\left(B,Z\right)\right\}$. Let the parameters A and B for the formalisms $K\left(A,Z\right)$ and $G\left(B,Z\right)$ are not defined accurately but are described, for example, in the terms of the theory of fuzzy sets by the set of membership functions of these parameters $\mu_A\left(Z\right)$, $\mu_B\left(Z\right)$. Using these membership functions can determine the membership function of the criterion $K\left(A,Z\right)$ for any set Z. Similarly, this membership function can be determined if the uncertainty about the values A and B is theoretic-probabilistic and is set by the set of distribution densities of these parameters' random values. Optimization problems under the conditions of uncertainty are solved using the next two-step computational scheme. The first step is to solve the original problem for any natural deterministic set of values for parameters A and B, such as a set of modal values of these parameters. The second step is to find a set Z that minimizes the integrated additive criterion that is formed. The first term of this criterion determines the numerical value of an important indicator of the effectiveness of the system, in terms of the problem considered. The value of this criterion's term is calculated through the resulting membership function of the problem's criterion. It can determine, for example, the compactness of the objective function or the probability that the criterion's numerical value is within the acceptable range. The second term assigns the norm of deviation of the required set of variables from the modal set. The resulting integrated criterion should be minimized.

4. 2. A formal description of the computational scheme of problem-solving

The main element of this scheme is the justification, selection, and formal description of an integrated criterion for the quality of operation taking into consideration uncertainty. The importance of this choice lies in the fact that it fundamentally affects the level of complexity of the analytical description of the first term in the integrated criterion and the corresponding optimization problem. It is clear that the problem obtained is easier than the original one. However, the need for a solution that satisfies the original limitations does not make it trivial. It should be noted that the additive structure of the integrated criterion makes it possible to take into consideration the possible difference in the importance of its components.

Let us move on to a more detailed description of the technology for solving optimization problems under the conditions of uncertainty.

The structural basis of the proposed method to solve optimization problems under the conditions of uncertainty is some specific function of the distribution of criterion values depending on the values for the fuzzy variables of the problem. If independent variables are random values, this function then is the theoretic-probabilistic density of distribution, if the variables are fuzzy numbers, it is then a membership function of the fuzzy criteria values. Since, as shown in [16], a theoretical-probabilistic analog can be built for the membership function of fuzzy numbers, our further presentation involves the terms from a fuzzy set theory.

The proposed method is implemented as follows. In the beginning, by using the membership functions of the fuzzy parameters' values, we set their values equal to modal and fit them in an analytical expression for the objective function. This raises the next deterministic problem of mathematical programming: it is required to find a set $X=(x_1,x_2,...,x_n)$, which maximizes the following objective function:

$$f\left(X,a_1^{(0)},a_2^{(0)},...,a_q^{(0)}\right) \tag{7}$$

and satisfies the following constraints:

$$G_i\left(X;b_{i1},b_{i2},...,b_{ip}\right)\le 0,\quad i=1,2,...,m, \tag{8}$$

where the parameters a_k, $k=1,2,...,q$, are the fuzzy numbers with the membership functions $\mu_k(a_k)$, accepting modal values a_k^0. Let $X^{(0)}$ be a solution to the resulting problem. Let us solve another deterministic problem of mathematical programming: it is required to find a set $X=(x_1,x_2,...,x_n)$, which minimizes the integrated criteria, which is to be built as follows. By using an analytical expression for the objective function, as well as the membership function of the problem's fuzzy parameters, based on the rules of operations over fuzzy numbers [15, 16], we find the membership function of the fuzzy value of the objective function of the problem $\mu(f(X,A))$. Next, we introduce the following criterion:

$$\Phi\left[\mu\left(f(X,A)\right),A^{(0)}\right]= \\ =\lambda F_1\left(X,A^{(0)}\right)+(1-\lambda)F_2\left(\mu\left(f(X,A)\right)\right). \tag{9}$$

In ratio (9), the first term determines the deviation of the solution X from the modal $X^{(0)}$, while the second term sets a measure of the compactness of the resulting membership function of the fuzzy value of the problem's objective function.

The measure of the compactness of the membership function is to be defined as the ratio of squares of two areas. The first is the area under the curve corresponding to the membership function of the fuzzy value of the problem's objective function, derived to solve X. The second is the value of this area, calculated for the modal set of $X^{(0)}$ variables. It is clear that the lower this ratio, the lower the level of uncertainty of the solution.

Given the above, we obtain the following expression for criterion (9).

$$\Phi_1\left[\mu\left(f(X,A)\right),A^{(0)}\right]=\lambda\frac{\left(X-X^{(0)}\right)^T\left(X-X^{(0)}\right)}{\left(X^{(0)}\right)^T X^{(0)}}+ \\ +(1-\lambda)\frac{\left(\int_{-\infty}^{\infty}\mu\left(f(X,A)\right)\mathrm{d}f(X,A)\right)^2}{\left(\int_{-\infty}^{\infty}\mu\left(f\left(X,A^{(0)}\right)\right)\mathrm{d}f\left(X,A^{(0)}\right)\right)^2}. \tag{10}$$

The regularization parameter λ establishes the desired ratio between the conflicting requirements to the criterion: the maximum compactness of the membership function of the fuzzy value of the problem's objective function and the minimal deviation of the solution derived from the modal one. The typically low computational complexity of the proposed method makes it possible to use it effectively when solving a large number of practical tasks.

The technology of implementing the method is especially simple if the objective function of the problem is separable. In this case, the task of assessing the compactness of the criterion is simplified by the possibility of its decomposition into additive components.

At the same time, a direct method of obtaining this solution can be used to solve problems under the conditions of uncertainty. Let us consider the technology of applying this method using a transport problem of linear programming as an example.

The canonical transport problem of linear programming is stated as follows. The production points of some product $A_1, A_2, ..., A_m$, and the consumption points of this product $B_1, B_2, ..., B_n$ are assigned. For each production point A_i, the volume of production a_i is defined, $i=1, 2, ..., m$, and for each point B_j – the volume of consumption b_j, $j=1, 2, ..., n$. It is also assumed that the routes of transportation of the product from manufacturers to consumers are known and the corresponding matrix C_{ij} of the averages of transportation cost per unit of the product $i=1,\ 2,...,m$, $j=1,\ 2,...,n$ is assigned. It is required to find a matrix $X=(x_{ij})$ of the values of planned transportation volumes, which minimizes:

$$L(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij} \tag{11}$$

and satisfies the following constraints:

$$\sum_{j=1}^{n}x_{ij}=a_i,\quad i=1,\ 2,...,m, \tag{12}$$

$$\sum_{i=1}^{m}x_{ij}=b_j,\quad j=1,\ 2,...,n, \tag{13}$$

$$\sum_{i=1}^{m}a_i=\sum_{j=1}^{n}b_j. \tag{14}$$

In this case, to take into consideration the stochastic nature of the cost of transportation, a set of averages $(\bar{c}_{ij})$ of these costs is introduced for each pair supplier-consumer, $i=1, 2, ..., m$, $j=1, 2, ..., n$. Then the objective function (11), which determines the total average cost of transportation, takes the following form:

$$L(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}\bar{c}_{ij}x_{ij}. \tag{15}$$

In this situation, the canonical transport problem is transformed to the following: it is required to find a transportation plan $X=(x_{ij})$, minimizing (15) and satisfying constraints (12) to (14). The resulting problem is solved by the standard method of potentials. The drawbacks of this problem's model are obvious. First, given the high values of variances in the original random parameters of the problem, it may be unsatisfactory to solve the problem in the form (2), (4) to (6) for each specific implementation of the plan. Second, the variance in the evaluation of the result (the average criterion value) under these conditions would be unacceptably large and, therefore, the criterion could prove to be little informative.

We shall improve the model (12) to (15) by using the probability that random total costs do not exceed a given threshold as a more informative criterion for the effectiveness of transportation. It is clear that this probability depends simultaneously on the values of mathematical expectations and variances in the random transportation costs for each route, it will be the greater, the smaller these statistical characteristics of the corresponding random variables.

Let the results of preliminary studies for each pair (I, j) (producer-consumer) determine the estimates of mathematical expectation m_{ij} and variance σ_{ij}^2 of the random value of the cost of transporting a unit of product c_{ij}. Let us assume that the density of distribution of the corresponding random values is Gaussian. Then, for the transportation plan $X=(x_{ij})$, the random value of the total cost of transportation $L(x)=\sum_{i=1}^{m}\sum_{j=1}^{n}A_{ij}x_{ij}$ follows a gaussian distribution with parameters $m_\Sigma=\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}x_{ij}$ and $\sigma_\Sigma^2=\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2x_{ij}^2$. In this case, the probability that the random total cost $L(x)$ exceeds the allowable threshold d_Π is determined by the following ratio:

$$P\left(L(x)\ge d_\Pi\right)=$$

$$=\int_{d_\Pi}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_\Sigma}\exp\left\{-\frac{\left(L(x)-m_\Sigma\right)^2}{2\sigma_\Sigma^2}\right\}\mathrm{d}L(x).$$

Transform the resulting ratio:

$$\int_{d_\Pi}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_\Sigma}\exp\left\{-\frac{\left(L(x)-m_\Sigma\right)^2}{2\sigma_\Sigma^2}\right\}\mathrm{d}L(x)\underset{\frac{L(x)-m_\Sigma}{\sigma_\Sigma}=u}{=}$$

$$=\int_{\frac{d_\Pi-m_\Sigma}{\sigma_\Sigma}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}\mathrm{d}u. \qquad (16)$$

The problem of minimization (16) is equivalent to the problem of maximization:

$$J(x)=\frac{d_\Pi-m_\Sigma}{\sigma_\Sigma}=$$

$$=\frac{d_\Pi-\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}x_{ij}}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2x_{ij}^2\right)^{\frac{1}{2}}}=\frac{\frac{d_\Pi}{A}\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij}-\sum_{i=1}^{m}\sum_{j=1}^{n}m_{ij}x_{ij}}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2x_{ij}^2\right)^{\frac{1}{2}}}=$$

$$=\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(\frac{d_\Pi}{A}-m_{ij}\right)x_{ij}}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2x_{ij}^2\right)^{\frac{1}{2}}}=\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij}x_{ij}}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2x_{ij}^2\right)^{\frac{1}{2}}};$$

$$A=\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ij},\ s_{ij}=\frac{d_\Pi}{A}-m_{ij}. \qquad (17)$$

Thus, a non-linear optimization problem has been obtained, with a fractional-quadratic objective function and linear constraints of the transportation type [22]. It is convenient to replace the problem of function maximization (17) with the equivalent problem of function minimization:

$$G(x)=\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2x_{ij}^2}{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij}x_{ij}\right)^2}. \qquad (18)$$

To solve the resulting problem in the fractional-quadratic programming form, we introduce a new variable:

$$y_0=\frac{1}{\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij}x_{ij}},$$

hence:

$$y_0\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij}x_{ij}=1. \qquad (19)$$

Formulate a new set of variables:

$$y_{ij}=y_0x_{ij},\quad i=1,\ 2,...,m,\quad j=1,\ 2,...,n. \qquad (20)$$

In this case, ratios (12) to (14), (18), (19) take the following form:

$$G(y)=\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2y_{ij}^2, \qquad (21)$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}s_{ij}y_{ij}=1.$$

$$\sum_{j=1}^{n}x_{ij}=\frac{1}{y_0}\sum_{j=1}^{n}y_{ij}=a_i,\quad i=1,\ 2,...,m, \qquad (22)$$

$$\sum_{i=1}^{m}x_{ij}=\frac{1}{y_0}\sum_{i=1}^{m}y_{ij}=b_j,\quad j=1,\ 2,...,n,$$

hence:

$$\sum_{j=1}^{n}y_{ij}=y_0a_i,\quad i=1,\ 2,...,m, \qquad (23)$$

$$\sum_{i=1}^{m}y_{ij}=y_0b_j,\quad j=1,\ 2,...,n. \qquad (24)$$

Now, the original problem is reduced to the following form: it is required to find a set $Y=(y_{ij})$, $i=1, 2, ..., m$, $j=1, 2, ..., n$, which minimizes (21) and satisfies constraints (22) to (24). Let us solve the resulting problem of quadratic programming.

Introduce a Lagrange function:

$$\Phi(y)=\sum_{i=1}^{m}\sum_{j=1}^{n}\sigma_{ij}^2y_{ij}^2-\sum_{ij}^{m}\lambda_i\left(\sum_{j=1}^{n}y_{ij}-y_0a_i\right)-$$

$$-\sum_{j=1}^{n}\mu_j\left(\sum_{i=1}^{m}y_{ij}-y_0b_j\right).$$

Next:

$$\frac{d\Phi(y)}{dy_{ij}} = 2\sigma_{ij}^2 y_{ij} - \lambda_i - \mu_j = 0,$$

$$i = 1, 2, ..., m, \quad j = 1, 2, ..., n,$$

$$y_{ij} = \frac{1}{2\sigma_{ij}^2}(\lambda_i + \mu_j), \quad i = 1, 2, ..., m, \quad j = 1, 2, ..., n. \qquad (25)$$

By fitting (25) in (22) to (24), we obtain:

$$\sum_{j=1}^{n} \frac{1}{2\sigma_{ij}^2}(\lambda_i + \mu_j) = y_0 a_i, \quad i = 1, 2, ..., m,$$

$$\sum_{i=1}^{m} \frac{1}{2\sigma_{ij}^2}(\lambda_i + \mu_j) = y_0 b_j, \quad j = 1, 2, ..., n.$$

By solving the resulting system of linear algebraic equations, we derive expressions for $\{\lambda_i\}$, $\{\mu_j\}$ via y_0 and the initial data $\{a_i\}$, $\{b_j\}$. By substituting these expressions in (25), we obtain ratios for y_{ij} through y_0. Next, we find the value of y_0 from (19). Finally, we calculate the values of the original variables x_{ij} by using (20). The problem has been solved.

Let us consider the simplest example of applying the proposed procedure. Let the following transport problem be set. There are two supply points for some product in the quantities of a_1 and a_2 units, as well as two points of consumption of the product in the volume of b_1 and b_2 units. Let us set the parameters for the relevant transportation network: $a_1=90$, $a_2=120$, $b_1=80$, $b_2=130$, and the matrix of average shipping costs of a unit from suppliers to consumers:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 12 & 10 \\ 13 & 11 \end{pmatrix}.$$

Let us introduce the matrix $X=(x_{ij})$, which assigns the desired transportation plan. In this case, the problem's objective function, the average total cost of transportation, takes the following form:

$$\bar{R}(x) = \sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij} x_{ij}.$$

Constraints for the variables x_{ij} are defined by the following expressions:

$$x_{11} + x_{12} = a_1 = 90, \quad x_{21} + x_{22} = a_2 = 120,$$

$$x_{11} + x_{21} = b_1 = 80, \quad x_{12} + x_{22} = b_2 = 130.$$

It is required to find a plan $X=(x_{ij})$ that minimizes the objective function $\bar{R}(x)$ and satisfies the introduced constraints. A solution to this simple problem, derived by the method of the minimum element of the M matrix, takes the following form:

$$x_{11}^{(0)} = 0, \quad x_{12}^{(0)} = 90,$$

$$x_{21}^{(0)} = 80, \quad x_{22}^{(0)} = 40.$$

The total average cost of transportation corresponding to the resulting plan $X(0)$ is:

$$\bar{R}\left(X^{(0)}\right) = 90\cdot 10 + 80\cdot 13 + 40\cdot 11 = 2340 \text{ unit.}$$

Next, solve this problem for the case when a matrix of variances of random shipping costs of a unit of cargo is set:

$$\sigma^2 = \begin{pmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{pmatrix} = \begin{pmatrix} 7.5 & 20 \\ 17.5 & 5 \end{pmatrix}.$$

In accordance with the proposed methodology, we find a constraint-satisfying transportation plan that minimizes the probability that the accidental value of the total cost of transportation exceeds the specified threshold. Build a Lagrange function:

$$\Phi(y) = \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}^2 y_{ij}^2 - \sum_{ij}^{2} \lambda_i \left(\sum_{j=1}^{2} y_{ij} - y_0 a_i \right) - \\ - \sum_{j=1}^{2} \mu_j \left(\sum_{i=1}^{2} y_{ij} - y_0 b_j \right).$$

and solve the resulting system of linear algebraic equations relative to the variables introduced y_{ij}. These equations take the following form:

$$2\sigma_{ij}^2 y_{ij} - \lambda_i - \mu_j = 0, \quad i = 1, 2, \quad j = 1, 2.$$

Hence:

$$y_{ij} = \frac{1}{2\sigma_{ij}^2}(\lambda_i + \mu_j), \quad i = 1, 2, \quad j = 1, 2.$$

By substituting the resulting expressions for y_{ij} and the numerical values of the problem's parameters in the equations for constraints, we obtain the following system of algebraic equations:

$$y_{11} + y_{12} = \frac{1}{15}(\lambda_1 + \mu_1) + \frac{1}{40}(\lambda_1 + \mu_2) = 90 y_0,$$

$$y_{21} + y_{22} = \frac{1}{35}(\lambda_2 + \mu_1) + \frac{1}{10}(\lambda_2 + \mu_2) = 120 y_0,$$

$$y_{11} + y_{21} = \frac{1}{15}(\lambda_1 + \mu_1) + \frac{1}{35}(\lambda_2 + \mu_1) = 80 y_0,$$

$$y_{12} + y_{22} = \frac{1}{40}(\lambda_1 + \mu_2) + \frac{1}{10}(\lambda_2 + \mu_2) = 130 y_0.$$

The solution to this system produces a description of the parameters λ_i, μ_j via y_0: $\lambda_1 = 300 y_0$, $\lambda_2 = 100 y_0$, $\mu_1 = 600 y_0$, $\mu_2 = 900 y_0$.

Hence:

$$y_{11} = \frac{1}{15}(\lambda_1 + \mu_1) = 60 y_0, \quad y_{12} = \frac{1}{40}(\lambda_1 + \mu_2) = 30 y_0,$$

$$y_{21} = \frac{1}{35}(\lambda_2 + \mu_1) = 20 y_0, \quad y_{22} = \frac{1}{10}(\lambda_2 + \mu_2) = 100 y_0.$$

Next, considering (20), we find the desired transportation plan.

$$x_{11}^{(1)} = 60, \quad x_{12}^{(1)} = 30, \quad x_{21}^{(1)} = 20, \quad x_{22}^{(1)} = 100.$$

The solution is derived.

We shall evaluate the practical usefulness of the proposed procedure for optimizing a transportation plan under the conditions of uncertainty. A scalar preference criterion used, in accordance with the recommendations from [23], is the probability that a random value of the total cost of transportation exceeds the assigned threshold. As a threshold, we choose a value exceeding the average total cost of transportation by 15 %.

For plan $X^{(0)}$, derived from optimizing on average without taking into consideration the variances in the cost of transportation, we obtain:

$$\bar{R}_0=\sum_{i=1}^{2}\sum_{j=1}^{2}m_{ij}x_{ij}^{(0)}=10\cdot 90+13\cdot 80+11\cdot 40=2340,$$

$$\sigma_{\Sigma}^{(0)}=\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_{ij}^{2}\left(x_{ij}^{(0)}\right)^{2}\right)^{\frac{1}{2}}=$$

$$=\left(20\cdot(90)^2+17.5\cdot(80)^2+5\cdot(40)^2\right)^{\frac{1}{2}}=531.04.$$

$$Bep\left\{R_0\geq R_n\right\}=Bep\left\{R_0\geq 1.15\bar{R}_0\right\}=0.251.$$

For plan $X^{(1)}$, derived by considering the variances in the cost of transportation, we obtain:

$$\bar{R}_1=\sum_{i=1}^{2}\sum_{j=1}^{2}m_{ij}x_{ij}^{(1)}=12\cdot 60+10\cdot 30+$$
$$+13\cdot 20+11\cdot 100=2380,$$

$$\sigma_{\Sigma}^{(1)}=\left(\sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_{ij}^{2}\left(x_{ij}^{(1)}\right)^{2}\right)^{\frac{1}{2}}=$$

$$=\begin{pmatrix}8\cdot(50)^2+10\cdot(70)^2+\\+6\cdot(70)^2+12\cdot(10)^2\end{pmatrix}^{\frac{1}{2}}=315.2.$$

$$Bep\left\{R_1\geq R_n\right\}=Bep\left\{R_1\geq 1.15\bar{R}_1\right\}=0.127.$$

Thus, the probability of exceeding the acceptable threshold in the event of uncertainty in the original data is twice less than the previous one.

It should be noted that the fundamental advantage of the proposed method of solving a transportation problem under the conditions of uncertainty is the possibility to derive an accurate and rapid result due to a one-time solution to the problem of fractional-quadratic programming.

5. Discussion of results of devising a method for solving mathematical programming problems under the conditions of uncertainty

Thus, a universal method has been proposed for solving the problems of mathematical programming under conditions where the original data are random values or fuzzy numbers with known distribution densities or membership functions, respectively. The practical independence of the computational scheme to solve a problem on the type of uncertainty is a fundamental difference and an undoubted advantage of the method. The method is based on the developed new concept about solving optimization problems, the implementation of which is not tied to the structure and features of the problem under consideration, the nature of the objective function, and constraints for its variables. It is important, however, that if the level of uncertainty of the original data is such that there is no analytical description of their distribution densities (or membership functions), then the problem is solved by using the technology of continual linear programming [24]. In addition, it is significant that in a very common private case, when a problem's objective function is separable, the computational procedure for solving the problem is significantly simplified. At the same time, the desired solution can be obtained by applying the method of fractional-nonlinear optimization.

The reported results can be interpreted and explained as follows.

First, the possible difficulties in solving the original problem of mathematical programming and its complexity due to the uncertainty of the original data do not affect the structure and technology of the proposed method. The formulated concept and the process of implementing the developed computational scheme reduce the original problem of mathematical programming, regardless of its complexity, to a simpler problem of minimizing the integrated criterion being constructed.

Second, the proposed method resolves the issue related to solving problems of mathematical programming under the conditions of uncertainty, regardless of its type.

The limitations that may arise in the process of implementing the method are determined by the difficulty of the procedure for constructing a membership function of the fuzzy value of the original criterion. In this regard, the direction of further research is to advance the method in order to simplify the computational procedure of its application in cases where the criterion of the original problem is complexly dependent on parameters. A possible approach to solving the problem, in this case, is to linearize the criterion by expanding it into a series. It is clear that the accuracy of a relevant solution to the problem could be compromised.

6. Conclusions

1. A concept for building a universal method to solve optimization problems under the conditions of uncertainty has been proposed. The fundamental basis of this concept is the reduction of the original optimization problem, stated under the conditions of uncertainty, to the deterministic problem to be solved by the appropriate method of mathematical programming. In this case, the fuzzily set variables for the original problem are described by their modal values.

2. A computational scheme for implementing a universal optimization method, consisting of two stages, has been developed. In the first stage, the original problem is reduced to a deterministic problem, which is solved by the appropriate method. In the second stage, the desired solution to the problem is found by minimizing the integrated criterion, which takes into consideration the compactness of the membership function of the problem's objective function, and the rate of difference between a given solution and a modal one.

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