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CONSTRUCTING THE MODELS OF PROGRAMMED FLIGHT FOR PATH CALCULATION IN DESIGNING TACTICAL AND ANTI-AIRCRAFT GUIDED MISSILES

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Several models of programmed flight have been constructed to perform calculations on flight path optimization in designing tactical and anti-aircraft-guided missiles. The developed models are based on the determination of interrelated programmed values of altitude and the flight path angle depending on the range which have a differential relationship. The combination of flight altitude and flight-path angle programs allows the users to simulate the steady flight of a guided missile to the predicted intercept endpoint using the methods of proportional control.

Good correspondence of the developed models to the physics of flight was shown by assessing the quality of approximation of the developed models of flight paths of anti-aircraft guided missiles obtained using other known models. The obtained approximation error was less than 5 % which indicates a good correspondence of the developed models to the physics of flight.

Compliance of the developed models of programmed flight with the intended purpose and the advantage over the most common known models were proved by optimizing the flight paths of the anti-aircraft-guided missile. In most of the considered calculation cases, the value of the objective function was improved to 2.9 %. The flight path was optimized using a genetic algorithm.

The developed models have a simple algebraic form and a small number of control parameters are presented in a ready-to-use form and do not require refinement for a concrete task. This allows them to be implemented in design practice without spending much time to speed up the calculation of optimal design variables and optimal flight paths of tactical and anti-aircraft-guided missiles

Keywords: missile, programmed flight model, flight path, optimization, optimal path, calculation

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1. Introduction

A surface-to-air missile (SAM) is a missile of the type used in anti-aircraft missile systems and designed to hit air targets [1].

A tactical missile (TM) is a type of ballistic missile designed to hit targets directly in the field of hostilities [2].

The TM and SAM designing practice shows that the choice of optimal reference paths in the process of determining design variables (DP) of missiles of these types is extremely important.

Besides, the problem of calculating the optimal paths is an important problem in aircraft designing.

The TM and SAM, as design objects, are united by the fact that their flight takes place in dense layers of the atmosphere in aero-ballistic paths. Purely ballistic paths for such missiles are not optimal as the path length increases when flying in such paths and, accordingly, the flight time increases as well. In contrast to ballistic paths, their rectifying results in large speed losses because of aerodynamic drag.

Elaboration of effective methods of optimization of design variables of TM and SAM while taking into account optimal flight paths is an urgent problem. A solution to this problem will significantly speed up the TM and SAM design process. To solve the problem of this type, it is necessary to apply analytical methods of setting various paths, i.e. the models of programmed flight. An optimal flight path is sought by varying the control parameters.

2. Literature review and problem statement

In the framework of classical aircraft design, initial DP are assessed on the basis of previous experience in design and/or analysis of analogs followed by ballistic calculations and DPs are defined more exactly based on this assessment [3]. As a rule, because of the complexity of these two problems, they are solved by different specialists. The process of solving these problems is often largely creative, rather than formalized. In addition, a series of such iterations may be required to obtain an acceptable result which significantly lengthens the process of developing technical proposals and/or preliminary projects.

The systems approach [3] is widely used in missile design. From the point of view of the systems approach, the problem of DP choice should be subordinated (to be solved within the frames) to the problem of effective flight to the predicted intercept point (PIP). The predicted intercept point is a calculated point of meeting with the target or the beginning of terminal guidance. In English literature, flight to PIP is called midcourse guidance, i.e. guidance in cruise or middle sections of the path to the moment of the start of self-guidance.

Since parameters of flight paths depend on the missile DPs and vice versa, calculation of optimal DPs and optimal flight path of the missile should be carried out simultaneously within the framework of one complex problem.

This is an optimization problem, i.e. it is necessary to choose such DPs and such path of the missile flight which will provide extremum of one or more criteria of optimality when meeting specified constraints. A solution to such a problem must be formalized because it is impossible to prove the optimality of the solution obtained using a trivial (or partially trivial) approach as the optimization problems cannot be trivial by definition.

In its turn, the problem of calculating the optimal path is the problem of optimal control to select the best in terms of some criterion (or several criteria) of the control program to achieve the required final parameters of the missile flight. In the general case, the problems of this class include the following [4]:

– description of the control object, namely a set of characteristics and differential equations of the missile motion which are widely known and are not the subject of this study;

– initial and necessary final characteristics of the control object;

– a set of admissible controls which, in fact, is specified to the model of programmed flight (its construction is this study objective);

– optimization criteria which give a quantitative assessment of control efficiency (maximum flight range, minimum flight time to the PIP, and maximum speed at the PIP are considered most often).

Approaches to specifying paths during optimization were analyzed and two groups of methods were singled out in [5]:

– direct methods based on a parameterization of the law of control and turning the problem of optimal control into the problem of optimizing the control parameters of the law;

– indirect methods based, as a rule, on the analytical solution of the problems of guiding a missile to a moving target or PIP using methods of the theory of optimal control.

The direct methods reduce the complexity of software implementation of calculation and therefore are more acceptable at the design stages [5]. When developing indirect methods, the problem of guidance requires significant simplifications to ensure an analytical solution [5, 6]. This brings about the fact that specific indirect methods should be used only in appropriate duel situations. Therefore, it is rational to use direct methods for a more global analysis at the initial design stages.

The study [7] has classified models of programmed aircraft flight which differ in the initial motion parameter for which the program is set and the value of the control angle of attack is calculated on its basis. It is the angle of attack that directly affects the magnitude of the aerodynamic forces acting on the aircraft during its flight in the atmosphere and thus determines its flight path. Thus, direct specification of the law of change of the angle of attack, e.g. depending on time [5] is the simplest direct method of controlling the missile flight in the atmosphere. In addition, this method is the fastest in terms of the time required to integrate the system of differential equations of motion by software means using numerical methods. This method is often used in practice, however, it has a number of significant drawbacks that complicate its practical application:

– the problem of hitting the PIP is not solved automatically which leads to the additional need to solve either the boundary-value problem or the problem of minimizing the final miss which significantly increases the calculation time as this process is iterative as well;

– values of the target functions are hypersensitive to changes in any control parameter of the law of change of the attack angle which significantly complicates the optimization calculations;

– the program of changing the angle of attack suitable for one set of aircraft characteristics and flight conditions is unsuitable for another set of characteristics.

These shortcomings lead to the following consequences:

– too much time is spent on the calculation of required paths;

– for further studies (controllability, stability, development of control systems), other parameters calculated in the process of flight modeling by the program of the angle of attack (for example, the angles of aircraft orientation) are still accepted as primary program parameters;

– the need to solve the boundary problem or the problem of minimizing the final miss makes this method impractical for application together with calculation of optimal aircraft DPs.

All models of the programmed flight of the aircraft given in [7] have one essential drawback: in the process of modeling, the required values of aerodynamic forces at a certain point in time may be not realized for one reason or another. In this case, the aircraft will irreversibly deviate from the programmed flight path because of the fact that the required current values of the angle of attack do not adapt to changes in the flight conditions.

There are two main ways to cope with this problem:

1) application of self-guidance methods (indirect methods);

2) when applying the direct methods, the required value of the angle of attack should be determined based on the primary control parameter and application of the methods of proportional control widely known from the theory of automated control [8].

The proportional navigation method (PNM) is a classic method of SAM guidance both on moving (in the general case) and stationary targets. The PNM can be applied both as a method of terminal guidance and as a method that completely determines the flight path.

The PNM is widely used in many present-day designs and is described in numerous studies [3, 7, 9]. Even the studies of recent years are associated with its improvement, modification, and designing optimal control systems based on it [10, 11].

The PNM advantage is in an automatic solution to the problem of hitting the predicted interception point (in the general case, with a non-zero but allowable miss), regardless of changes in flight conditions if an aircraft has enough energy and maneuverability.

A significant disadvantage consists in that the physical essence of the PNM consists in reducing to zero the angular velocity of rotation of the aircraft-target line (or aircraft-PIP line). Accordingly, the angular velocity of rotation of the aircraft velocity vector is approaching zero and resulting in a straightened, not optimal path [3].

The PNM disadvantage in terms of calculations consists in the need to determine the control angle of attack at each iteration in the integration of the differential equations of the aircraft motion using a nonlinear algebraic equation. The methods that specify control in a form of accelerations and direction or through the path curvature have the same disadvantage [6]. This leads to an increase in the calculation time as this equation must also be solved by a numerical iterative method. In addition, the root of the equation may not be found sometimes because of the dependence of numerical methods on an initial approximation which will lead to the calculation failure.

To apply the methods of proportional control, it would be possible to use a model of programmed flight with a specified flight-path angle according to the classification given in [7]. However, the program of the flight-path angle alone is not enough for the application of this method. This is explained by the high risk of "lying" on a parallel course on which the flight-path angle corresponds to the program and altitude does not correspond to the required value.

The model of programmed flight with a set altitude seems obvious but it was not presented in the classification given in [7]. This can be caused by the fact this model cannot calculate the exact value of the required angle of attack at a variable altitude by means of purely algebraic methods. In addition, the use of altitude alone as the primary software parameter together with the proportional control methods will not provide a sufficiently stable flight. More stable flight can be provided by proportional derivative control [8]. In the absence of a programmed differential component (e.g., the flight-path angle), the fluctuations of actual altitude relative to the programed one which will occur in a case of deviations will not be damping.

Polynomials [3, 12, 13] are often used to specify the flight path shape including taking into account the restrictions on the terminal flight-path angle [13]. The advantage of polynomials consists in the ability of easy scaling to obtain more complex shapes of the flight paths by increasing the polynomial degree. However, it should be noted that, as a rule, the coefficients of polynomials in a conventional form differ by order of magnitude at different argument orders. In turn, this can negatively affect the performance of numerical optimization methods.

It is impossible to provide angles close to 90° at the launch section of flight with the help of polynomials at relatively moderate values of coefficients as well as correctly present the launch section of the flight path as a whole. This fact also complicates the optimization process when using numerical methods.

A model of setting the path in a form of a polynomial is described in [3] for which an expression for determining the flight-path angle is given. However, the described model is built for a coordinate system with its x-axis directed to the PIP and is inconvenient for application in the coordinate systems that are more conventional for flight simulation.

Methods of constructing flight paths using Bezier curves were also considered in [14] where coordinates of the control points of the curve are the controlling parameters. Bezier curves, like polynomials, are easily scalable but the practical application advantages over polynomials require further study.

Thus, one of the following disadvantages is characteristic for the models of programmed flight considered in this section:

– they do not specify interrelated programmed values of altitude and flight-path angle;

– they do not provide for the automatical guidance of the aircraft to the PIP;

– they do not provide the ability to explicitly set the launch angle and the terminal flight-path angle;

– they have different orders of magnitude of the control parameters.

3. The aim and objectives of the study

This study objective is to simplify the process of optimization of the TM and SAM flight paths proper and comprehensive optimization of DP and flight paths of the missiles of these types at the design stages. This will speed up the TM and SAM design process as well as improve the quality of its results.

To achieve this objective, it was necessary to solve the following problems:

– develop models of programmed flight for calculation of the TM and SAM flight paths which will set interrelated values of altitude and flight-path angle in a parameterized form, provide automatic hitting the PIP and a possibility to set angles of launch and approach to the PIP and have identical or close orders of values of the control parameters;

– analyze the possibility of approximation of the constructed models of flight paths calculated using other known models;

– compare the results of optimization of flight paths with the use of known and constructed models of programmed flight.

4. The study materials and methods

In the general case, the type of control law for the construction of the TM and SAM flight paths is selected based on the ideas of the types of typical flight paths. The only difference between typical TM and SAM flight paths in terms of choosing the type of the control law consists in that the PIP altitude is always zero for the TM.

It is known that the functions setting the dependences of altitude and the flight path angle on the range (the range means the ground range) have a differential relationship:

$$\frac{dH(L)}{dL}=tg\big(\theta(L)\big), \tag{1}$$

where L is the range, H is the altitude, θ is the flight-path angle.

To derive the interconnected algebraic relationships for $H(L)$ and $\theta(L)$, two fundamentally opposite approaches can be used:

1) specify the function type $\theta(L)$, then derive such relationships between the function parameters that the function $H(L)$ passes through the origin (0, 0) and a given endpoint (H_k, L_k) and thus partially takes into account initial and necessary final characteristics of the controlled object). Linear models were developed using this approach;

2) specify the function $H(L)$ in such a way that it passes through the origin (0, 0) and the given endpoint (H_k, L_k), then find the derivative of the function by L. Polynomial models were developed using this approach.

The bilinear model.

A procedure of derivation of relationships for the models of the programed flight was explained based on the example of the bilinear model because this model is the simplest. Only the relationships taken from Section 5 will be presented for other models.

Let the function $\theta(L)$ be piecewise linearly specified and consist of two segments while the function form changes at some transition point with coordinate $L=L_n$:

$$\theta(L)=\begin{cases} a\cdot L+b \text{ at } L\le L_n, \\ c\cdot L+d \text{ at } L>L_n, \end{cases} \tag{2}$$

where *a, b, c,* and *d* are some coefficients, then:

$$\frac{dH(L)}{dL}=\begin{cases} \operatorname{tg}(a\cdot L+b) \text{ at } L\le L_n, \\ \operatorname{tg}(c\cdot L+d) \text{ at } L>L_n. \end{cases} \tag{3}$$

Integrate these expressions, provided that the coordinates of the launch point $L=0$ and $H=0$ to obtain the dependence:

$$H(L)=\begin{cases} \frac{1}{a}\cdot\ln\left(\frac{\cos(b)}{\cos(a\cdot L+b)}\right) \text{ at } L\le L_n, \\ \frac{1}{a}\cdot\ln\left(\frac{\cos(b)}{\cos(a\cdot L_n+b)}\right)+ \\ +\frac{1}{c}\cdot\ln\left(\frac{\cos(c\cdot L_n+d)}{\cos(c\cdot L+d)}\right) \text{ at } L>L_n. \end{cases} \tag{4}$$

Let θ_0 be the angle of inclination of the path at the launch point; θ_n be the flight path angle at the transition point; θ_k be the flight path angle at PIP with coordinates $L=L_k$ and $H=H_k$, and values of the angles are known. Suppose also that the coordinate of range L_n of the transition point is known Then values of the coefficients *a, b, c,* and *s* can be determined using the systems of equations:

$$\begin{cases} a\cdot 0+b=\theta_0 \\ a\cdot L_n+b=\theta_n \end{cases}, \quad \begin{cases} c\cdot L_n+d=\theta_n, \\ c\cdot L_k+d=\theta_k. \end{cases} \tag{5}$$

It is possible to solve these systems of equations for *a, b, c,* and *d* by a substitution method or by a matrix method. As a result, expressions for finding *a, b, c,* and *d:* are obtained:

$$a=\frac{\theta_n-\theta_0}{L_n}, \quad b=\theta_0,$$

$$c=\frac{\theta_k-\theta_n}{L_k-L_n}, \quad d=\frac{L_k\cdot\theta_n-L_n\cdot\theta_k}{L_k-L_n}. \tag{6}$$

To determine coordinates of range of the transition point L_n, write the expression to determine the PIP altitude H_k:

$$H_k=H(L_k)=\frac{1}{a}\cdot\ln\left(\frac{\cos(b)}{\cos(a\cdot L_n+b)}\right)+ \\ +\frac{1}{c}\cdot\ln\left(\frac{\cos(c\cdot L_n+d)}{\cos(c\cdot L_k+d)}\right). \tag{7}$$

Substitute the expressions for *a, b, c,* and *d* into the given expression and perform necessary mathematical transformations to obtain the expression for determining L_n depending on H_k, L_k, θ_0, θ_n and θ_k:

$$L_n= \\ =\frac{(\theta_0-\theta_n)\cdot\left[H\cdot(\theta_n-\theta_k)+L_k\cdot\ln\left(\frac{\cos(\theta_n)}{\cos(\theta_k)}\right)\right]}{\theta_0\cdot\ln\left(\frac{\cos(\theta_n)}{\cos(\theta_k)}\right)+\theta_n\cdot\ln\left(\frac{\cos(\theta_k)}{\cos(\theta_0)}\right)+\theta_k\cdot\ln\left(\frac{\cos(\theta_0)}{\cos(\theta_n)}\right)}. \tag{8}$$

H_k, L_k, θ_0, θ_n and θ_k are control parameters for this model and L_n, *a, b, c,* and *d* are intermediate parameters.

Because the value of the expression in formula (4)

$$\frac{1}{a}\cdot\ln\left(\frac{\cos(b)}{\cos(a\cdot L_n+b)}\right)$$

is constant and independent of L, denote it as C, thus simplifying formula (4)

$$H(L)=\begin{cases} \frac{1}{a}\cdot\ln\left(\frac{\cos(b)}{\cos(a\cdot L+b)}\right) \text{ at } L\le L_n, \\ C+\frac{1}{c}\cdot\ln\left(\frac{\cos(c\cdot L_n+d)}{\cos(c\cdot L+d)}\right) \text{ at } L>L_n. \end{cases} \tag{9}$$

Expressions of type

$$C=\frac{1}{a}\cdot\ln\left(\frac{\cos(b)}{\cos(a\cdot L_n+b)}\right) \tag{10}$$

will be called auxiliary expressions.

For this construction of the flight path model, the curve $H(L)$ always passes through the points (0, 0), i.e. the origin, and (L_k, H_k,), i.e. PIP for $L_n>0$.

The algorithm of the practical application of the model relationships is as follows. First, values of intermediate parameters are calculated from the given (input) values of the control parameters H_k, L_k, θ_0, θ_n and θ_k: first, for L_n, then for the coefficients *a, b, c,* and *d*, after finally, these values are substituted in expressions for $H(L)$ and $\theta(L)$.

When simulating the TM or SAM flight, the value of the control angle of attack is determined from the relationship:

$$\alpha_n=k_H\cdot\left(H(L_n)-H_n\right)+k_\theta\cdot\left(\theta(L_n)-\theta_n\right), \tag{11}$$

where α_n is the current value of the angle of attack; k_h is the coefficient of proportionality of altitude control, k_θ is the coefficient of proportionality of control by the flight path angle; L_n, H_n, θ_n are current values of range, altitude, and flight path angle, respectively. Values of the coefficients k_h and k_θ are chosen in terms of ensuring acceptable rigidity of the problem and the allowable final miss. In this study, k_h=0.02 and k_θ=10.

According to a similar principle, the author has developed trilinear, biparabolic, and triparabolic models. The last two models define the shape of the path in a form of two and three parabolic segments, respectively, and have similar control parameters. Expressions for parabolic models are simpler than those for linear models. However, for the reasons described in Section 6, parabolic models are not presented.

Expressions for dependences $H(L)$ of the polynomial model were obtained by adding to the linear function passing through the points (0, 0) and (L_k, H_k) of the polynomial function passing through the points (0, 0) and (L_k, 0).

The shortcoming of the polynomial model described in Section 2 is eliminated by dividing the argument (L) of a corresponding degree by the value of distance to the PIP raised to the same degree. This form is the most convenient for practical application in solving problems of optimization of paths and DPs of the SAM and TM.

To eliminate the shortcoming of ensuring the correct control of the launch section, two modified options have been developed in the polynomial model by introducing auxiliary terms: power and fractionally rational.

The effectiveness of the constructed models of the programed flight was assessed in two ways:

– by assessment of the possibility of approximation using the constructed models of some pre-calculated reference paths using known models. This characterizes compliance of the model with the physics of flight;

– by comparison of extreme values of target functions for the constructed models and for the paths calculated with the use of known models. This characterizes conformity of the model to its intended purpose.

The flight was simulated for a hypothetical long-range SAM using a system of differential equations of aircraft flight in the atmosphere of a spherical non-rotating planet [15]. The lift-off mass of the SAM is 1055 kg. The SAM cruise motor is a dual-thrust solid fuel motor. The weight of solid fuel is 580 kg. Mixed fuel type: HTPB. The specific impulse of thrust in a vacuum: 260 s. The SAM was designed according to the normal aerodynamic scheme with wings located near the center of mass and aerodynamic rudders located in the tail.

The paths calculated using the method of setting the law of change of the angle of attack and PNM were used for comparison.

Dependence of the angle of attack on time was set in a piecewise constant form

$$\alpha(t)=\begin{cases}\alpha_1 \text{ at } t\le t_1,\\ \alpha_2 \text{ at } t_1<t\le t_2,\\ \alpha_3 \text{ at } t>t_2,\end{cases} \tag{12}$$

where α_1, α_2 and α_3 are the values of the angle of attack at corresponding intervals; α_1, α_2, α_3, t_1, and t_2 are the control parameters.

The coefficient of proportionality varying depending on the inclined distance to the target that is set by three linear segments was used for the PNM according to the formula:

$$k_p(L_s)=\min\left(\max\left(\begin{matrix}\dfrac{L_s-l_1\cdot L_{s0}}{(l_2-l_1)\cdot L_{s0}}\times\\ \times(k_2-k_1)+k_1;k_1\end{matrix}\right);k_2\right)\times$$

$$\times\left(1+(k_0-1)\cdot\frac{L_{s0}-L_s}{L_{s0}}\right), \tag{13}$$

where k_p is the coefficient of proportionality of PNM; L_s is the inclined distance to the target; k_0, k_1, and k_2 are some positive numbers; L_{s0} is the value of the inclined distance to the target at the time of launch.

To make an equivalent comparison, the number of control parameters for all models (including МПН and the method of setting the law of change of the angle of attack) was taken equal to the number of control parameters for the three-line model, i.e. equal to five without taking into account H_k and L_k. The bilinear and biparabolic models for which the maximum possible number of control parameters is three without taking into account H_k and L_k are exceptions.

The approximation was performed using the method of least squares [16, 17].

The following is conventionally used as the criteria for assessing the quality of approximation:

– mean approximation error (MAPE) [18, 19] (Table 1):

$$\text{MAPE}=\frac{1}{n}\cdot\sum_{i=1}^{n}\left|\frac{H_{ri}-H_a(L_{ri})}{H_{ri}}\right|\cdot 100\,\%; \tag{14}$$

– coefficient of determination (R^2) [16, 17] (Table 2):

$$R^2=1-\frac{\sum_{i=1}^{n}\left(H_{ri}-H_a(L_{ri})\right)^2}{\sum_{i=1}^{n}\left(H_{ri}-\overline{H_r}\right)^2}. \tag{15}$$

where n is the number of points of the approximated path; H_r is the value of altitudes of the points of the approximated path; L_r is the value of ranges of the points of the approximated path; $H_a(L_r)$ is the value of the approximating function at points L_r; the upper dash denotes the arithmetic mean.

Optimization of the SAM flight paths was performed using a genetic algorithm. The results are given in Table 3.

5. The results of the construction of programmed flight models

5. 1. Correlation of the programed flight models

The three-line model.

This model is a scaled bilinear model and consists of three segments. In addition to the control parameters H_k, L_k, θ_0 and θ_k similar to the bilinear model, the following is also added:

– θ_{n1}: the angle of path inclination at the first transition point;

– θ_{n2}: the angle of path inclination at the second transition point;

– k_L: the coefficient of the ratio of the coordinate of the distance of the second transition point (L_{n2}) to the coordinate of the distance of the first transition point (L_{n1}), $k_L > 1$.

Control parameters: H_k, L_k, θ_0, θ_{n1}, θ_{n2}, θ_k and k_L. Intermediate parameters: L_{n1}, L_{n2}, a_1, a_2, a_3, b_1, b_2, b_3.

The first group of auxiliary expressions:

$$\lambda=\theta_{n2}\cdot\ln\left(\frac{\cos(\theta_{n1})}{\cos(\theta_k)}\right)-$$

$$-\theta_{n1}\cdot\ln\left(\frac{\cos(\theta_{n2})}{\cos(\theta_k)}\right)-\theta_k\cdot\ln\left(\frac{\cos(\theta_{n1})}{\cos(\theta_{n2})}\right),$$

$$\vartheta=\theta_{n1}\cdot\ln\left(\frac{\cos(\theta_0)}{\cos(\theta_{n2})}\right)-$$
$$-\theta_{n2}\cdot\ln\left(\frac{\cos(\theta_0)}{\cos(\theta_{n1})}\right)-\theta_0\cdot\ln\left(\frac{\cos(\theta_{n1})}{\cos(\theta_{n2})}\right). \qquad (16)$$

Coordinates of the transition points:

$$L_{n1}=\frac{(\theta_{n1}-\theta_{n2})\cdot\left[H_k-\frac{L_k}{(\theta_k-\theta_{n2})}\cdot\ln\left(\frac{\cos(\theta_{n2})}{\cos(\theta_k)}\right)\right]}{\lambda\cdot\frac{k_L}{(\theta_k-\theta_{n2})}-\vartheta\cdot\frac{1}{(\theta_0-\theta_{n1})}},$$

$$L_{n2}=k_L\cdot L_{n1}. \qquad (17)$$

Values of the coefficients:

$$a_1=\frac{\theta_{n1}-\theta_0}{L_{n1}},\quad a_2=\frac{\theta_{n1}-\theta_{n2}}{L_{n1}-L_{n2}},\quad a_3=\frac{\theta_k-\theta_{n2}}{L_k-L_{n2}},$$

$$b_1=\theta_0,\quad b_2=\frac{L_{n1}\cdot\theta_{n2}-L_{n2}\cdot\theta_{n1}}{L_{n1}-L_{n2}},\quad b_3=\frac{L_k\cdot\theta_{n2}-L_{n2}\cdot\theta_k}{L_k-L_{n2}}. \qquad (18)$$

The second group of auxiliary expressions:

$$C_1=\frac{1}{a_1}\cdot\ln\left(\frac{\cos(b_1)}{\cos(a_1\cdot L_{n1}+b_1)}\right),$$

$$C_2=C_1+\frac{1}{a_2}\cdot\ln\left(\frac{\cos(a_2\cdot L_{n1}+b_2)}{\cos(a_2\cdot L_{n2}+b_2)}\right). \qquad (19)$$

Dependences for $H(L)$ and $\theta(L)$:

$$H(L)=\begin{cases}\frac{1}{a_1}\cdot\ln\left(\frac{\cos(b_1)}{\cos(a_1\cdot L+b_1)}\right) \text{ at } L\le L_{n1},\\ C_1+\frac{1}{a_2}\cdot\ln\left(\frac{\cos(a_2\cdot L_{n1}+b_2)}{\cos(a_2\cdot L+b_2)}\right) \text{ at } L_{n1}<L\le L_{n2},\\ C_2+\frac{1}{a_3}\cdot\ln\left(\frac{\cos(a_3\cdot L_{n2}+b_3)}{\cos(a_3\cdot L+b_3)}\right) \text{ at } L>L_{n2},\end{cases}$$

$$\theta(L)=\begin{cases}a_1\cdot L+b_1 \text{ at } L\le L_{n1},\\ a_2\cdot L+b_2 \text{ at } L_{n1}<L\le L_{n2},\\ a_3\cdot L+b_3 \text{ at } L>L_{n2}.\end{cases} \qquad (20)$$

The polynomial model.

Control parameters: H_k, L_k, and A is the vector of the polynomial coefficients which consists of n elements while the polynomial order is n+1.

Dependences for $H(L)$ and $\theta(L)$:

$$H(L)=\frac{H_k}{L_k}\cdot L+(L-L_k)\cdot\sum_{i=1}^{n}A_i\cdot\left(\frac{L}{L_k}\right)^i,$$

$$\theta(L)=\operatorname{arctg}\left[\frac{H_k}{L_k}+\sum_{i=1}^{n}\begin{matrix}A_i\cdot\left(\frac{L}{L_k}\right)^i+\\+\left(\frac{L}{L_k}+1\right)\cdot\sum_{i=1}^{n}i\cdot A_i\cdot\left(\frac{L}{L_k}\right)^{i-1}\end{matrix}\right]. \qquad (21)$$

The angle θ_0 is related to the coefficient A_1 by the dependence

$$A_1=\frac{H_k}{L_k}-\operatorname{tg}(\theta_0), \qquad (22)$$

and θ_k with vector A by the dependence

$$\sum_{i=1}^{n}A_i=tg(\theta_k)-\frac{H_k}{L_k}. \qquad (23)$$

If it is necessary to set angles θ_0 and θ_k simultaneously, then the dependence (23) will take the form:

$$\sum_{i=2}^{n}A_i=\operatorname{tg}(\theta_k)-\frac{H_k}{L_k}-A_1. \qquad (24)$$

The polynomial-power model.

In accordance with the recommendations given in [12], the auxiliary summand in the polynomial-power model (PPM) has the form of a power function. Change of values of the parameters a and k provides control of the launch section while k>1, a>0.

Control parameters: H_k, L_k, A, k, a.

Dependences for $\theta(L)$ and $H(L)$:

$$\theta(L)=\operatorname{arctg}\left[\begin{matrix}\frac{H_k}{L_k}+\sum_{i=1}^{n}A_i\cdot\left(\frac{L}{L_k}\right)^i+\left(\frac{L}{L_k}+1\right)\sum_{i=1}^{n}i\cdot A_i\cdot\left(\frac{L}{L_k}\right)^{i-1}-\\-\frac{a}{L_k}\cdot\left(\frac{L}{L_k}\right)^{\frac{1}{k}-1}\cdot\left(L+\frac{L-L_k}{k}\right)\end{matrix}\right],$$

$$H(L)=\frac{H_k}{L_k}\cdot L+(L-L_k)\cdot\left[\sum_{i=1}^{n}A_i\cdot\left(\frac{L}{L_k}\right)^i-a\cdot\left(\frac{L}{L_k}\right)^{\frac{1}{k}}\right]. \qquad (25)$$

If it is necessary to set θ_k (θ_0 is always equal to 90°), it is necessary to use the dependence:

$$\sum_{i=1}^{n}A_i=\operatorname{tg}(\theta_k)+a-\frac{H_k}{L_k}. \qquad (26)$$

It should be noted that the derivative of the power function at the point L=0 is equal to ∞) (it has a singularity) which may cause a problem when working in some mathematical packages (for example, Mathcad or MATLAB (USA). Such recommendations were also made during this study.

The polynomial rational model.

The auxiliary addend in the polynomial rational model (PRM) has the form of a fractionally rational function. Parameters a and k are also used to control the launch section. For most situations, 3<k<0, a>0 because the summand value approaches L at larger values of k and it becomes excessively sharp at smaller inclinations in the launch section.

Control parameters: H_k, L_k, A, k, a.

Dependences for $H(L)$ and $\theta(L)$:

$$H(L)=\frac{H_k}{L_k}\cdot L+$$
$$+(L-L_k)\cdot\left[\sum_{i=1}^{n}A_i\cdot\left(\frac{L}{L_k}\right)^i-a\cdot\frac{L\cdot(1+10^k)}{L+10^k\cdot L_k}\right],$$

$$\theta(L)= =\operatorname{arctg}\left[\begin{array}{l}\frac{H_k}{L_k}+\sum_{i=1}^{n}A_i\cdot\left(\frac{L}{L_k}\right)^i+\left(\frac{L}{L_k}+1\right)\times \\ \times\sum_{i=1}^{n}i\cdot A_i\cdot\left(\frac{L}{L_k}\right)^{i-1}-\frac{a}{L+10^k\cdot L_k}\times \\ \times\left[\left(1+10^k\right)\cdot\left(L-\left(L-L_k\right)\cdot\left(\frac{L}{L+10^k\cdot L_k}-1\right)\right)\right]\end{array}\right]. \quad (27)$$

If it is necessary to set the angle θ_0, the following dependence should be used:

$$A_1=\frac{H_k}{L_k}-\operatorname{tg}\left(\theta_0\right)+a\cdot\left(1+\frac{1}{10^k}\right). \quad (28)$$

If it is necessary to set the angle θ_k, dependence (26) similar to the PPN should be used and if it is necessary to set θ_0 and θ_k simultaneously, the above dependence will take the form:

$$\sum_{i=2}^{n}A_i=\operatorname{tg}\left(\theta_k\right)+a-\frac{H_k}{L_k}-A_1. \quad (29)$$

The launch angle in the PRM, in contrast to the PPM where the launch angle is always 90°, can take values close to 90° but not 90° in any case.

5. 2. The results of approximation of reference paths by means of the models of programmed flight

The reference designations of paths in Fig. 1 and Tables 1–3) mean the following:

– the capital letter is for the path type: V for the optimal path according to the criterion of maximum speed in PIP; T for the optimal path according to the criterion of minimum flight time in PIP; B for the ballistic path; I for atypical flight path;

– the number is for the coordinates of the final point in the range and altitude: 13 is for L_k=13 km; H_k=0.02 km; 70 is for L_k=70 km; L_k=2 km; 200 is for L_k=200 km and H_k=10 km;

– the lowercase letter is for the type of reference model of the programed flight: 'a' is for the law of change of the angle of attack, 'p' is for PNM.

Color designation of cells in Tables 1–3 is used to compare the quality of the result: red for a relatively bad result, green for a relatively good result.

Fig. 2–4 help to visually assess the quality of path approximations.

Table 1

Mean approximation error (MAPE)

Path model	Reference path					
	V13a	V70a	T200a	T200p	B200a	I200p
Bilinear	7.644 %	6.878 %	7.339 %	8.082 %	1.964 %	8.949 %
Biparabolic	9.174 %	9.654 %	9.258 %	8.789 %	1.288 %	9.668 %
Trilinear	1.721 %	2.187 %	1.223 %	1.697 %	1.398 %	3.484 %
Triparabolic	2.419 %	3.103 %	1.646 %	2.461 %	0.694 %	3.669 %
Polynomial	3.674 %	2.875 %	2.835 %	4.707 %	3.261 %	5.167 %
PPM	1.203 %	2.244 %	2.162 %	0.775 %	0.956 %	2.284 %
PRM	1.531 %	1.588 %	2.031 %	0.607 %	0.336 %	3.598 %

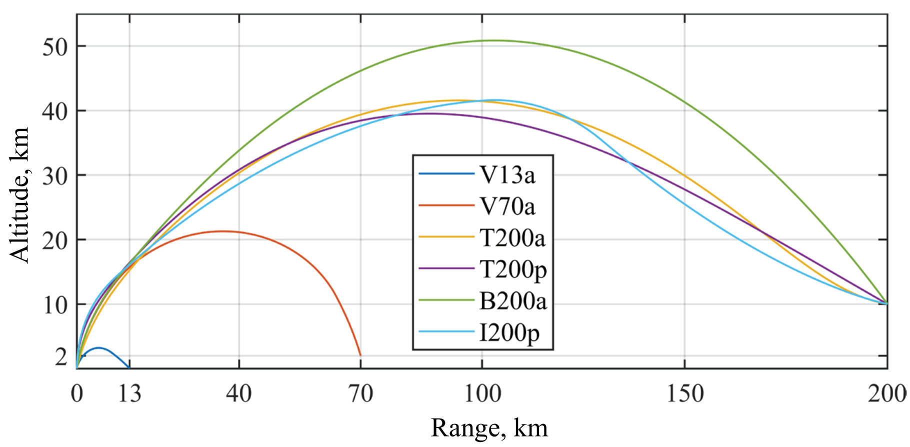

Fig. 1. Reference SAM flight paths

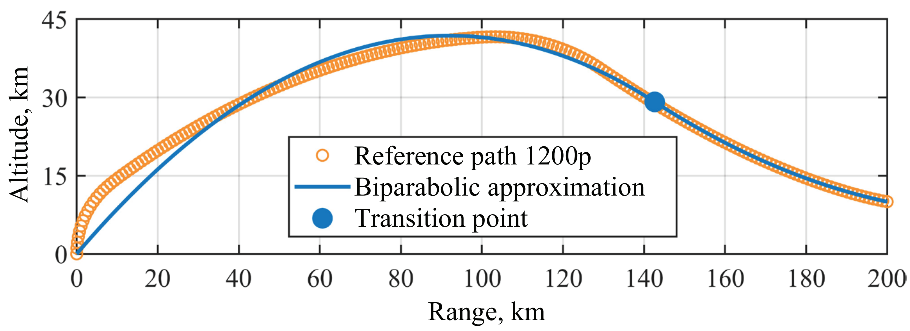

Fig. 2. An example of poor-quality approximation of the I200p path by the bilinear model

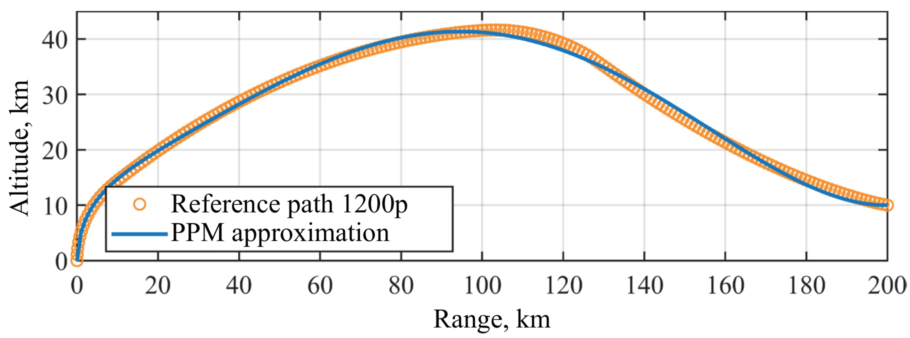

Fig. 3. Example of high-quality PPM approximation of the I200p path

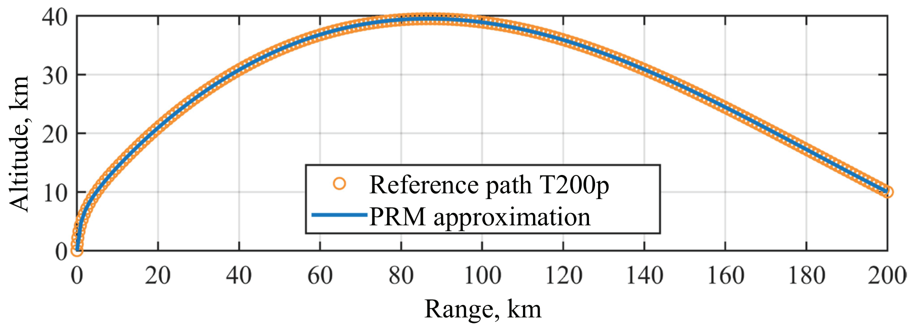

Fig. 4. An example of an almost perfect PRM approximation of the T200p path

Table 2

Coefficient of determination (R^2) of the approximated paths

Path model	Reference path					
	V13a	V70a	T200a	T200p	B200	I200
Bilinear	0.9905	0.9761	0.9865	0.9811	0.9991	0.9718
Biparabolic	0.9867	0.9504	0.9773	0.9771	0.9999	0.9642
Trilinear	0.9996	0.9992	0.9988	0.9993	0.9991	0.9908
Triparabolic	0.9991	0.9983	0.9995	0.9986	0.99995	0.9924
Polynomial	0.9983	0.9987	0.9983	0.9935	0.9981	0.9929
PPM	0.9998	0.9988	0.9987	0.9999	0.9999	0.9966
PRM	0.9998	0.9989	0.99904	0.99995	0.99998	0.996

5. 3. The results of path optimization using known and developed models of programmed flight

The results of optimization of flight paths of long-range SAMs described in Section 4 using various models of programmed flight are given in Table 3.

Fig. 5 helps make a visual assessment of the similarity of the flight paths that are optimal regarding the criterion of maximum final speed (column V70, Table 3) obtained by using known and developed models of programmed flight.

It can be stated that the obtained paths shown in Fig. 5 are quite close to some theoretically optimal flight path.

Table 3

Extreme values of objective functions

Model	Objective function/calculation case					
	Min. time, s			Max. speed, m/s		
	T13	T70	T200	V13	V70	V200
Angle of attack	18.221	78.592	193.963	1214.04	803.165	1099.14
PNM	18.554	79.866	200.369	1216.97	796.232	1066.48
Bilinear	18.266	78.622	202.722	1207.63	796.849	1104.68
Biparabolic	18.268	78.782	206.662	1203.92	779.135	982.65
Trilinear	18.230	78.609	196.619	1209.67	805.480	1120.40
Triparabolic	18.269	78.668	198.035	1208.37	795.642	1100.68
Polynomial	18.272	78.596	197.617	1214.29	807.389	1085.88
PPM	18.199	78.382	194.956	1218.22	806.513	1111.73
PRM	18.201	78.409	194.732	1215.70	803.492	1131.39

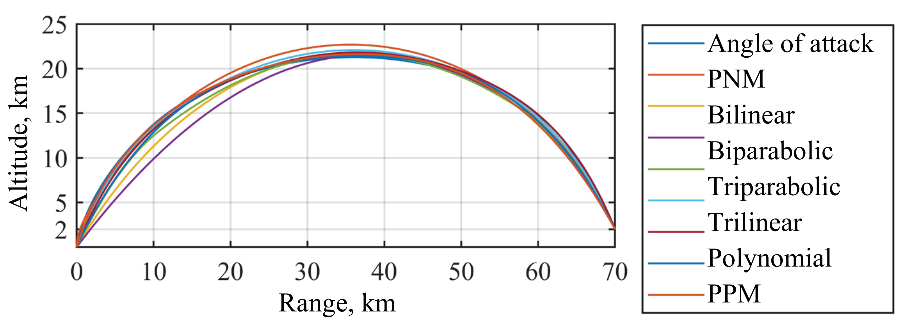

Fig. 5. The V70 flight paths optimal regarding the criterion of maximum final speed for various models

6. Discussion of the results obtained in the development and study of compliance with the purpose of the models of programmed flight

When conducting a comparative analysis (the results are given in Tables 1–3), the linear models showed better results compared to the parabolic models. Because of the absence of advantages, the parabolic models are not recommended for practical application, so their expressions are not presented in this study.

The advantage of linear models of programmed flight of this type consists in the minimum number of control parameters. All control parameters have a physical meaning: in addition to the obvious ones, i.e. the PIP coordinates (L_k and H_k), flight path angles at the launch points, transitions, and PIP are also set. If necessary, this makes it possible to set limits for values of the angles of launch and terminal flight-path in an explicit form during optimization of the flight paths.

It is believed that the value of the mean approximation error less than five percent indicates a good fit of the approximation model to the physics of this process. The mean error of approximation of reference paths (Fig. 1) for three-segment and polynomial models does not exceed 5 %. The only exception is approximation by the polynomial model of the non-standard path I200p which equals 5.167 % (Table 1).

Among the developed models of the segment-type programmed flight, the trilinear model showed the best results. It was better than the polynomial model in most cases. Therefore, the trilinear model should be recommended for its use in further studies.

The presence of knees in the function $\theta(L)$ of the linear segment models is not a problem in simulation because the actual realized values of the flight path angle are smoothed due to the dynamic characteristics of the aircraft.

From the point of view of the possibility of approximation of pre-calculated reference paths, the modified polynomial models showed almost identical results (Tables 1, 2). In terms of comparing the extreme values of the objective functions for the calculated paths, the PPM showed a slightly better result than the PRM (Table 3). However, when using the PPM, some difficulties appear during the calculation. These difficulties could not be eliminated in a way that would not lead to a significant increase in the calculation time. Therefore, according to the results of this study, among the polynomial models considered in this study, it is recommended to give preference to the PRM for further practical application.

Modified polynomial models showed the best results due to the ability to separately control the launch section of the flight.

In comparison with the applied existing models of programmed flight, the model of setting the law of change of the angle of the attack showed the best result in optimization just in a single case (T200), Table 1, but it differed from the result shown by the PRM by only 0.4 % (Table 3).

The visual similarity of flight paths (Fig. 5) obtained with the help of various models also indicates a good correspondence to the physics of flight.

The results are explained by:

– the principle of constructing models of programmed flight which provides a stable flight due to setting the interconnected programmatic values of altitude and angle of the flight path using the proportional derivative control;

– variety of constructed options of the programed flight models used in the study (seven models were constructed); this has allowed us to make a comprehensive comparison and determine which models are worth further practical application and which should not be recommended.

The setting of interrelated programmed values of altitude and the flight path angle is the main difference of the developed models of programmed flight from those presented in [7, 12]. The use of fractionally-rational summand to ensure control of the launch section of the flight is the main difference between the developed polynomial rational model and those presented in [12, 13].

It is recommended to use the bilinear model only for estimation calculations when it is necessary to obtain the result as quickly as possible because the minimum number of control parameters makes the optimization process as fast as possible. It is recommended to use the trilinear model when it is necessary to set limits on values of the launch and terminal flight-path angles in an explicit form. In all other cases, it is recommended to use modified polynomial models.

It should be noted that not all flight paths from those prescribed by the models can be implemented by SAM or TM because of finite maneuverability. However, such flight paths are detected and excluded from consideration in the process of numerical flight simulation (on the reason that they do not meet certain restrictions).

Setting of the programed flight of missiles only in the vertical plane (the firing plane) is the main limitation in the application of the developed models of programmed flight. This simplification is conventionally used at the design stages. The models of lateral programmed flight must be developed separately. Besides, the developed models specify the motion of highly maneuverable missiles in the atmosphere on aero-ballistic paths. For the missiles with limited aerodynamically determined maneuverability, it is necessary to use fundamentally different models of programmed flight.

The fact that all models were designed for zero coordinates of the launch point (for simplification) can be considered the main disadvantage of this study. Further improvement of this study may consist in the elimination of this shortcoming. This refinement will make it possible to apply the models in the problems of optimizing the flight paths of air-to-ground missiles as well as adjusting the flight task during the flight (re-targeting).

7. Conclusions

1. Several models of programmed flight of the TM and SAM were developed. They set interrelated values of programmed altitude and flight path angle and provide automatic hit in the PIP. The developed models have a minimum number of control parameters, a simple algebraic form, and allow the user to set limits on the launch and terminal flight-path angles. The developed models are presented in a form ready for practical application in the problems of actual optimization of flight paths of the TM and SAM and complex optimization of DP and flight paths of missiles of these types at design stages.

2. Possibilities of approximation of some pre-calculated basic paths with application of other known models with the help of the developed models of programmed flight were analyzed. In all cases, the mean approximation error did not exceed 5 % for most models which is a good result. For

trilinear and modified polynomial models, this figure did not exceed 3.6 % which is due to the ability to correctly set the launch section of the flight. This result allows us to recommend the trilinear and modified polynomial models for further practical application.

3. Extreme values of objective functions for the paths calculated according to the developed models and for the paths calculated using other known models were compared. Similar to the approximation analysis, the best results were shown by the trilinear and modified polynomial models due to the same features. Only in a single case, a known model of setting the law of change of the angle of the attack showed the best result. This result confirms the recommendation for further practical application of trilinear and modified polynomial models.

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