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ANALYTICAL STUDY OF AUTO-BALANCING WITHIN THE FRAMEWORK OF THE FLAT MODEL OF A ROTOR AND AN AUTO-BALANCER WITH A SINGLE CARGO

This paper reports the analytically established conditions for the onset of auto-balancing for the case of a flat rotor model on isotropic elastic-viscous supports and an auto-balancer with a single load. The rotor is statically unbalanced, the rotation axis is vertical. The auto-balancer has a single cargo – a pendulum, a ball, or a roller. The balancing capacity of the cargo is equal to the rotor imbalance.

The physical-mathematical model of the system is described. The differential equations of motion are recorded in dimensionless form relative to the coordinate system that rotates synchronously with the rotor. The so-called main movement has been found; in it, the cargo synchronously rotates with the rotor and balances it. The differential equations of motion are linearized in the neighborhood of the main movement. A characteristic equation has been constructed. It helped investigate the stability of the main movement (an auto-balancing mode) for the cases of the absence and presence of resistance forces in the system.

It was established that in the absence of resistance forces in the system:

– the rotor has three characteristic rotational speeds, and the first always coincides with the resonance frequency;

– auto-balancing occurs when the rotor rotates at speeds between the first and second ones, and above the third characteristic speed;

– the value of the second and third characteristic speeds is significantly influenced by the ratio of weight to the mass of the system;

– the second and third characteristic speeds monotonously increase with an increase in the ratio of cargo weight to the mass of the system.

Resistance forces significantly affect both the values of the second and third characteristic speeds and the conditions of their existence. Small resistance forces do not change the quality behavior of the system. With high resistance forces, the number of characteristic speeds decreases to one.

The paper reports the results applicable to an auto-balancer with many cargoes when it balances the imbalance that equals the balancing capacity of the auto-balancer

Keywords: passive auto-balancer, rotor, automatic balancing, static balancing, motion stability, static imbalance

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1. Introduction

To apply passive auto-balancers for automatic balancing on the stroke of high-speed rotors, a deep theoretical study of the phenomenon of auto-balancing [1–15] is required. The most general information is given by the results of analytical studies [2–15]. Given the large number of degrees of freedom of the auto-balancing system, it is extremely difficult to analytically investigate the auto-balancing process. Therefore, most studies analytically derive differential equations of system movement, and further research is carried out by numerical methods. In this case, however, the results obtained are not general in nature.

The alternative is to solve a series of so-called model problems. On the one hand, they should represent somewhat simplified models of rotary systems with auto-balancers, which can be relatively deeply investigated analytically. On the other hand, the results of solving model problems should make it possible to apply them to more complex rotary systems with auto-balancers.

Below we investigate the stability of an auto-balancing mode regarding a flat model of the statically unbalanced rotor on isotropic elastic-viscous supports. The rotor balances a ball, a roller, or a pendulum auto-balancer with a single cargo. This is a model problem. It is distinguished not only by the relative mathematical simplicity but also by the possibility of applying the obtained results to auto-balancers with many cargoes. Thus, for the case of the greatest rotor imbalance, cargoes in an auto-balancer with many cargoes come together and are built opposite the imbalance. In this case, one composite cargo is conditionally formed.

Thus, it is important to solve the specified model problem both in order to construct an analytical theory of passive auto-balancing and to practically implement it in order to design these devices for specific rotors.

2. Literature review and problem statement

The structure, principle of operation, and examples of using the ring, pendulum, ball (roller) auto-balancers are described in [1]. In [2], a flat rotor model on isotropic elastic supports and a two-ball auto-balancer are applied to establish steady movements at which the balls synchronously rotate with the rotor. Resistance forces were not taken into consideration. The existence of one main movement was identified at which the balls balance the rotor, as well as three side ones at which the rotor is unbalanced. At the first side movement the balls are on the heavy side of the rotor, at the second – from the light one, and, at the third, one ball is on the heavy side, the other one is on the light side of the rotor. An energy approach was used to establish that the first side movement is stable at pre-resonant speeds of rotor rotation, and the main movement is at the above-the-resonance speeds. In [3], a similar result was obtained using an approximate method – the synchronization of dynamic systems. The results reported in works [2, 3] create an idea that a complete bifurcation theory was constructed for the considered model of the rotor and auto-balancer. According to the results of that theory, when the main movement is stable, the side movements are unstable. That makes it possible not to investigate the stability of side movements but to concentrate on studying the stability of the main movement. The authors of [4] considered a model of the rotor on isotropic elastic-viscous supports and a two-ball auto-balancer. The movement of balls relative to the auto-balancer is prevented by the forces of viscous resistance. They studied the stability of the main movement in the small using the first method by Lyapunov based on the equations of the first approximation. A characteristic equation was derived in the form of a polynomial of the 8th power. Given a significant number of parameters and the high power of the polynomial, the cited study was carried out both analytically, in some particular cases, and by numerical methods. The decomposition of roots based on the powers of the small parameter (under certain imbalances) established that the auto-balancing would occur at the above-the-resonant speeds of rotor rotation. Numerical methods have established that the onset of auto-balancing is affected by both the imbalance of the rotor (the auto-balancing positions of balls in an auto-balancer) and the forces of viscous resistance.

Work [5] discovered the new modes of steady movements of the rotor with a (pendulum) auto-balancer – the modes of cargo jamming. These modes are caused by the Sommerfeld effect [6] and occur with low resistance forces in the system. According to the results of [5], under the new modes of movement, cargoes get stuck at one of the rotor's resonant speeds.

It should be noted that there are many studies that analytically and by numerical methods investigated different modes of movement of the rotor and auto-balancer with two balls, rollers, or pendulums. Analytically, the stability of the main movement is most fully investigated in work [7], where the methods and approaches reported in [4] were applied. Decomposition of the roots of the characteristic equation and characteristic speeds (when transitioning which auto-balancing occurs or is lost) was found at different ratios of smallness between the parameters of the system. It was found that the characteristic speeds monotonously (and rapidly) increase with an increase in the ratio of weight of cargoes to the mass of the system.

The stability of the main, side movements and jamming modes was most completely investigated by numerical methods in work [8]. Paper [9] applied numerical methods to study the influence of viscous resistance forces on the speed of auto-balancing onset. Analytically approximate methods were employed to study the modes of cargo jamming (two pendulums) in [10]. In [11], mostly numerical methods investigated jamming modes in a two-ball auto-balancer.

The common disadvantages of the studies reported in [1–11] are the construction of the theory for auto-balancers with two cargoes, as well as the insufficient depth of the analytical solution to problems. Also, the theory is built separately for ball and pendulum auto-balancers.

The authors of work [12] derived a characteristic equation that determines the stability of the main movements when balancing the rotor with a multi-ball auto-balancer. Paper [13] analytically investigated the main, side steady movements and modes of cargo jamming for the case of an auto-balancer with many cargoes. The coordinates of the rotor and the parameters of the total imbalance of the rotor were taken as generalized coordinates.

It should be noted that even in the case of an auto-balancer with two cargoes, the problems are too complex and cumbersome for analytical solution. To overcome these difficulties, some studies solved model problems. Thus, the phenomenon of a single pendulum getting stuck was investigated by approximate methods in [14]. It was found that the pendulum gets stuck at one of the resonance frequencies of rotor fluctuations. The result is applicable for auto-balancers with many cargoes. This is because when the cargoes get stuck, they are tightly pressed against each other and rotate together relative to the rotor as a single conditional cargo. But that applies only to the conditions of the existence and stability of the jamming mode. In an auto-balancer with many cargoes, the composite cargo can disintegrate and, because of this, there are other steady movements (side, main). The disadvantage of the cited study is the solution to the problem by approximate methods. Because of this, not all possible jamming modes were found.

In [15], analytical methods were used to investigate the phenomenon of ball jamming (roller, pendulum) within a flat model of the balanced rotor on isotropic supports. Analytically, it was found that the system has only such steady modes of movement under which a (single) cargo gets stuck. For a specific rotor rotational speed, there may be one or three angular jamming speeds. Only odd speeds can be stable if they are numbered in ascending order. The results reported in [14, 15] demonstrate the feasibility of using model problems in the construction of an analytical theory of passive auto-balancers.

Based on the results of studies [7, 13], we can conclude that the auto-balancer would balance any imbalance if it balances the maximum and minimum imbalance. With the greatest imbalance, the cargoes come together, move to the light side of the rotor, and form a conditional composite cargo there. This can be modeled by an auto-balancer with a single cargo. The analytical problem of balancing the rotor with an auto-balancer with a single cargo has not yet been solved.

3. The aim and objectives of the study

The aim of this study is to analytically investigate the stability of an auto-balancing mode (the main movement stability) within a model of the flat rotor on isotropic elastic supports and an auto-balancer with a single cargo. This is important both for the construction of an analytical theory of passive auto-balancing and for practical tasks related to the design of these devices for specific rotors.

To accomplish the aim, the following tasks have been set:

– to mathematically state the problem of investigating the stability of an auto-balancing mode;

– to find conditions for the stability of an auto-balancing mode in the absence of resistance forces in the system;

– to determine the influence of resistance forces on the stability of an auto-balancing mode.

4. Methods to study the stability of an auto-balancing mode

The theoretical studies employ elements from the theory of automatic balancing of rotors [13], classical mechanics [16], the theory of stability of movements of mechanical systems [17].

The stability of an auto-balancing mode is investigated by the first method of Lyapunov using the equations of the first approximation [13, 17]. In this case, a characteristic equation is built. Due to the high order of the corresponding polynomial, the studies apply a method of decomposition of polynomial roots according to the powers of the small parameter [18]. At the same time, different ratios of smallness between the parameters of the system are considered. In more detail, research methods are revealed in the solution of the set study tasks.

5. Results of investigating the auto-balancing mode stability

5. 1. The physical-mathematical model of the system

5. 1. 1. Description of the flat rotor model and auto-balancer

To study the stability of the auto-balancing mode, a well-known flat model of the rotor and auto-balancer is used (Fig. 1) [13, 14]. The rotor of mass M has the shape of a disk. The center of the disk hosts a weightless absolutely rigid shaft located vertically. The shaft is held by isotropic elastic-viscous supports with stiffness and damping coefficients c and $\tilde{b}$, accordingly. The rotor moves in parallel to the horizontal plane. The rotational speed of the rotor is constant ω=const. The auto-balancer hosts a single cargo weighing m. The center of the cargo mass moves relative to the auto-balancer in a circle of radius l. Cargo in the form of a ball or a roller has a radius R (Fig. 1, b), and rolls without slipping along a circular treadmill. Cargo in the form of a pendulum (Fig. 1, c) is freely mounted onto the shaft. The main central axial moment of inertia of the pendulum is I_C.

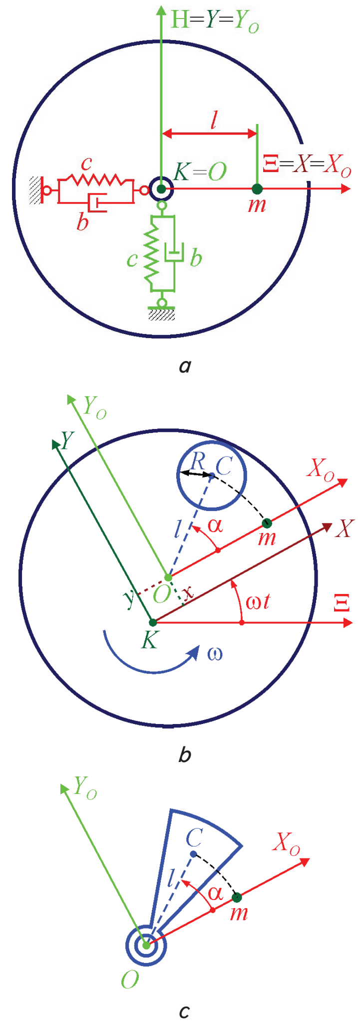


Fig. 1. The flat model of a rotor and an auto-balancer [13, 14]: a – statically unbalanced rotor on isotropic elastic-viscous supports; b – rotor and ball movement (roller); c – pendulum movement

Static imbalance of the rotor is created by the point mass m located at a distance l from the longitudinal axis of the rotor.

If the system is stationary, the shaft (point O) is combined with the rotation axis (point K). When the system moves, the shaft deviates from the axis of rotation.

The movement of the system is described using the following systems of right rectangular axes:

– $K\Xi H$ – fixed axes;

– OX_OY_O – moving axes that are rigidly connected to the rotor; they begin in the center of the disk (point O), and the X_O axis passes through an unbalanced point mass;

– KXY – moving axes parallel to the OX_OY_O axes.

The angle of rotor rotation (KXY axes) around the K point is ωt, where t is the time. The position of cargo relative to the OX_OY_O is set by angle α. When moving a ball (roller) relative to the rotor, the force of viscous resistance $\tilde{\beta} l\alpha'$, acts on it, where $\tilde{\beta}$ is the coefficient of viscous resistance forces, $l\alpha'$ is the speed of movement of the center of a ball (roller) relative to the rotor: the stroke indicates a time derivative. When moving a pendulum relative to the rotor, the pendulum is exposed to a moment of the force of viscous resistance $\tilde{\beta} l^2\alpha'$, where $\tilde{\beta}$ is the coefficient of forces of viscous resistance, α' is the angular speed of rotation of the pendulum around the shaft.

The mechanical system has the following mass and resonance frequency

$$M_\Sigma = M + 2m,\ \ \omega_0 = \sqrt{c / M_\Sigma}. \qquad (1)$$

Note that the balancing capacity of the cargo is equal to the static imbalance of the rotor.

5. 1. 2. The dimensionless differential equations of system movement, an auto-balancing mode

The dimensionless differential equations of system motion are as follows:

$$\tilde{L} = \ddot{\alpha} + \beta\dot{\alpha} - \begin{bmatrix} \left(\ddot{u} - 2n\dot{v} - n^2 u\right)\sin\alpha - \\ -\left(\ddot{v} + 2n\dot{u} - n^2 v\right)\cos\alpha \end{bmatrix} = 0,$$

$$L_u = \ddot{u} - 2n\dot{v} - n^2 u + b\left(\dot{u} - nv\right) + u -$$
$$-\varepsilon\left[n^2 + \ddot{\alpha}\sin\alpha + \left(\dot{\alpha} + n\right)^2\cos\alpha\right] = 0,$$

$$L_v = \ddot{v} + 2n\dot{u} - n^2 v + b\left(\dot{v} + nu\right) + v -$$
$$-\varepsilon\left[-\ddot{\alpha}\cos\alpha + \left(\dot{\alpha} + n\right)^2\sin\alpha\right] = 0. \qquad (2)$$

In (2), the following dimensionless time (τ), constants (u, v), parameters (n, β, b) are introduced:

$$\tau = \omega_0 t,\ u = \frac{x}{\kappa l},\ v = \frac{y}{\kappa l},\ n = \frac{\omega}{\omega_0},$$

$$\beta = \frac{\tilde{\beta}}{\kappa m\omega_0},\ b = \tilde{b}\big/\left(\omega_0 M_\Sigma\right),\ \varepsilon = m\big/\left(\kappa M_\Sigma\right), \qquad (3)$$

the point indicates a derivative from dimensionless time; for a roller $\kappa = 7/5$, a ball $\kappa = 3/2$, a pendulum $\kappa = 1 + I_C/\left(ml^2\right)$.

On stationary movements

$$\tilde{\alpha}, \tilde{u}, \tilde{v} = \text{const} \qquad (4)$$

equations (2) take the following form:

$$\tilde{L}^{(0)} = n^2\left(\tilde{u}\sin\tilde{\alpha} - \tilde{v}\cos\tilde{\alpha}\right) = 0,$$

$$L_u^{(0)} = \tilde{u}\left(1 - n^2\right) - bn\tilde{v} - \varepsilon n^2\left(1 + \cos\tilde{\alpha}\right) = 0,$$

$$L_v^{(0)} = \left(1 - n^2\right)\tilde{v} + bn\tilde{u} - \varepsilon n^2\sin\tilde{\alpha} = 0. \qquad (5)$$

The system of equations (5) permits the following solution:

$$\tilde{u} = \tilde{v} = 0,\ \tilde{\alpha} = \pi. \qquad (6)$$

This is the main steady movement. During it, a cargo synchronously rotates with the rotor, occupies a position opposite the unbalanced mass, and balances it. In this case, the shaft is combined with the rotation axis.

5. 1. 3. The linearization of differential motion equations, a characteristic equation

We consider (6) as an unperturbed movement. Introduce the perturbed movement into consideration

$$u,\ v,\ \alpha = -\pi + \gamma, \qquad (7)$$

where u, v, γ are the perturbations, and $|u|$, $|v|$, $|\gamma|$, $|\dot{u}|$, $|\dot{v}|$, $|\dot{\gamma}| \ll 1$.

Substitute (7) in (2). Upon linearization, we find the following equations of the first approximation

$$\tilde{L} = \ddot{\gamma} + \beta\dot{\gamma} - \left(\ddot{v} + 2n\dot{u} - n^2 v\right) = 0,$$

$$L_u = \ddot{u} - 2n\dot{v} - n^2 u + b\left(\dot{u} - nv\right) + u + 2\varepsilon n\dot{\gamma} = 0,$$

$$L_v = \ddot{v} + 2n\dot{u} - n^2 v + b\left(\dot{v} + nu\right) + v - \varepsilon\left(\ddot{\gamma} - n^2\gamma\right) = 0. \qquad (8)$$

Introduce the following designations

$$a_{11} = \lambda^2 + \beta\lambda,\ a_{12} = -2n\lambda,\ a_{13} = -\left(\lambda^2 - n^2\right),$$

$$a_{21} = -\varepsilon a_{12},\ a_{22} = \lambda^2 + b\lambda + 1 - n^2,\ a_{23} = -n\left(2\lambda + b\right),$$

$$a_{31} = \varepsilon a_{13},\ a_{32} = -a_{23},\ a_{33} = a_{22}. \qquad (9)$$

Then, the characteristic equation of the system takes the following form:

$$\Delta(\lambda) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -\varepsilon a_{12} & a_{22} & a_{23} \\ \varepsilon a_{13} & -a_{23} & a_{22} \end{vmatrix} =$$
$$= a_{11}\left(a_{22}^2 + a_{23}^2\right) + \varepsilon\left[a_{22}\left(a_{12}^2 - a_{13}^2\right) + 2a_{12}a_{13}a_{23}\right] = 0. \qquad (10)$$

The characteristic equation in the form of a polynomial

$$\Delta(\lambda) = a_0\lambda^6 + a_1\lambda^5 + \cdots a_5\lambda + a_6 = 0, \qquad (11)$$

where

$$a_0 = 1 - \varepsilon,\ a_1 = \beta + \left(2 - \varepsilon\right)b,$$

$$a_2 = \left(2 - \varepsilon\right)\left(n^2 + 1\right) + b\left(b + 2\beta\right),$$

$$a_3 = 2bn^2\varepsilon + \beta b^2 + 2\left(n^2 + 1\right)\left(b + \beta\right),$$

$$a_4 = \left(n^2 - 1\right)^2 + \varepsilon n^2\left(n^2 + 6\right) + 2\beta b\left(n^2 + 1\right) + b^2 n^2,$$

$$a_5 = \beta\left(n^2 - 1\right)^2 + bn^2\left(3\varepsilon n^2 + \beta b\right),\ \ a_6 = \varepsilon n^4\left(n^2 - 1\right). \qquad (12)$$

One can see from (12) that when the rotor rotates at a resonance angular speed, one zero root appears in characteristic equation (11). Therefore, the angular velocity n=1 is suspected of the first characteristic speed. When passing it, the main movement should acquire stability. This is subject to that the remaining roots of characteristic equation (11) would have a negative real part.

One can see from (12) that if there are other characteristic speeds greater than 1, then when rotating the rotor at these speeds, characteristic equation (11) should have at least one pair of purely imaginary roots.

5. 2. Investigating the stability of the main movement in the absence of resistance forces

In the absence of resistance forces β=*b*=0, characteristic equation (11) takes the following form

$$\Delta = a_0 x^3 + a_2 x^2 + a_4 x + a_6, \tag{13}$$

where

$$a_0 = 1-\varepsilon,\ \ a_2 = (2-\varepsilon)(n^2+1),\ \ a_4 = (n^2-1)^2 + \varepsilon n^2 (n^2+6),$$

$$a_6 = \varepsilon n^4 (n^2-1),\ \ x = \lambda^2. \tag{14}$$

The necessary condition for the stability of the main movement is the negative condition for all roots of polynomial (13). It is well known that in order for all the roots of polynomial (13) to be negative, it is necessary and sufficient that the following conditions are met:

$$a_j > 0\ \ (j = 0,2,4,6);$$

$$p(n,\varepsilon) = -a_2^2 a_4^2 + 4a_2^3 a_6 + 27a_0^2 a_6^2 - $$
$$-18a_0 a_2 a_4 a_6 + 4a_0 a_4^3 \le 0. \tag{15}$$

We find the first critical speed from the first group of conditions in (15)

$$n_1(\varepsilon) = 1, \tag{16}$$

which coincides with the resonance frequency. Note that the last condition in (15) is met because

$$\forall \varepsilon > 0:\ p(1,\varepsilon) = -196\varepsilon^2 (8\varepsilon^2 - 11\varepsilon + 4) < 0. \tag{17}$$

Other speeds will be determined in the form of series by using the method of decomposition of polynomial roots by the powers of the small parameter [18]. Explicitly, the polynomial $p(n, \varepsilon)$ takes the following form

$$p(n,\varepsilon) = c_0 n^{10} + c_2 n^8 + \dots + c_8 n^2 + c_{10}, \tag{18}$$

where

$$c_0 = -16(1-2\varepsilon)^4 < 0,\ \ c_2 = 8(8-\varepsilon-14\varepsilon^2)(1-2\varepsilon)^2,$$

$$c_4 = -8(12 - 55\varepsilon + 140\varepsilon^2 - 192\varepsilon^3 + 104\varepsilon^4),$$

$$c_6 = 64 - 344\varepsilon + 567\varepsilon^2 - 296\varepsilon^3 - 32\varepsilon^4,$$

$$c_8 = -2(8 - 20\varepsilon + 11\varepsilon^2 + 6\varepsilon^3),\ \ c_{10} = -\varepsilon^2. \tag{19}$$

The decomposition of real roots of polynomial (18) takes the following form

– for $0 < \varepsilon < 0.05$

$$n_2(\varepsilon) = 1 + \frac{49}{32}\varepsilon + \frac{12397}{4096}\varepsilon^2 + \frac{12955649}{2097152}\varepsilon^3 \dots,$$

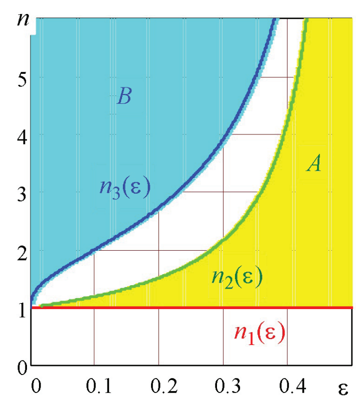
$$n_3(\varepsilon) = 1 + 2\left(\frac{\varepsilon}{4}\right)^{\frac{1}{3}} + \frac{7}{3}\left(\frac{\varepsilon}{4}\right)^{\frac{2}{3}} + \frac{29}{8}\frac{\varepsilon}{4} + \frac{5119}{648}\left(\frac{\varepsilon}{4}\right)^{\frac{4}{3}} + \cdots; \tag{20}$$

– for $0.05 \le \varepsilon < 0.5$

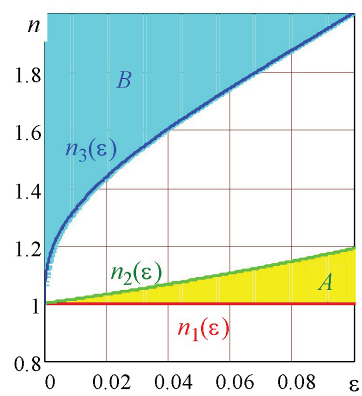
$$n_{2/3}(\varepsilon) = \frac{1}{8q^2}\left(1 \mp 2q + \frac{11}{2}q^2 \pm \frac{37}{4}q^3 - \frac{155}{8}q^4 + \cdots\right),$$

$$q = \sqrt{(1-2\varepsilon)/8}. \tag{21}$$

In Fig. 2, in the plane (n, ε) of the dimensionless parameters of the system, we built precise charts of the characteristic speeds of rotor rotation. Characteristic speeds are the boundaries of the two regions of stability *A* and *B* of the main motion in the plane.



a



b

Fig. 2. Characteristic speeds and regions of stability of the main movement in the absence of resistance forces: *a* – over the entire change interval of change in ε (0<ε<0.5); *b* – for 0<ε<0.1; *A* – the first, *V* – the second region of stability of the main movement

Our calculations show that the decomposition error in (20) or (21) does not exceed 7 %. At the same time, the biggest error is at a distance of 0 or 0.5.

Thus, for any $\varepsilon \in (0, 0.5)$, the system has three characteristic speeds. When ε approaches 0.5, the second and third characteristic speeds increase unlimitedly. In this case, the case of ε=0.5 has only theoretical significance because it cannot be implemented, and, in practice, $\varepsilon << 1$.

5. 3. Investigating the stability of the main movement in the presence of resistance forces

5. 3. 1. Studying the stability of the main movement based on the decomposition of the roots of the characteristic equation

Below we perform the decomposition of the roots of characteristic equation (11) at different ratios of smallness between dimensionless parameters of the system. This method of research is standard [13].

1. The case when the weight of the cargo is much smaller than the rotor mass $\varepsilon << 1$:

$$\lambda_{1/2} = -\frac{b}{2} \pm i\sqrt{1-\left(\frac{b}{2}\right)^2} - in + O(\varepsilon),$$

$$\lambda_{3/4} = \bar{\lambda}_{1/2}, \quad \lambda_5 = -\beta + O(\varepsilon),$$

$$\lambda_6 = -\frac{\varepsilon n^4\left(n^2-1\right)}{\beta\left[\left(n^2-1\right)^2 + b^2 n^2\right]} + O\left(\varepsilon^2\right). \tag{22}$$

2. The case of large forces of internal resistance $\beta >> 1$:

$$\lambda_{1/2} = -\frac{b}{2} \pm i\sqrt{1-\left(\frac{b}{2}\right)^2} - in + O\left(\frac{1}{\beta}\right),$$

$$\lambda_{3/4} = \bar{\lambda}_{1/2}, \quad \lambda_5 = -\frac{\beta}{1-\varepsilon} + O(1),$$

$$\lambda_6 = -\frac{\varepsilon n^4\left(n^2-1\right)}{\beta\left[\left(n^2-1\right)^2 + b^2 n^2\right]} + O\left(\frac{1}{\beta^2}\right). \tag{23}$$

3. The case when the weight of the cargo is much smaller than the rotor mass and the small forces of internal resistance $\varepsilon << 1,\ \beta \sim \sqrt{\varepsilon}$:

$$\lambda_{1/2} = -\frac{b}{2} \pm i\sqrt{1-\left(\frac{b}{2}\right)^2} - in + O\left(\sqrt{\varepsilon}\right), \quad \lambda_{3/4} = \bar{\lambda}_{1/2},$$

$$\lambda_{5/6} = -\frac{\beta}{2} \pm \left\{\left(\frac{\beta}{2}\right)^2 - \frac{\varepsilon n^4\left(n^2-1\right)}{\left(n^2-1\right)^2 + b^2 n^2}\right\}^{\frac{1}{2}} + O(\varepsilon). \tag{24}$$

4. The case when the weight of the cargo is much smaller than the rotor mass and the rotor rotates rapidly $\varepsilon << 1, n \sim 1/\varepsilon$:

$$\lambda_{1/2} = -\frac{b}{2} \pm i\sqrt{1-\left(\frac{b}{2}\right)^2} - in + O(\varepsilon), \quad \lambda_{3/4} = \bar{\lambda}_{1/2},$$

$$\lambda_{5/6} = -\frac{\beta}{2} \pm i\sqrt{\varepsilon} n + O(\varepsilon). \tag{25}$$

5. The case when the weight of the cargo is much smaller than the rotor mass and the small resistance forces $\varepsilon << 1$, $\beta, b \sim \varepsilon$:

$$\lambda_{1/2} = -\frac{b}{2} + i(1 \pm n) + \frac{i\varepsilon}{4(1\pm n)^2} + O\left(\varepsilon^2\right), \quad \lambda_{3/4} = \bar{\lambda}_{1/2},$$

$$\lambda_{5/6} = -\frac{\beta}{2} \pm i\sqrt{\frac{\varepsilon}{n^2-1}} n^2 + O\left(\varepsilon^{\frac{3}{2}}\right). \tag{26}$$

6. The case when the rotor rotates rapidly $n>>1$:

$$\lambda_{1/2} = -\frac{b}{2} \pm i\sqrt{1-\left(\frac{b}{2}\right)^2} - in + O\left(\frac{1}{n}\right), \quad \lambda_{3/4} = \bar{\lambda}_{1/2},$$

$$\lambda_{5/6} = -\frac{\beta + \varepsilon b}{2(1-\varepsilon)} \pm in\sqrt{\frac{\varepsilon}{1-\varepsilon}} + O\left(\frac{1}{n}\right). \tag{27}$$

The obtained decompositions are applicable for the approximate determination of the roots of characteristic equation (11) at a distance from the resonance velocity ($|n-1|>0$). According to the decompositions, the auto-balancing mode is asymptotically stable at the above-the-resonance speeds of rotor rotation ($n>1$), and, at pre-resonance – unstable.

Decomposition of roots does not make it possible to determine the number and magnitude of the characteristic speeds of rotor rotation but allow us to draw the following conclusions:

– there may be a single or an odd number of characteristic speeds;

– when ε tends to zero, all characteristic speeds approach the resonance speed of rotor rotation ($n=1$);

– when the speed of rotor rotation passes an odd characteristic speed, the auto-balancing mode acquires stability while, when passing the even one, it loses it.

5. 3. 2. The impact of resistance forces on characteristic speeds

When the rotor rotates at the second or third characteristic speeds, a pair of purely imaginary roots appears in characteristic equation (11). We put in (11) the desired imaginary root $\lambda=i\mu$, where μ is a real number. Select separately the real and imaginary components. We obtain the following system of algebraic equations:

$$\mathrm{Re}(x) = a_0 - a_2 x + a_4 x^2 - a_6 x^3,$$

$$\mathrm{Im}(x)/x = a_1 - a_3 x + a_5 x^2, \quad x = \mu^2 > 0. \tag{28}$$

Using the system of equations (28), we determine the real positive x and the corresponding real positive critical speeds. The solution is derived in the following form:

$$x = x_0 + x_q \varepsilon^q + x_{2q}\varepsilon^{2q} + \cdots, \quad n = 1 + n_p \varepsilon^p + n_{2p}\varepsilon^{2p} + \cdots, \tag{29}$$

where q, p are the rational numbers.

In the case when the weight of the cargo is much smaller than the rotor mass and the resistance forces are small $(\varepsilon, b, \beta << 1)$, the following decompositions are found:

$$x_2 = \varepsilon\frac{3b}{4(b+\beta)} + O\left(\varepsilon^2\right),$$

$$n_2(\varepsilon, b, \beta) = 1 + \varepsilon\left[\frac{49}{32} - \frac{(b+7\beta)^2}{32(b+\beta)}\right] + O\left(\varepsilon^2\right);$$

$$x_3 = \left[\frac{\beta(b+\beta)}{4b^2}\right]^{\frac{1}{3}} \varepsilon^{\frac{2}{3}} + O(\varepsilon),$$

$$n_3(\varepsilon,b,\beta)=1+\left[\frac{(b+\beta)^2}{2b\beta}\right]^{\frac{1}{3}}\varepsilon^{\frac{1}{3}}+O\left(\varepsilon^{\frac{2}{3}}\right). \qquad (30)$$

Our comparison of decompositions (30) and (20) shows that the small forces of viscous resistance do not change the order of smallness of the second and third characteristic speeds relative to the small parameter ε. In this case, small resistance forces reduce the second and increase the third characteristic speed, and the magnitude of the change depends on the ratio between *b* and β (and not each parameter separately).

Further search for decompositions shows that in the case when the resistance forces are finite (equivalent to 1:*b*, β~1), there are no decompositions in form (30) and corresponding characteristic speeds. The auto-balancing mode is asymptotically stable at the resonance rotor rotation speeds.

6. Discussion of results obtained in the analytical study of the conditions for the onset of auto-balancing

We have built a mathematical model of the considered mechanical system, which is the same for pendulums, balls, or rollers. The dynamics of the mechanical system are characterized by 4 dimensionless parameters (3). The system has a single auto-balancing mode (6). Its (conventional) stability in the absence of resistance forces is influenced by the dimensionless speed of rotor rotation *n* and the ratio of the weight of the cargo to the mass of the entire system – ε. In the presence of resistance forces in the system, the dimensionless coefficients *b*, β characterizing the values of external and internal forces of viscous resistance also affect the stability of movement.

In the absence of resistance forces in the system (*b*=β=0), it has three characteristic rotor rotation speeds; the auto-balancing mode is stable when the rotor rotation speed is between the first and second or above the third characteristic speeds. The first characteristic speed coincides with the resonance frequency. With a decrease in ε, the second and third characteristic speeds tend to the resonance speed of rotor rotation.

It should be noted that:

– the third characteristic speed with an increase in the small parameter is rapidly increasing due to proportionality $\varepsilon^{\frac{1}{3}}$;

– due to the small first region of stability (*A*) at ε<<1 an auto-balancer can be used only at the speeds of rotor rotation exceeding the third characteristic speed (Fig. 2, *b*).

Small resistance forces in the system do not change the quality behavior of the system but reduce the second and increase the third characteristic speed. In this case, the magnitude of the change depends on the ratio between *b* and β (and not each parameter separately). With increasing resistance forces, the number of characteristic speeds decreases to one, and this speed is equal to the resonance speed of rotor rotation.

The resulting conditions for the onset of auto-balancing are also applicable for auto-balancers with many cargoes. However, they become necessary (but not sufficient) due to the fact that the composite cargo can kinematically disintegrate into separate cargoes, which is not taken into consideration by the model used.

The problem in question can be solved analytically even more fully. This is the point of the model problem. The new results would deepen the already obtained theoretical knowledge about the phenomenon of auto-balancing.

In the future, it is planned to analytically solve a series of other model problems that are important for understanding the auto-balancing process.

7. Conclusions

1. The considered rotor system with an auto-balancer has a single auto-balancing mode of movement. Its stability in the absence of resistance forces is influenced by the speed of rotor rotation and the ratio of the weight of the cargo to the mass of the entire system. In the presence of resistance forces in the system, the (dimensionless) coefficients that characterize the values of external and internal resistance forces also affect the stability of movement.

2. In the absence of resistance forces in the system, it has three characteristic speeds of rotor rotation, and the smallest one coincides with the resonance speed of rotor rotation. The auto-balancing mode is stable when rotating the rotor at speeds between the first and second or above the third characteristic speeds. With a decrease in the ratio of the weight of the cargo to the mass of the system, the second and third characteristic speeds tend to the resonance speed of rotor rotation.

3. Small resistance forces in the system do not change the quality behavior of the system. In this case, small resistance forces reduce the second and increase the third characteristic speed, and the magnitude of the change in speeds depends on the ratio between *b* and (and not each parameter separately). With increasing resistance forces, the number of characteristic speeds decreases to one, and this speed coincides with the resonance speed of rotor rotation.

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