UDC 539.3
DOI: 10.15587/1729-4061.2021.229428

SOLVING A ONE MIXED PROBLEM IN ELASTICITY THEORY FOR HALF-SPACE WITH A CYLINDRICAL CAVITY BY THE GENERALIZED FOURIER METHOD

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Designing and constructing underground structures for various purposes, such as tunnels, mines, mine workings, necessitate the development of procedures for calculating their strength and reliability. The physical model of such objects worth considering is a homogeneous isotropic half-space that contains an infinitely long hollow cylinder, located parallel to its border. One can explore problems related to the mechanics of deformable solids for such a multiply connected body.

This paper reports the proofs of addition theorems for the basic solutions to the Lamé equation regarding the half-space and cylinder recorded, respectively, in the Cartesian and cylindrical coordinate systems. This result is important from a theoretical point of view in order to substantiate a numerical-analytical method – the generalized Fourier method. This method makes it possible to solve spatial boundary problems from the theory of elasticity and thermo-elasticity for isotropic and transversal-isotropic multiply connected bodies. Similar to the classical Fourier method, the general solutions to equilibrium equations have been used here but in several coordinate systems rather than one.

From a practical point of view, this method has made it possible to investigate the combined problem of elasticity theory regarding the multiply-connected body described above. The analysis of the stressed-strained state of this elastic body has made it possible to draw conclusions on determining those regions that are most vulnerable to destruction. The highest values are accepted by normal stresses in the region between the boundaries of the half-space and the cylinder. Changing the σ_y component along the Ox axis corresponds to the displacements assigned on the half-space. The τ_{xy} component contributes less to the distribution of stresses than σ_x and σ_y. The applied aspect of using the reported results is the possibility to apply them when designing underground structures

Keywords: addition theorems, Lamé equation, generalized Fourier method, half-space, cylindrical cavity

Received date 09.02.2021
Accepted date 16.04.2021
Published date 26.04.2021

***How to Cite:** Ukrayinets, N., Murahovska, O., Prokhorova, O. (2021). Solving a one mixed problem in elasticity theory for half-space with a cylindrical cavity by the generalized fourier method. Eastern-European Journal of Enterprise Technologies, 2 (7 (110)), 48–57. doi: https://doi.org/10.15587/1729-4061.2021.229428*

1. Introduction

Spatial problems related to the theory of elasticity are well studied for simply connected bodies. However, in fields such as mining, geomechanics, construction mechanics, the most interesting are those spatial regions whose boundary consists of several non-intersecting surfaces of different orthogonal curvilinear coordinate systems. These can include underground tunnels, mines, mine workings, gas and oil storage facilities. When designing such structures, there are many different factors to consider in terms of their strength. Various cavities, inclusions, cracks produce a special effect while their geometric arrangement is of particular importance. In this regard, the boundary problems from the theory of elasticity are considered for respective multiply connected bodies. A model of those underground structures that is worth considering is a homogeneous isotropic half-space with an infinite circular cylindrical cavity, located parallel to its border.

The results from studying the boundary value problems related to the theory of elasticity are of interest at the stage of designing underground construction objects. An important stage is the construction and study of the corresponding model of such structures, as well as the analysis of its stress-strain state under different types of loads at boundary surfaces. Of particular interest in the practical sense is the distribution of stresses near cavities, as well as the identification of regions in an elastic body where the stress is maximal. Analyzing the stress-strain state of multiply connected bodies makes it possible to determine those regions where the stresses are concentrated. Therefore, such studies are relevant; data to be obtained could be used by engineers in construction mechanics and mining at the stage of modeling underground structures.

2. Literature review and problem statement

Paper [1] examines the axisymmetric problem from the theory of elasticity for a semi-infinite body with a spherical cavity. The solution to equilibrium equations is written in the Papkovic-Neuber form and expressed through harmonic functions. Work [2] addresses the identification of stress and deformation fields around the stress-free surface of an ellipsoidal cavity, located in a strained elastic medium . Such a problem is important for assessing the structural safety of underground mine workings. An analytical solution recorded in ellipsoid coordinates has been obtained. Problems on stress concentration in the vicinity of a spheroidal nanoscale cavity, located near the free surface, are considered in [3]. The boundary problem solving methods described in papers [1–3] are applicable to unlimited elastic regions with finite cavities.

Monograph [4] examines the main boundary problems from the theory of elasticity for a half-space weakened by a cavity located at an arbitrary depth from the flat boundary. To solve them, a method of potential in a modified form and a method of fictitious regions are used. The surface of the cavity is considered to belong to the Lyapunov surface class. This method cannot be applied to an elastic medium with an infinite cavity. The pressure of a plate on the half-space with a circular cylindrical cavity, the surface of which is reinforced by elastic elements, is considered in work [5]. The problem is reduced to the Fredholm integral equation of the second kind. An analysis of the distribution of stresses around the underground hole, which is subjected to asymmetrical surface load, was carried out in [6]. The holes with sharp angles were considered. The study involves the method of boundary integral equations and the Neumann series. Reducing the problems related to the theory of elasticity to integral equations, described in works [5, 6], is possible purely for the specified regions. An analytical method to study the concentration of stresses around the cavity of an arbitrary form is proposed in [7]. This method involves the possibility of modeling the effect of the cavity on the redistribution of internal forces by introducing fictitious forces acting on its surface. The elastic half-space with a cavity in the form of a rectangular parallelepiped and a quadrangle pyramid was considered. The half-space is loaded with concentrated force applied to its free surface. A given method is applicable to non-limited regions with finite-size cavities.

In work [8], a finite-element method is applied to solve a problem on the deformation of underground mine workings of a circular cross-section in a mountain range. The problem of the concentration of stresses in an infinite medium in a hydrostatic compression field, with two spherical inclusions, is solved in [9]. However, a finite-element method used in works [8, 9] is not applicable to infinite regions.

Paper [10] reports an accurate solution to a non-stationary problem for an infinite elastic layer containing a rigid cylindrical inclusion, with smooth contact conditions superimposed on the cylindrical surface. To build a solution, the axisymmetric equations of motion were treated with the integral transformations by Laplace and Weber.

A generalized Fourier method (GFM) is successfully applied to solve the problems related to the theory of elasticity in multiply connected bodies [11, 12]. This method makes it possible to find solutions to the basic and mixed boundary problems from the theory of elasticity and thermo-elasticity for isotropic and transversal-isotropic multiply connected canonical bodies. Article [13] investigates the contact problem of thermo-elasticity for the elastic half-space with a rigid spherical inclusion. To solve it, the authors used addition theorems for the solutions to Lamé equations for the ball and cylinder. Paper [14] explores a problem on the effect of axial-concentrated force on the elastic transversal-isotropic half-space with a still inclusion in the form of a rotation paraboloid. The problem was solved by the generalized Fourier method with the help of addition theorems for the solutions to the equilibrium equations of a transversal-isotropic rotation paraboloid and solutions for the half-space. The authors of work [15] use the generalized Fourier method to solve a boundary problem from the theory of elasticity for a cylinder with cylindrical cavities forming a hexagonal structure. To meet the boundary conditions, they applied addition theorems of solutions to the Lamé equation for a cylinder, recorded in cylindrical coordinate systems that are shifted relative to each other. The generalized Fourier method employed in works [13–15] to solve boundary problems for half-space with inclusions, as well as for a cylinder with cavities, could be applied to solving problems related to the theory of elasticity for the half-space with an infinite cylindrical cavity.

Solutions to the second, first, and mixed problems from the theory of elasticity for isotropic half-space with an infinite circular cylindrical cavity parallel to its boundary are reported in works [16–18]. Paper [18] considers a case where stresses are set at the boundary of the half-space, and displacements on the cylindrical surface. The generalized Fourier method was used to solve the problems. At the same time, vector theorems of the addition of basis solutions to a Lamé equation for the half-space and cylinder were used to satisfy the boundary conditions. However, works [16–18] report the addition theorems without proofs. Therefore, in order to fully substantiate the generalized Fourier method for solving the boundary problems related to the theory of elasticity in a half-space with an infinite cylindrical cavity, it is necessary to prove the addition theorems for the half-space and cylinder.

The authors of work [19] solved a boundary problem for the half-space with two cylindrical cavities, on the boundary surfaces of which the contact type conditions are assigned. The stress-strain of the layer with a cylindrical cavity on a hard base is investigated in [20]. he addition theorems of the solutions to the Lamé equation for the half-space and cylinder were also used to solve the problems in [19, 20].

Hence, it follows that different methods have solved the boundary problems related to the theory of elasticity for unlimited elastic bodies with cavities of finite size. Works [16–20] report solving boundary problems for the half-space and layer with an infinite cylindrical cavity or cavities in different statements. However, the mixed problem in elasticity theory for half-space with an infinite cylindrical cavity in the case where displacements are assigned at the boundary of the half-space, and stresses on the surface of the cylinder, is not solved. Therefore, it is advisable to consider a solution to this boundary problem.

3. The aim and objectives of the study

The aim of this work is to solve the mixed problem in elasticity theory for half-space with an infinite circular cylindrical cavity using the generalized Fourier method. In practical terms, this would make it possible to investigate the stress-strain state of this spatial region, in particular, near a cylindrical cavity.

To accomplish the aim, the following tasks have been set:

– to prove the addition theorems of solutions of the Lamé equation for the half-space and cylinder, wrote in Cartesian and cylindrical coordinate systems;

– to propose an analytical-numerical algorithm to solve the mixed problem of elasticity theory for half-space with a cylindrical cavity parallel to its boundary.

4. Materials and methods to investigate the mixed problem in elasticity theory for the half-space with a cavity

To solve the mixed problem of the theory of elasticity in the half-space with an infinite cylindrical cavity parallel to its boundary, a generalized Fourier method was used. It is based on the application of addition theorems of basis solutions to the Lamé equation for the respective canonical surfaces that make up the boundary of a multi-connected body. The addition theorems are used to solve problems in different scientific fields but, in most studies, they are of a private nature. In order to substantiate the generalized Fourier method regarding a given problem, we have proven the addition theorems of solutions to the Lamé equation for the half-space and cylinder.

Consider two equally oriented coordinate systems with the combined centers – the Cartesian $\{x, y, z\}$ and the cylindrical $\{\rho, \varphi, z\}$, with a center at point O. The connection between the coordinates is set by the following ratios: $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, $z=z$, where $0\le\rho<\infty$, $0\le\varphi<2\pi$, $-\infty<z<\infty$.

The elastic medium is to be considered homogeneous and isotropic. Then the equilibrium Lamé equation in displacements in the absence of volumetric forces takes the following form

$$\Delta\vec{u}+(1-2\sigma)^{-1}\,\mathrm{grad}\,\mathrm{div}\vec{u}=0, \tag{1}$$

where $\vec{u}$ is the vector of elastic displacements, σ is the Poisson coefficient.

Consider the sets of linearly independent particular solutions to equation (1) in the specified coordinate systems (k=1, 2, 3, m=0, ±1, ±2,...):

$$\vec{u}_1^{(\pm)}(x,y,z;\lambda,\mu)=N_1^{(1)}u^{(\pm)}(x,y,z;\lambda,\mu), \tag{2}$$

$$\vec{u}_2^{(\pm)}(x,y,z;\lambda,\mu)= =\lambda^{-1}\left(4(\sigma-1)\,u^{(\pm)}\,\vec{e}_2^{(1)}+\mathrm{grad}\left(y\,u^{(\pm)}(x,y,z;\lambda,\mu)\right)\right), \tag{3}$$

$$\vec{u}_3^{(\pm)}(x,y,z;\lambda,\mu)=N_3^{(1)}u^{(\pm)}(x,y,z;\lambda,\mu), \tag{4}$$

$$\vec{R}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(2)}r_m(\rho,\varphi,z;\lambda), \tag{5}$$

$$\vec{S}_{k,m}(\rho,\varphi,z;\lambda)=N_k^{(2)}s_m(\rho,\varphi,z;\lambda), \tag{6}$$

$$N_1^{(\tau)}=\lambda^{-1}\mathrm{grad},\quad N_3^{(\tau)}=i\lambda^{-1}rot\left(\vec{e}_3^{(\tau)}\cdot\right),\quad \tau=1,2,$$

$$N_2^{(2)}=\lambda^{-1}\left(\mathrm{grad}(\rho\partial/\partial\rho)+4(\sigma-1)\left(\mathrm{grad}-\vec{e}_3^{(2)}\,\partial/\partial z\right)\right).$$

In formulae (2) to (6), $\vec{e}_k^{(\tau)}$ (k=1,2,3, τ=1.2) denote the orts of the Cartesian (τ=1) and cylindrical (τ=2) coordinate systems. Functions $u^{(\pm)}(x, y, z; \lambda, \mu)$, $r_m(\rho, \varphi, z; \lambda)$, $s_m(\rho, \varphi, z; \lambda)$, where m=0, ±1, ±2, ..., are the Cartesian and cylindrical basis solutions to the Laplace equation:

$$u^{(\pm)}(x,y,z;\lambda,\mu)=e^{i\lambda z\pm\gamma y+i\mu x},\quad \gamma=\sqrt{\lambda^2+\mu^2}, \tag{7}$$

$$r_m(\rho,\varphi,z;\lambda)=e^{i\lambda z+im\varphi}I_m(\lambda\rho), \tag{8}$$

$$s_m(\rho,\varphi,z;\lambda)=(sign\,\lambda)^m\,e^{i\lambda z+im\varphi}K_m(|\lambda|\rho). \tag{9}$$

Here, $I_m(\lambda\rho)$ and $K_m(\lambda\rho)$ are the modified Bessel's functions of the 1st and 2nd kind, m=0, ±1, ±2,..., are the parameters of λ, μ∈(–∞, ∞). The harmonic functions for a cylinder (8) and (9) are considered in work [21], and functions (7) are presented there in a slightly different form. The functions $\vec{R}_{k,m}(\rho,\varphi,z;\lambda)$, $\left(\vec{S}_{k,m}(\rho,\varphi,z;\lambda)\right)$, regular in the region $\{\rho<R\}$ ($\{\rho<R\}$), where $R>0$ are the internal (external) basis solutions to a Lamé equation for a cylinder. The functions $\vec{u}_k^{(+)}(x,y,z;\lambda,\mu)$, $\left(\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)\right)$, regular in the region $\{y<h\}$ ($\{y>h\}$), are the internal (external) basis solutions to a Lamé equation for the half-space.

An elastic half-space with an infinite cylindrical cavity parallel to its boundary is a two-connected body bounded by the canonical surfaces of the Cartesian and cylindrical coordinate systems. To solve the boundary problem of the theory of elasticity regarding this elastic body, a generalized Fourier method [11] has been used. The method implies the following:

– for each boundary surface of a multiply connected canonical body, a system of basis solutions to the homogeneous Lamé equation is introduced;

– a general solution to the problem is constructed in the form of a superposition of the basis solutions to a Lam equation for simply connected bodies in the corresponding coordinate systems;

– using addition theorems, a general solution to the problem is written in the coordinate system associated with each boundary surface;

– substituting a general solution into the boundary conditions leads to an infinite system of linear algebraic equations with a completely continuous operator within space l_2 and the right-hand sides belonging to l_2. This makes it possible to solve the system by the reduction method.

5. Results of investigating the mixed problem of elasticity theory for half-space with a cylindrical cavity

5. 1. Proving the addition theorems of basis solutions to a Lamй equation for half-space and cylinder

Theorem 1. For the arbitrary λ∈**R**, the decomposition of the internal basis solutions to a Lamé equation for the half-space $\vec{u}_p^{(+)}(x,y,z;\lambda,\mu)$ into internal basis solutions for the cylinder $\vec{R}_{p,m}(\rho,\varphi,z;\lambda)$ (p=1, 2, 3) is valid:

$$\vec{u}_p^{(+)}(x,y,z;\lambda,\mu)= =\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m\vec{R}_{p,m}(\rho,\varphi,z;\lambda),\ p=1,3, \tag{10}$$

$$\vec{u}_2^{(+)}(x,y,z;\lambda,\mu)=\lambda^{-2}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m\times \times\left[\begin{array}{l} m\mu\,\vec{R}_{1,m}(\rho,\varphi,z;\lambda)+ \\ +\gamma\,\vec{R}_{2,m}(\rho,\varphi,z;\lambda)+4\mu(1-\sigma)\vec{R}_{3,m}(\rho,\varphi,z;\lambda)\end{array}\right], \tag{11}$$

where $\omega_1(\lambda, \mu)=(\mu-\gamma)/\lambda$.

Proof. We use a ratio linking harmonic functions recorded in the Cartesian and cylindrical coordinate systems [21].

$$e^{\gamma y+i\mu x}=\sum_{m=-\infty}^{\infty}\left(i(\mu-\gamma)\lambda^{-1}\right)^m I_m(\lambda\rho)\,e^{im\varphi}, \tag{12}$$

and the fact that the series in (12) is evenly converging at any λ∈**R**. Taking into consideration (12), we transform the function $u^{(+)}(x, y, z; \lambda, \mu)$:

$$u^{(+)}= e^{i\lambda z}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m I_m(\lambda\rho)\,e^{im\varphi}=$$
$$=\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m r_m(\rho,\varphi,z;\lambda)\,. \qquad (13)$$

Then, considering expressions (5) and (6) wrote for k=1.3, we obtain:

$$\vec{u}_1^{(+)}(x,y,z;\lambda,\mu)=\mathrm{grad}\,u^{(+)}(x,y,z;\lambda,\mu)/\lambda=$$
$$=\mathrm{grad}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m r_m(\rho,\varphi,z;\lambda)/\lambda=$$
$$=\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m \vec{R}_{1,m}(\rho,\varphi,z;\lambda),$$

$$\vec{u}_3^{(+)}(x,y,z;\lambda,\mu)=i\lambda^{-1}\mathrm{rot}\left(u^{(+)}(x,y,z;\lambda,\mu)\cdot\vec{e}_z\right)=$$
$$=i\lambda^{-1}\mathrm{rot}\left(\vec{e}_z\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m r_m\right)=$$
$$=\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m \vec{R}_{3,m}(\rho,\varphi,z;\lambda). \qquad (14)$$

Prove formula (11). Considering decomposition (13) and assuming $y=\rho\sin\varphi=-i\rho(e^{i\varphi}-e^{-i\varphi})/2$, record $yu^{(+)}(x, y, z; \lambda, \mu)$:

$$y\,u^{(+)}(x,y,z;\lambda,\mu)=$$
$$=\rho\sin\varphi\;e^{i\lambda z}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m I_m(\lambda\rho)\,e^{im\varphi}=$$
$$=-0.5i\rho\,e^{i\lambda z}\left(\sum_{m=-\infty}^{\infty}\begin{matrix}\left(i\,\omega_1(\lambda,\mu)\right)^m\times\\ \times I_m(\lambda\rho)\left(e^{i(m+1)\varphi}-e^{i(m-1)\varphi}\right)\end{matrix}\right). \qquad (15)$$

In (15), the expression in brackets will be marked through Φ; transform it:

$$\Phi=\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^{m-1} I_{m-1}(\lambda\rho)\,e^{im\varphi}-$$
$$-\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^{m+1} I_{m+1}(\lambda\rho)\,e^{im\varphi}=$$
$$=\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^{m-1} e^{im\varphi}\times$$
$$\times\left(I_{m-1}(\lambda\rho)+\omega_1^2(\lambda,\mu)I_{m+1}(\lambda\rho)\right).$$

Considering $\omega_1^2(\lambda,\mu)=2\mu\omega_1(\lambda,\mu)/\lambda+1$, and using a formula for the derivative given in [22]

$\partial I_m(\lambda\rho)/\partial\rho=\lambda(I_{m-1}(\lambda\rho)+I_{m+1}(\lambda\rho))/2$,

write Φ in the following form:

$$\Phi=2\lambda^{-1}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^{m-1} e^{im\varphi}\times$$
$$\times\left(\partial I_m(\lambda\rho)/\partial\rho+\mu\,\omega_1(\lambda,\mu)\,I_{m+1}(\lambda\rho)\right).$$

Using the formula given in [22]

$\rho I_{m+1}(\lambda\rho)=\rho(\partial I_m(\lambda\rho)/\partial\rho)/\lambda-mI_m(\lambda\rho)/\lambda$,

where m=0, ±1, ±2,..., transform $\rho\Phi$ to the following form:

$$\rho\,\Phi=2i\lambda^{-2}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m e^{im\varphi}\times$$
$$\times\left(\gamma\rho\partial I_m(\lambda\rho)/\partial\rho+\mu\,m\,I_m(\lambda\rho)\right). \qquad (16)$$

Substitute the ratio for $\rho\Phi$ from (16) into expression (15)

$$y\,u^{(+)}(x,y,z;\lambda,\mu)=\lambda^{-2}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m\times$$
$$\times\left(\gamma\rho\,\partial/\partial\rho+\mu\,m\right)r_m(\rho,\varphi,z;\lambda)$$

and, taking into consideration representation (5) recorded for k=1.2, find

$$\mathrm{grad}\left(y\,u^{(+)}(x,y,z;\lambda,\mu)\right)=$$
$$=\mu\lambda^{-2}\sum_{m=-\infty}^{\infty}\left(i\omega_1(\lambda,\mu)\right)^m m\;\mathrm{grad}\,r_m(\rho,\varphi,z;\lambda)+$$
$$+\gamma\lambda^2\sum_{m=-\infty}^{\infty}\left(i\omega_1(\lambda,\mu)\right)^m\mathrm{grad}\left(\rho\,\partial/\partial\rho\right)r_m(\rho,\varphi,z;\lambda)=$$
$$=\frac{\mu}{\lambda}\sum_{m=-\infty}^{\infty}\left(i\omega_1(\lambda,\mu)\right)^m m\,\vec{R}_{1,m}(\rho,\varphi,z;\lambda)+$$
$$+\frac{\gamma}{\lambda}\sum_{m=-\infty}^{\infty}\left(i\omega_1(\lambda,\mu)\right)^m \vec{R}_{2,m}(\rho,\varphi,z;\lambda)-$$
$$-4(\sigma-1)\gamma\lambda^{-2}\times$$
$$\times\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m\left(\mathrm{grad}-\vec{e}_z\,\partial/\partial z\right)r_m(\rho,\varphi,z;\lambda). \qquad (17)$$

Substitute the resulting expression for the term $\mathrm{grad}(yu^{(\pm)}(x, y, z; \lambda, \mu))$ from (17) into the right-hand side of (3). In this case, one of the terms would contain the following series

$$\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m\begin{pmatrix}\vec{e}_y-\gamma\lambda^{-2}\times\\ \times\begin{pmatrix}\vec{e}_x\,\partial/\partial x+\\ +\vec{e}_y\,\partial/\partial y\end{pmatrix}\end{pmatrix}r_m(\rho,\varphi,z;\lambda).$$

Transform it using ratios (13), (7), (4) and formula (14) recorded for k=3:

$$\left(\vec{e}_y-\gamma\lambda^{-2}\left(\vec{e}_x\,\partial/\partial x+\vec{e}_y\,\partial/\partial y\right)\right)u^{(+)}(x,y,z;\lambda,\mu)=$$
$$=i\mu\lambda^{-2}\left(-\gamma\,\vec{e}_x+i\mu\,\vec{e}_y\right)u^{(+)}=$$
$$=-\,i\mu\lambda^{-2}\;rot\left(u^{(+)}\;\vec{e}_z\right)=-\mu\lambda^{-1}\,\vec{u}_3^{(+)}=$$
$$=-\mu\lambda^{-1}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m\vec{R}_{3,m}(\rho,\varphi,z;\lambda).$$

Substitute the resulting expression into (3), finally, write:

$$\vec{u}_2^{(+)}(x,y,z;\lambda,\mu)=$$
$$=\lambda^{-2}\sum_{m=-\infty}^{\infty}\left(i\,\omega_1(\lambda,\mu)\right)^m\begin{bmatrix}\mu\,m\vec{R}_{1,m}(\rho,\varphi,z;\lambda)+\\ +\gamma\,\vec{R}_{2,m}(\rho,\varphi,z;\lambda)+\\ +4\mu(1-\sigma)\vec{R}_{3,m}(\rho,\varphi,z;\lambda)\end{bmatrix}.$$

Theorem 1 is proven.

Theorem 2. At $y>0$, the integral representations of the external basis solutions to a Lamé equation for the cylinder $\vec{S}_{k,m}(\rho,\varphi,z;\lambda)$ (k=1, 2, 3, m=0, ±1, ±2,...) through external basis solutions for the half-space $\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)$ (k=1, 2, 3), where $\omega_2(\lambda,\mu)=(\mu-\gamma)/|\lambda|$ are valid:

$$\vec{S}_{k,m}(\rho,\varphi,z;\lambda)=0.5(-i\,\mathrm{sign}\lambda)^m\times$$
$$\times\int_{-\infty}^{\infty}\gamma^{-1}\omega_2^m(\lambda,\mu)\,\vec{u}_k^{(-)}(x,y,z;\lambda,\mu)d\mu,\ k=1,3, \tag{18}$$

$$\vec{S}_{2,m}(\rho,\varphi,z;\lambda)=0.5(-i\,\mathrm{sign}\lambda)^m\times$$
$$\times\int_{-\infty}^{\infty}\omega_2^m(\lambda,\mu)\begin{bmatrix}(m\mu-\lambda^2\gamma^{-1})\vec{u}_1^{(-)}(x,y,z;\lambda,\mu)-\\ -\lambda^2\vec{u}_2^{(-)}(x,y,z;\lambda,\mu)+\\ +4\mu(1-\sigma)\vec{u}_3^{(-)}(x,y,z;\lambda,\mu)\end{bmatrix}\gamma^{-2}d\mu, \tag{19}$$

Proof. Using the following formula from [21] that holds for $y>0$ and $m\in\mathbf{R}$,

$$K_m(|\lambda|\rho)e^{im\varphi}=0.5(-i)^m\times$$
$$\times\int_{-\infty}^{\infty}\left((\mu-\gamma)|\lambda|^{-1}\right)^m\gamma^{-1}e^{-\gamma y+i\mu x}d\mu, \tag{20}$$

transform the functions $s_m(\rho,\varphi,z;\lambda)$ (m=0,±1,±2,...):

$$s_m(\rho,\varphi,z;\lambda)=0.5(-i\,\mathrm{sign}\,\lambda)^m\times$$
$$\times\int_{-\infty}^{\infty}\gamma^{-1}\omega_2^m(\lambda,\mu)u^{(-)}(x,y,z;\lambda,\mu)d\mu. \tag{21}$$

Considering ratios (2) and (4), record functions $\vec{S}_{k,m}(\rho,\varphi,z;\lambda)$ in the following form:

$$\vec{S}_{1,m}(\rho,\varphi,z;\lambda)=\lambda^{-1}\mathrm{grad}s_m=$$
$$=0.5(-i\,\mathrm{sign}\lambda)^m\int_{-\infty}^{\infty}\gamma^{-1}\omega_2^m(\lambda,\mu)\,\vec{u}_1^{(-)}d\mu,$$

$$\vec{S}_{3,m}(\rho,\varphi,z;\lambda)=i\lambda^{-1}\mathrm{rot}(s_m\vec{e}_z)=$$
$$=0.5(-i\,\mathrm{sign}\lambda)^m\int_{-\infty}^{\infty}\gamma^{-1}\omega_2^m(\lambda,\mu)\,\vec{u}_3^{(-)}d\mu.$$

Prove formula (19). Consider function $\vec{S}_{2,m}(\rho,\varphi,z;\lambda)$ and, using representation (21), transform the first term in formula (6) at k=2:

$$\mathrm{grad}\left(\left(\rho\,\partial/\partial\rho\right)s_m(\rho,\varphi,z;\lambda)\right)=$$
$$=\mathrm{grad}\left(\left(x\,\partial/\partial x+y\,\partial/\partial y\right)s_m(\rho,\varphi,z;\lambda)\right)=$$
$$=0.5(-i\,\mathrm{sign}\lambda)^m\begin{bmatrix}\mathrm{grad}\left(ix\int_{-\infty}^{\infty}\omega_2^m u^{(-)}\mu\gamma^{-1}d\mu\right)-\\ -\mathrm{grad}\left(y\int_{-\infty}^{\infty}\omega_2^m u^{(-)}d\mu\right)\end{bmatrix}. \tag{22}$$

Introduce the designation $H_m(\lambda,\mu)=\mu\omega_2^m(\lambda,\mu)/\gamma$ and, by applying the integration for parts, transform the first integral in (22) to the following form:

$$ix\int_{-\infty}^{\infty}e^{i\mu x-\gamma y+i\lambda z}H_m(\lambda,\mu)d\mu=$$
$$=\int_{-\infty}^{\infty}H_m(\lambda,\mu)\frac{\mu y u^{(-)}}{\gamma}d\mu-\int_{-\infty}^{\infty}\frac{\partial H_m(\lambda,\mu)}{\partial\mu}u^{(-)}d\mu. \tag{23}$$

Considering (23), by introducing a gradient under the sign of integral, write (22) in the following form:

$$\mathrm{grad}\left(\left(\rho\,\partial/\partial\rho\right)s_m\right)=0.5(-i\,\mathrm{sign}\lambda)^m\times$$
$$\times\begin{pmatrix}\int_{-\infty}^{\infty}H_m(\lambda,\mu)\mathrm{grad}\left(yu^{(-)}\right)\mu\gamma^{-1}d\mu-\\ -\int_{-\infty}^{\infty}\partial H_m(\lambda,\mu)/\partial\mu\ \mathrm{grad}u^{(-)}d\mu-\\ -\int_{-\infty}^{\infty}\omega_2^m(\lambda,\mu)\mathrm{grad}\left(yu^{(-)}\right)d\mu\end{pmatrix}=$$
$$=0.5(-i\,\mathrm{sign}\lambda)^m\times$$
$$\times\begin{pmatrix}\int_{-\infty}^{\infty}\left(H_m(\lambda,\mu)\mu\gamma^{-1}-\omega_2^m(\lambda,\mu)\right)\mathrm{grad}\left(yu^{(-)}\right)d\mu-\\ -\int_{-\infty}^{\infty}\partial H_m(\lambda,\mu)/\partial\mu\ \mathrm{grad}\,u^{(-)}(x,y,z;\lambda,\mu)d\mu\end{pmatrix}. \tag{24}$$

Transform the following expressions:

$$H_m(\lambda,\mu)\mu\gamma^{-1}-\omega_2^m(\lambda,\mu)=-\lambda^2\gamma^2\omega_2^m(\lambda,\mu), \tag{25}$$

$$\partial H_m(\lambda,\mu)/\partial\mu=\gamma^{-2}\omega_2^m(\lambda,\mu)\left(\lambda^2\gamma^{-1}-\mu m\right) \tag{26}$$

and substitute them into (24). Expressing the following functions from formulae (2), (3)

$$\mathrm{grad}u^{(-)}(x,y,z;\lambda,\mu)=\lambda\,\vec{u}_1^{(-)}(x,y,z;\lambda,\mu),$$

$$\mathrm{grad}\left(y\,u^{(-)}(x,y,z;\lambda,\mu)\right)=$$
$$=\lambda\,\vec{u}_2^{(-)}(x,y,z;\lambda,\mu)-4(\sigma-1)\,u^{(-)}(x,y,z;\lambda,\mu)\vec{e}_y,$$

substitute them into (24):

$$\mathrm{grad}\left(\left(\rho\,\partial/\partial\rho\right)s_m\right)=0.5(-i\,\mathrm{sign}\lambda)^m\times$$
$$\times\int_{-\infty}^{\infty}\gamma^{-2}\omega_2^m(\lambda,\mu)\begin{pmatrix}\left(\mu m-\lambda^2\gamma^{-1}\right)\lambda\,\vec{u}_1^{(-)}-\\ -\lambda^3\vec{u}_2^{(-)}(x,y,z;\lambda,\mu)-\\ -4(1-\sigma)\lambda^2u^{(-)}(x,y,z;\lambda,\mu)\vec{e}_y\end{pmatrix}d\mu. \tag{27}$$

Using (21), the second term in formula (6) at k=2 is written in the following form:

$$(\mathrm{grad}-\vec{e}_z\,\partial/\partial z)s_m(\rho,\varphi,z;\lambda)=$$
$$=\left(\vec{e}_x\,\partial/\partial x+\vec{e}_y\,\partial/\partial y\right)s_m(\rho,\varphi,z;\lambda)=$$
$$=0.5(-i\,\mathrm{sign}\lambda)^m\times$$
$$\times\int_{-\infty}^{\infty}\gamma^{-1}\omega_2^m(\lambda,\mu)\left(i\mu\,\vec{e}_x-\gamma\,\vec{e}_y\right)u^{(-)}(x,y,z;\lambda,\mu)d\mu. \tag{28}$$

Substituting expressions (27), (28) into the same formula, as well as the following function

$$i\,\mathrm{rot}\left(u^{(-)}(x,y,z;\lambda,\mu)\,\vec{e}_z\right)=\lambda\,\vec{u}_3^{(-)}(x,y,z;\lambda,\mu),$$

we obtain:

$$\vec{S}_{2,m}(\rho,\varphi,z;\lambda)=0.5(-i\,\mathrm{sign}\,\lambda)^m\times$$
$$\times\int_{-\infty}^{\infty}\gamma^{-2}\omega_2^m(\lambda,\mu)\begin{pmatrix}(\mu\,m-\gamma\lambda^2)\,\vec{u}_1^{(-)}(x,y,z;\lambda,\mu)-\\-\lambda^2\vec{u}_2^{(-)}(x,y,z;\lambda,\mu)+\\+4\mu\,(1-\sigma)\,\vec{u}_3^{(-)}(x,y,z;\lambda,\mu)\end{pmatrix}\mathrm{d}\mu,$$

$$m=0,\pm1,\pm2,....$$

Theorem 2 is proven.

5. 2. An analytical-numerical algorithm for solving the mixed problem of elasticity theory for half-space with a cylindrical cavity parallel to its boundary

Consider the region Ω, a half-space filled with a homogeneous isotropic medium and containing an infinite circular cylindrical cavity parallel to its boundary. The Cartesian $\{x, y, z\}$ and cylindrical $\{\rho, \varphi, z\}$ coordinate systems are to be associated to the boundary surfaces of the region so that the Oy axis is perpendicular to the boundary of the half-space, and the Oz axis is directed along the axis of the cylinder. Denote the cylinder's radius through a, h is the distance from the cylinder axis to the boundary of the half-space. The boundary surfaces of the half-space and the cylinder, set by the equations $y=h$ and $\rho=a$, would be denoted through S_1 and S_2. The region Ω can be described by a system of inequalities: $\{y<h, \rho>a, h>a\}$.

Consider the mixed problem in elasticity theory for the region Ω in the following statement. Search for a solution to the homogeneous Lamé equation (1), which, on boundary surfaces, satisfies the boundary conditions:

$$\vec{u}\big|_{S_1}=\vec{u}_{01}(\mathrm{x},z),\tag{29}$$

$$\overrightarrow{Fu}\Big|_{S_2}=\overrightarrow{Fu}_{02}(\varphi,z).\tag{30}$$

Here $\overrightarrow{Fu}$ is the vector of elastic stresses, and the assigned functions $\vec{u}_{01}(\mathrm{x},z)$ and $\overrightarrow{Fu}_{02}(\varphi,z)$ are represented in the form of absolutely converging series and integrals:

$$\vec{u}_{01}(x,z)=\sum_{j=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_j(\lambda,\mu)e^{i\lambda z+i\mu x}\mathrm{d}\lambda\,\mathrm{d}\mu\,\vec{e}_j^{(1)},\tag{31}$$

$$\overrightarrow{Fu}_{02}(\varphi,z)=\sum_{j=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}A_j^m(\lambda)e^{i\lambda z+im\varphi}\mathrm{d}\lambda\,\vec{e}_j.\tag{32}$$

At the same time, functions $C_j(\lambda, \mu)$ are limited, the $\sum_{m=-\infty}^{\infty}A_j^m(\lambda)$ series is absolutely converging at all $\lambda\in\mathbf{R}$, and vectors $\vec{e}_j$ (j=1, 2, 3) are expressed by the following formulae:

$$\vec{e}_1=0.5\left(\vec{e}_1^{(2)}+i\,\vec{e}_2^{(2)}\right)e^{i\,\varphi},$$
$$\vec{e}_2=0.5\left(\vec{e}_1^{(2)}-i\,\vec{e}_2^{(2)}\right)e^{-i\,\varphi},\quad \vec{e}_3=\vec{e}_3^{(2)}.\tag{33}$$

To solve the problem, a generalized Fourier method is applicable that employs the addition theorems 1 and 2 of the solutions to a Lamé equation for the half-space and cylinder proven above in chapter 5. 1. The solution $\vec{u}$ to problem (1), (29), (30) is written in the form of a linear combination of external solutions for the cylinder $\vec{S}_{k,m}(\rho,\varphi,z;\lambda)$ and internal solutions for the half-space $\vec{u}_p^{(+)}(x,y,z;\lambda,\mu)$:

$$\vec{u}=\sum_{k=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}B_{km}(\lambda)\vec{S}_{k,m}\,\mathrm{d}\lambda+$$
$$+\sum_{p=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_p(\lambda,\mu)\,\vec{u}_p^{(+)}\,\mathrm{d}\mu\,\mathrm{d}\lambda.\tag{34}$$

Functions $\vec{S}_{k,m}(\rho,\varphi,z;\lambda)$ and $\vec{u}_p^{(+)}(x,y,z;\lambda,\mu)$ are assigned by formulae (6) and (2) to (4), and the unknown integral densities $B_{km}(\lambda)$ and $H_p(\lambda,\mu)$ are to be found as a result of meeting the boundary conditions of the problem.

Satisfy boundary condition (32) on surface S_2. To this end, we use formulae (10), (11) in theorem 1 that expressing the internal basis solutions for the half-space $\vec{u}_p^{(+)}(x,y,z;\lambda,\mu)$ through the internal basis solutions for the cylinder $\vec{R}_{k,m}(\rho,\varphi,z;\lambda)$, and write the term in expression (34) in the cylindrical system of coordinates:

$$\vec{u}=\sum_{k=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}B_{km}(\lambda)\vec{S}_{k,m}d\lambda+$$
$$+\sum_{p=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_p(\lambda,\mu)\left(\sum_{k=1}^{3}\sum_{m=-\infty}^{\infty}f_{pk}^m\vec{R}_{k,m}\right)\mathrm{d}\mu\,\mathrm{d}\lambda,\tag{35}$$

where functions $f_{pk}^m(\lambda,\mu)$ (m=0, ±1, ±2,...) are defined in [16]. Using the ratio given in [23] for the stress vector $F\vec{u}$, operating at a certain surface with an external normal vector $\vec{n}$,

$$F\vec{u}=2G\begin{pmatrix}\sigma(1-2\sigma)^{-1}\vec{n}\,\mathrm{div}+\\+(\vec{n},\mathrm{grad})+0.5\,\vec{n}\times\mathrm{rot}\end{pmatrix}\vec{u},$$

where G is the shift module, in expression (35) move on to the stresses:

$$\overrightarrow{Fu}\Big|_{S_2}=\sum_{k=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}B_{km}(\lambda)\,\overrightarrow{FS}_{k,m}(\rho,\varphi,z;\lambda)\Big|_{S_2}\mathrm{d}\lambda+$$
$$+\sum_{p=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_p(\lambda,\mu)\left(\sum_{k=1}^{3}\sum_{m=-\infty}^{\infty}\begin{matrix}f_{pk}^m(\lambda,\mu)\times\\\times\overrightarrow{FR}_{k,m}(\rho,\varphi,z;\lambda)\Big|_{S_2}\end{matrix}\right)\mathrm{d}\mu\,\mathrm{d}\lambda.\tag{36}$$

Here, functions $\overrightarrow{FS}_{k,m}(\rho,\varphi,z;\lambda)\Big|_{S_2}$, $\overrightarrow{FR}_{k,m}(\rho,\varphi,z;\lambda)\Big|_{S_2}$ are the stresses that act at surface S_2 with the normals $\vec{n}_2^{(+)}=-\vec{e}_\rho$ and $\vec{n}_2^{(-)}=\vec{e}_\rho$ [17]:

$$\overrightarrow{FS}_{k,m}\Big|_{S_2}=\sum_{j=1}^{3}z_{kj}^m(|\lambda|\,a)e^{i\lambda z+im\varphi}\vec{e}_j,$$

$$\overrightarrow{FR}_{k,m}\Big|_{S_2}=\sum_{j=1}^{3}h_{kj}^m(\lambda a)e^{i\lambda z+im\varphi}\vec{e}_j.\tag{37}$$

Using representations (37), we write down the vector $\overrightarrow{Fu}$ from (36) in the coordinate form at surface S_2 and, by satisfying boundary condition (30) at it, we obtain a system of linear algebraic equations relative to $B_{km}(\lambda)$:

$$\sum_{k=1}^{3}B_{km}(\lambda)z_{kj}^m(|\lambda|a)=$$
$$=A_j^m(\lambda)-\sum_{p=1}^{3}\int_{-\infty}^{\infty}H_p(\lambda,\mu)q_{jp}^m(\lambda,\mu)\mathrm{d}\mu,\tag{38}$$

where $q_{jp}^{m}(\lambda,\mu)=\sum_{\alpha=1}^{3} h_{\alpha j}^{m}(\lambda a) f_{p\alpha}^{m}(\lambda,\mu)$, j=1,2,3. the j'1,2,3. The system's (38) determiner at σ∈[0, 1/2) is not zero, and, at $|m|\geq 2$, it is limited at the bottom by the quantity $C(\sigma)$ $(\lambda^2 a^2+m^2)K_{m-1}(|\lambda|a)\, K_m(|\lambda|a)\, K_{m+1}(|\lambda|a)$ [24].

Satisfy boundary condition (29) at surface S_1. Using integral representations (18), (19) of the external solutions for the cylinder $\vec{S}_{k,m}(\rho,\varphi,z;\lambda)$ through the external solutions for the half-space $\vec{u}_l^{(-)}(x,y,z;\lambda,\mu)$, proven in theorem 2, transform the first term in expression (34):

$$\vec{u}=\sum_{k=1}^{3}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} B_{km}(\lambda)\sum_{l=1}^{3}\int_{-\infty}^{\infty} g_{kl}^{m}(\lambda,\mu)\vec{u}_l^{(-)}\mathrm{d}\mu\,\mathrm{d}\lambda+$$
$$+\sum_{p=1}^{3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_p(\lambda,\mu)\vec{u}_p^{(+)}d\mu\, d\lambda. \quad (39)$$

The $g_{kl}^{m}(\lambda,\mu)$ (m=0, ±1, ±2, ...) functions have been defined in [16]. Using the coordinate representations of functions from [16]:

$$\vec{u}_p^{(+)}(x,y,z;\lambda,\mu)\Big|_{S_1}=\sum_{\xi=1}^{3} d_{p\xi}(\lambda,\mu)e^{i\lambda z+i\mu x}\vec{e}_\xi^{(1)},\quad p=1,2,3,$$

$$\vec{u}_l^{(-)}(x,y,z;\lambda,\mu)\Big|_{S_1}=\sum_{\xi=1}^{3}\tilde{d}_{l\xi}(\lambda,\mu)e^{i\lambda z+i\mu x}\vec{e}_\xi^{(1)},\quad l=1,2,3,$$

record the function $\vec{u}$ from (39) at the surface S_1 in the following form:

$$\vec{u}\Big|_{S_1}=$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{\xi=1}^{3}\left(\begin{array}{l}\sum_{k=1}^{3}\sum_{m=-\infty}^{\infty} B_{km}(\lambda)\sum_{l=1}^{3} g_{kl}^{m}(\lambda,\mu)\tilde{d}_{l\xi}(\lambda,\mu)+\\ +\sum_{p=1}^{3} H_p(\lambda,\mu)d_{p\xi}(\lambda,\mu)\end{array}\right)\times$$
$$\times e^{i\lambda z+i\mu x}\vec{e}_\xi^{(1)}\mathrm{d}\mu\,\mathrm{d}\lambda.$$

By satisfying boundary condition (29) at the surface S_1, we derive a system of linear algebraic equations relative to $H_p(\lambda,\mu)$ (p=1,2,3):

$$\sum_{p=1}^{3} H_p(\lambda,\mu)d_{p\xi}(\lambda,\mu)=$$
$$=\bar{C}_\xi(\lambda,\mu)-\sum_{j=1}^{3}\sum_{n=-\infty}^{\infty} B_{jn}(\lambda)\sum_{l=1}^{3} g_{jl}^{n}(\lambda,\mu)\tilde{d}_{l\xi}(\lambda,\mu), \quad (40)$$

where ξ=1, 2, 3. The system's (40) determinant is different from zero and satisfies the inequality $D_2(\lambda,\mu)=\gamma(3-4\sigma)e^{3\gamma h}/\lambda^2>\gamma e^{3\gamma h}/\lambda^2$ at σ∈[0, 1/2), similar to the problem of the theory of elasticity in displacements [16]. Express $H_p(\lambda,\mu)$ from system (40)

$$H_p(\lambda,\mu)=D_2^{-1}\times$$
$$\times\sum_{\xi=1}^{3}\nu_{\xi p}(\lambda,\mu)\left(C_\xi(\lambda,\mu)-\sum_{j=1}^{3}\sum_{n=-\infty}^{\infty}\begin{array}{c}B_{jn}(\lambda)\times\\ \times\sum_{l=1}^{3} g_{jl}^{n}(\lambda,\mu)\hat{d}_{l\xi}(\lambda,\mu)\end{array}\right),$$

where $\nu_{\xi p}(\lambda,\mu)$ (ξ, p=1,2,3) are the algebraic complements to elements of the system's matrix. By substituting $H_p(\lambda,\mu)$ into (38), we obtain three infinite systems of linear algebraic equations relative to $B_{km}(\lambda)$:

$$B_{km}(\lambda)=\sum_{j=1}^{3}\sum_{n=-\infty}^{\infty} G_{kj}^{mn}(\lambda)B_{jn}(\lambda)+Q_k^{m}(\lambda). \quad (41)$$

The coefficients and the right-hand sides of the system $G_{kj}^{mn}(\lambda)$, $Q_k^{m}(\lambda)$ take the following form:

$$G_{kj}^{mn}(\lambda)=D_1^{m}(|\lambda|a)^{-1}\times$$
$$\times\sum_{r=1}^{3}\varpi_{rk}^{m}(|\lambda|a)\int_{-\infty}^{\infty} F_{rj}^{mn}(\lambda,\mu)\mathrm{d}\mu,$$

$$Q_k^{m}(\lambda)=D_1^{m}(|\lambda|a)^{-1}\times$$
$$\times\sum_{r=1}^{3}\varpi_{rk}^{m}(|\lambda|a)\left(A_r^{m}(\lambda)-\int_{-\infty}^{\infty} L_r^{m}(\lambda,\mu)d\mu\right).$$

The system of equations (41) can be recorded as $(I+\tilde{G})\vec{b}=\vec{q}$, where $\tilde{G}$ is the system's operator. Provided that the boundary surfaces $a<h$ are not intersected, the operator $\tilde{G}$ is quite continuous within space l_2, and the right-hand side $\vec{q}\in l_2$. This follows from the convergence of the series $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|G_{kj}^{mn}(\lambda)\right|^2$, $\sum_{m=-\infty}^{\infty}\left|Q_k^{m}(\lambda)\right|^2$ (k,j=1,2,3) at $a<h$ for all λ∈(–∞, ∞). It follows from Hilbert's alternative that system (41) of the mixed problem in elasticity theory in the region Ω is solvable, and has the only solution belonging to l_2. The system can be solved approximately by the reduction method given in [25].

We have investigated the stress-strain state of the elastic half-space with an infinite circular cylindrical cavity parallel to its boundary, for the case where displacements are set on the surface of the half-space while the surface of the cylinder is free from stresses.

A numerical analysis of the problem was carried out for the functions $\vec{u}_{01}(x,z)/q=-\left(\cos(\lambda z)/1+(x/d)^2\right)\vec{e}_y$, $\overrightarrow{Fu}_{02}(\varphi,z)/(2G)=\vec{0}$, σ=0.25, q=1, λ=1, d=1, assigned at the borders of region Ω, and for different values of the quantity $\varepsilon=a/h$.

To estimate the convergence rate of the reduction method, the functions $\vec{u}$ and $\overrightarrow{Fu}$ were calculated at surfaces S_1 and S_2 at different values of the quantity ε and the order n of system (41). Table 1 gives the values of σ_ρ/E on the cylindrical surface S_2 at z=0. For the rest of the $\overrightarrow{Fu}$ vector components, we derive values of the same or lesser order. For the boundary condition (30) in the half-space, the accuracy of 10^{-7} is achieved at lower n values.

Table 1

Values of σ_ρ/E on cylindrical surface S_2 at different ε and n

n	ε=0.3	ε=0.5	ε=0.7	ε=0.9
n=3	10^{-1}	10^{-1}	10^{-2}	10^{-1}
n=7	10^{-5}	10^{-4}	10^{-4}	10^{-4}
n=12	10^{-7}	10^{-7}	10^{-6}	10^{-4}
n=15	10^{-8}	10^{-8}	10^{-7}	10^{-5}

Fig. 1 shows the displacements of u_ρ at the surface of the cylinder for ε=0.7. They correspond to the displacements set at the border of the half-space.

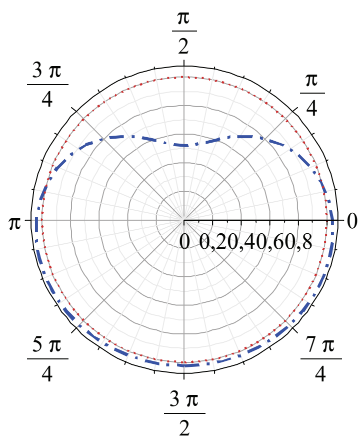

Fig. 1. The displacements of u_ρ at the surface of the cylinder

Fig. 2, *a* shows the distribution of stresses σ_ρ and $\tau_{\rho\varphi}$ within the region Ω on the concentric circles set by the equation $\rho=a+k(h-a)/3$, $k=1..3$. At the same time, the k value corresponds to the curve number. The stresses σ_ρ and $\tau_{\rho\varphi}$ at the surface of the cylinder are zero under the set condition. Fig. 3, *a, b* shows the distribution of stresses σ_φ and σ_z within the region Ω on the same circles and the surface of the cylinder ($k=0$).

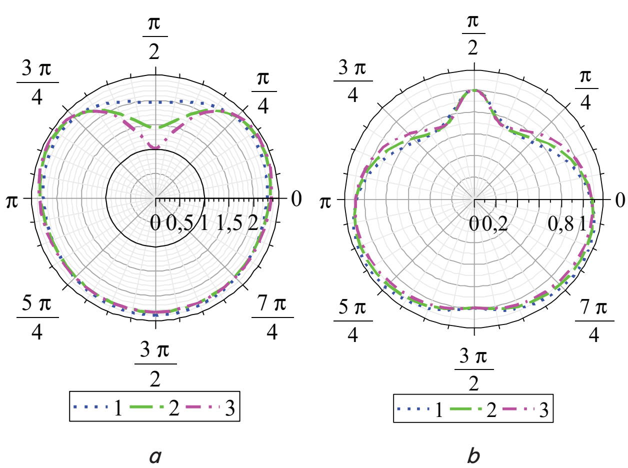

Fig. 2. The distribution of stresses on the circles: $a-\sigma_\rho/E$; $b-\tau_{\rho\varphi}/E$

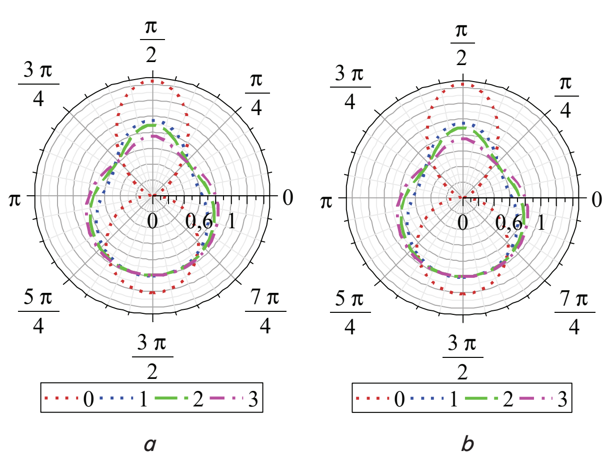

Fig. 3. The distribution of stresses on the circles: $a-\sigma_\varphi/E$; $b-\sigma_z/E$

The largest modulo values are accepted by the component σ_φ near a cavity at $\varphi\approx\pi/6$, the stresses $\tau_{\rho\varphi}$ are concentrated near the boundary of the half-space at $\varphi\approx\pi/3$. The component σ_ρ is significantly smaller than σ_φ and $\tau_{\rho\varphi}$.

Fig. 4, *a, b,* and Fig. 5 show the charts of the normal σ_x, σ_y, and tangent τ_{xy} stresses on parallel straight lines set by equations $y=a+j(h-a)/3$, $j=1..3$. The number $j=3$ corresponds to the surface of the half-space.

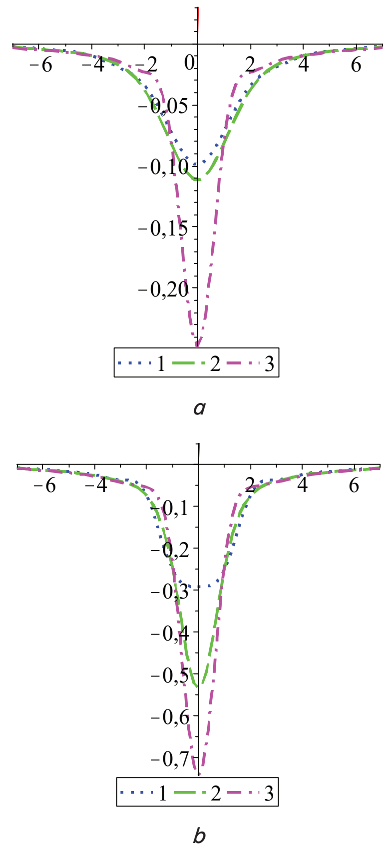

Fig. 4. The distribution of stresses on straight lines: $a-\sigma_x/E$; $b-\sigma_y/E$

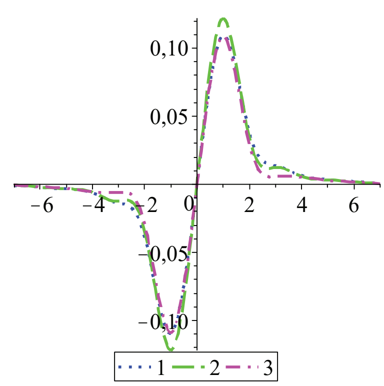

Fig. 5. The distribution of stresses τ_{xy}/E on straight lines

The highest values are accepted by the σ_y component at $x=0$, its change along the Ox axis corresponds to the displacements assigned on the half-space. The dependence of the σ_x and σ_y components on x and y is due to the influence of boundary conditions on the cylinder and half-space. The τ_{xy} component contributes less to the distribution of stresses than σ_x and σ_y.

6. Discussion of results of studying a mixed problem in elasticity theory for half-space with a cylindrical cavity

The results from solving a mixed problem in elasticity theory for half-space with a cylindrical cavity are explained within the framework of the linear theory of elasticity. Fig. 2, *a*, and Fig. 4, *b* show that the highest values are accepted by the normal stresses in the area between the boundaries of the half-space and the cylinder. This is due, first, to the presence of a cavity, and, second, to the form of the function assigned on the surface of the half-space.

Various analytical and numerical methods are used to solve the problems of the theory of elasticity in half-space with cavities of different shapes. All of them are particular in nature as they can be used to solve problems with a specific cavity geometry. Compared to them, the generalized Fourier method is a a theoretically based method, as well as an effective numerical-analytical technique to solve spatial problems in multiply connected bodies. First, it employs exact solutions to a Lamé equation associated with each boundary surface of a multiply connected body. Strictly proven addition theorems make it possible to satisfy boundary conditions at these surfaces. As a result, the problem is reduced to an infinite system of equations, the coefficients and right-hand sides of which decrease on infinity. That makes it possible to apply a reduction method to solve the system. Given this, the problem can be solved with any predetermined accuracy by increasing the number of equations of the system. To test the accuracy of the solution, the displacements and stresses components were calculated on boundary surfaces and compared with the specified values.

A finite-element method is often used to solve the practical problems that arise when designing underground structures. However, its scope is limited to the bodies of finite size. It is also ineffective for multiply connected bodies with closely spaced borders. When solving the problems by the generalized Fourier method, the latter problem is solved by increasing the order of the system. One can see it from Table 1, which gives the order of the system for different values of the geometric parameter ε, at which the accuracy of the problem solution is 10^{-6}.

The generalized Fourier method is used to solve boundary problems in multiply connected bodies, the boundaries of which consist of two or more coordinate surfaces of the curvilinear orthogonal coordinate systems. This method cannot be applied to regions whose boundary surfaces intersect or touch each other.

In the future, this method could be applied to investigate the main and mixed problems in elasticity theory for half-space with one or more inclusions. In this case, the conjugation conditions must be additionally set on the cylindrical surface.

7. Conclusions

1. We have proven the addition theorems of basis solutions to a Lamé equation for the half-space and cylinder, recorded in the Cartesian and cylindrical coordinate systems. This proof is necessary to strictly substantiate the application of the generalized Fourier method for solving boundary problems in the elastic half-space with an infinite circular cylindrical cavity. When proving the addition theorems, we used formulae linking harmonic functions in the Cartesian and cylindrical coordinate systems, as well as the ratios for the modified Bessel functions of the 1st and 2nd kind.

2. The mixed problem in elasticity theory for the half-space with an infinite circular cylindrical cavity parallel to its boundary has been solved by the generalized Fourier method. Specifically:

– the addition theorems of a half-space and a cylinder have made it possible to write down the solution to the problem in a coordinate system associated with each boundary surface of a doubly connected body. As a result, the specified boundary conditions were satisfied on the boundary between the half-space and the cylindrical surface;

– the problem has been reduced to an infinite system of linear algebraic equations relative to the integral densities $B_{km}(\lambda)$. The operator of the system is quite continuous within space l_2 under the condition $a<h$ that the boundary surfaces do not intersect. This has allowed us to solve the system by the reduction method. The stress-strain state of the elastic half-space containing an infinite cylindrical cavity parallel to its boundary has been investigated, for the case when displacements are set at the boundary of the half-space while the surface of the cylinder is free from stresses. The σ_ρ, σ_φ, σ_z and $\tau_{\rho\varphi}$ components were calculated on the concentric circles set by the equation $\rho=a+k(h-a)/3$, k=1..3, and the σ_φ and σ_z components on the surface of the cylinder (k=0) as well. The normal σ_x, σ_y, and tangent τ_{xy} stresses were determined on the parallel straight lines set by the equation $y=a+j(h-a)/3$, j=1..3. Numerical analysis reveals:

– the largest modulo values are accepted by the σ_φ component near the cavity at $\varphi\approx\pi/6$; the $\tau_{\rho\varphi}$ stresses are concentrated near the boundary of the half-space at $\varphi\approx\pi/3$. The σ_ρ component values are significantly less than the values of σ_φ and $\tau_{\rho\varphi}$;

– the highest values are accepted by the σ_y component at x=0; and its change along the Ox axis corresponds to the displacements set on the half-space. The dependence of the σ_x component on x and y is due to the influence of boundary conditions on the cylinder and half-space. The τ_{xy} component contributes less to the distribution of stresses than σ_x and σ_y. The reliability of our calculations has been confirmed by the analytical justification for the application of a reduction method to solve the system of equations and the accuracy of meeting boundary conditions on the surface of the half-space and cylindrical cavity.

Acknowledgments

We express our gratitude to the Doctor of Physical and Mathematical Sciences, Professor Vladimir Sidorovich Protsenko for the idea to prove the addition theorems and the statement of a mixed problem in elasticity theory for half-space with a cavity.

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