The synthesis of factorization and symmetry methods produced a general analytical solution to the fourth-order differential equation with variable coefficients. The form and structure of the variable coefficients correspond, in this case, to the problem of the oscillations of a concave beam of variable thickness. The solution to this equation makes it possible to study in detail the oscillations of such and similar, for example convex, beams at the different fixation of their ends' sections. A practical confirmation has been obtained that the beam whose thickness changes in line with the concave parabola law $H=a^2x^2+1$, where a is the concave factor, demonstrates an increase in the natural frequencies of its free oscillations with an increase in its rigidity. As an example, the object's maximum deflection dependence on the beam rigidity factor has been established. The nature of this dependence confirmed the obvious conclusion that the deflections had decreased while the rigidity had increased. The evidence from the calculation results can be a testament to the correctness of the reported procedure of problem-solving.

The considered problem and the analytical solution to it could serve as a practical guide to the optimal design of beam structures. In this case, it is very important to take into consideration the place and nature of the distribution of cyclical extreme operating stresses. The resulting ratios to solve the problem make it possible to simulate the required normal stresses in both the fixation and central zones when the rigidity parameter is changed. Designers could predict such a parabolic profile of the beam, which would ensure the required reduction of maximum stresses in the place of fixing the beam. The considered example of solving the problem of the natural oscillations of the beam with rigid fixation of the ends illustrates the effectiveness of the factoring and symmetry methods used. The developed solution algorithm could be extended to study the natural bending oscillations of the beam at other fixing techniques, not excluding a variant of a completely free beam

Keywords: free oscillations, variable thickness, symmetry method, factoring method, approximation, differential equation

UDC 534.8
DOI: 10.15587/1729-4061.2021.230820

ANALYTICAL STUDY OF THE NATURAL BENDING OSCILLATIONS OF A CONCAVE BEAM WITH PARABOLIC CHANGE IN THICKNESS

Kirill Trapezon
Corresponding author
PhD, Associate Professor
Department of Acoustic and Multimedia Electronic Systems
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"
Peremohy ave., 37, Kyiv, Ukraine, 03056
E-mail: kirill.trapezon@gmail.com

Alexandr Trapezon
Doctor of Technical Sciences, Leading Researcher
Laboratory No. 7.1
G. S. Pisarenko Institute for Problems of Strength of the National Academy of Sciences of Ukraine
Tymiriazievska str., 2, Kyiv, Ukraine, 01014

Received date 07.05.2021
Accepted date 09.06.2021
Published date 30.06.2021

How to Cite: *Trapezon, K., Trapezon, A. (2021). Analytical study of the natural bending oscillations of a concave beam with parabolic change in thickness. Eastern-European Journal of Enterprise Technologies, 3 (7 (111)), 15–23. doi: https://doi.org/10.15587/1729-4061.2021.230820*

1. Introduction

Beams with various cross-sections are widely used as components of machines and structures. For example, beam structures are used in the construction of bridges [1, 2] where metal span structures consist of two cut bearing beams. The adjacent application area of beam structures could be considered their application in the design of railroad tracks when taped beams are tested for strength as railroad sleepers at bending oscillations [3]. Of interest is the use of over-the-spring beam structures as supporting elements in the rolling stock on railroads [4].

Beams are the main bearing elements in the design of airplane turbine engine blades [5–7]. In aerospace technology [8, 9], the beam elements are used to create the aeroelastic vibrations of a cruise missile since the wing of such a missile could be considered as an elastic beam operated at bending. Of particular interest is also the application of beams in the structure of an ultrasonic sawmill frame [10]. In this case, the intensification of the cutting process and ensuring the effectiveness of the sawing process involve a curved waveguide in the scheme of creating longitudinal-bending-longitudinal oscillations. Beams could also be used as basic structural elements to ensure the process of ultrasonic welding [11]. In this case, to create the required transverse oscillations of the working element, the beam is welded to the concentrator of longitudinal oscillations at the right angle, so that it generates bending oscillations. Other innovative industries involving the widespread use of beam structures are robotics [6], aeronautics [8], and wind power [12, 13].

Typically, beam structures whose operation implies bending are exposed, under a resonance mode, to destructive stresses. Oscillations arising during the operation of beam structures pose a serious danger to them, especially when the oscillation frequencies approach their natural oscillations. In order to prevent possible resonance, it is necessary to have clear information about the critical values of the natural frequencies and the related shapes of oscillation in the most responsible structural elements. The natural frequencies and shapes of bending oscillations in the beams of variable rigidity have been studied in detail only for individual cases. Analysis of natural bending oscillations is necessary so that by knowing the natural frequencies and the distribution of normal stresses due to bending one could improve the technical resource of the beam structure, especially when different types of fixation are used in the beam scheme. In this regard, the task to derive a closed analytical solution to the problem of free oscillations of the beam with variable cross-section is still relevant.

The literature reports and describes the exact solutions to this problem only for limited geometric configurations of beams in the form of a wedge, a cone, and a pyramid [14]. Of practical interest are the beams of the rectangular cross-section whose rigidity is determined by the width $b(x)$ and the thickness $H(x)$. The most technological cases are b=const; $H=H(x)$, or $b=b(x)$; H=const. Paper [15] considers the cases of rigidity change, in which the problem of oscillations has a precise solution. For the case of b=const, reported in the cited paper, the earlier results of the precise solution to the problem are known only for $H=x$ (Kirchhoff, Mononobe) and $H=e^{mx}$ [15]. Often, researchers use approximate numerical methods to find a solution to a problem, which tend to be cumbersome and impossible to generalize for direct, broad practical applications. The relevance of this scientific issue, in addition to the extremely diverse technical application of beam structures, is that it is necessary to have an accurate analytical solution to the relevant boundary problem in order to analyze the natural bending oscillations of the beam of the variable cross-section. This requirement is explained by the fact that in order to ensure the desired operational resource of structures with such beams, at a different form of their fixation, it is necessary to have information on the distribution of deflections and stresses lengthwise a beam element. Such a requirement could only be met on the basis of theoretical analysis of the free oscillations of beams with variable cross-sections.

2. Literature review and problem statement

The results of detailed studies into variable cross-section beams under a free oscillation mode are very limited due to known mathematical difficulties arising when trying to analytically solve the problem of oscillations. This is directly emphasized by the authors of article [5] who state that obtaining an accurate analytical solution is a complex procedure due to the presence of variable coefficients in the main resolving equation. As a possible solution to the problem of free oscillations of the beam, they propose using an approximate method of perturbations. Only variants with linear changes in the height and width of the beam are considered; in addition, the numerical method of Poincare-Lindsted is proposed for finding natural frequencies. The considered cases could hardly be extended to the variant of the beam whose thickness varies according to the parabolic law.

Another example of trying to overcome the above mathematical issues in the search for an analytical solution to a given problem is given in article [16]. It is emphasized that the analytical solution for beams is very difficult to obtain, and precise solutions were obtained only for some special profiles of beams under certain boundary conditions. Bessel functions, hypergeometric series, "energy" series, and Beristale polynomials are offered as special functions as the main tool for the analytical solution. The approximate solutions to the problem, reported in the cited paper, were derived from the Rayleigh-Ritz methods, a finite-element method, the method of dynamic rigidity, a differential quadrature method, the differential method of transformation. The approaches considered concern however only a prism beam and a beam whose thickness changes exponentially. The extension of the above procedure to other types of beams is very difficult because differential transformations are used to solve the resolving equations based on a series of recurrent algebraic expressions. When recording a function of the oscillation shape, the length of the expression for a recurrent ratio could be way too long and technically difficult to find a frequency equation.

The problem of free oscillations of a prism beam was considered in work [17]. For a rigidly fixed permanent-section beam, the free oscillation analysis is based on six different sets of characteristic functions to describe the cross-section movements of the beam. In addition, in the case of a rectangular beam, in order to search for natural frequencies, it is proposed to use an approximate energy approach based on the Ritz method. It should be noted that the suggested procedure cannot be extended to the concave beams of the variable cross-section.

Article [6] reports a study into the free oscillations of beam structures with variable width. Based on the asymptotic method of perturbations, the authors obviously confirmed that the beam's natural frequencies decrease with an increase in its width at the free end at the same length. Only a clamped console beam was considered as an example; the reliability criterion of the reported results is a finite-element method and the experiment conducted.

Papers [8, 18] proposed using a Fourier harmonic series to construct the frequency equation for considering the oscillations of a free-supported beam with an arbitrary thickness [18], or with the thickness that changes in line with an exponential law [8]. However, in order to find unknown coefficients that are included in the frequency equation, the authors of [18] give an algebraic expression in a matrix form of 50×50. Only a beam of constant height and a beam with a linear change of height were considered as an example. It is obvious that those expressions could hardly be applied for a beam with a parabolic profile and rigidly clamped ends.

Work [19] outlines a variant of designing the optimal structure of a tank car support device where the structure itself is represented in the form of a beam of variable thickness. Numerical and graphic methods are used as a calculation tool for the optimal design of the support model. Based on the schemes of the support device, one can judge the achievement of reducing rigidity along the middle surface of the structure's sheet, reducing the maximum stress in the support zone, reducing the mass of the structure. Analytical approaches were not considered in addressing the specified issues.

Elements of variation calculus are used in articles [1, 20] where a numerical method of integrated-differential ratios was used to state the boundary problem based on the Euler-Bernoulli hypothesis. As a result, a polynomial with variable coefficients was applied to find the displacement function; the approximation of this function was performed under the condition for a minimum of the quadric functional. The proposed solution is based on the so-called energy and movement integral, as the distribution of energy along the beam structure could be analyzed only if one takes this integral on a rectangular area.

The review of the scientific literature reveals that the above cases of analytical solutions do not concern beams in the form of a symmetrical structure of the parabolic profile in the form of $H=a^2x^2+1$. Therefore, there is a need for a substantive study into the oscillations of such particular beam, especially since such structures (bridge spans, arches, ceilings, above-the-spring elements) are often for isolated purposes.

The above suggests that it is appropriate to conduct a study to solve a problem about the transverse oscillations of a variable thickness beam with varying degrees of concaveness. The scientific publications reviewed employ mathematical tools based only on approximate calculations. The accuracy and reliability of the results may be questionable. Analytical solutions, unlike approximate or numerical ones, make it possible to expand the existing estimation base for beams with variable thickness, supplementing it with new results obtained in the final form.

3. The aim and objectives of the study

The aim of this study is to derive a closed analytical solution to the problem of the natural oscillations of a beam whose thickness changes according to the parabolic law. The base structure is a concave beam with a profile that changes in line with the law $H=a^2x^2+1$.

To accomplish the aim, the following tasks have been set:

– by using the factorization method, transform the original differential equation of the fourth order to the form that would allow it to be decomposed into a system of two self-adjoint resolving equations of the second order;

– by using the symmetry method and the scheme of its application, find an approximating function, which could find precise solutions to the derived equations of the second order, and, therefore, a solution to the original equation of the fourth order;

– to explore the oscillations of a symmetrical concave beam, rigidly fixed at both ends. Calculate the frequency numbers k_i (i=1÷3); build, as an example, graphic illustrations for k_i and a series of the shapes of oscillations depending on the degree of beam concaveness.

4. Materials and methods to explore the problem about the oscillations of a concave beam

A mathematical model of this study is a differential equation of small oscillations for a beam with a rectilinear axis and a variable cross-section. The equation is based on the positions of the technical theory of rods (Euler-Bernoulli beam theory) under the following assumptions:

– oscillations occur in one of the main planes of the beam bending and the size of the cross-section is small compared to the length of the beam; the thickness of the beam is no more than 1/5 of its length;

– the cross-sections remain flat at deformation (the hypothesis of flat sections) and are perpendicular to the deformed axis of the beam;

–the normal stresses on the sites parallel to the axis are negligible; stretching the axis is also neglected.

To facilitate the search for an analytical solution, the method of factoring is used to decompose the original equation of the fourth order into two self-adjoint equations of the second order with variable coefficients.

To find a solution, the method of approximation of variable coefficients in the resolving equations of the second order is employed.

A symmetry method is applied to construct the required approximation function at which the resolving equations have precise solutions.

A method of sequential trial calculations is used to solve frequency equations.

The materials to which the results of our study are applicable must follow the Hook law.

5. Results of exploring a problem about the oscillations of a concave beam

5. 1. The original differential equation and its transformation

Consider the beam in the form of a symmetrical structure (Fig. 1) of the parabolic profile, the type of $H=a^2x^2+1$.

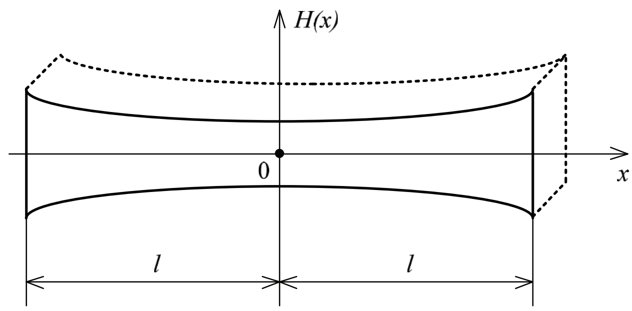


Fig. 1. Sketch of the beam element

The initial differential equation for the problem about the natural (free) oscillations of a variable thickness beam $H(x)$ is the equation of oscillation shapes in the form given in [21]:

$$\frac{d^2}{dx^2}\left(H^3\frac{d^2W}{dx^2}\right)-k^4HW=0, \tag{1}$$

where

$$k^4=12\omega^2\rho l^4/E, \tag{2}$$

$W=W(x)$ – beam deflections when it oscillates; $x=X/l$; l – half the length of the beam (Fig. 1); $H(x)$ – the variable thickness (height) of the beam; $w=2\pi f$ – circular frequency; f – frequency of oscillations; E, $\rho=\gamma/g$ – Young modulus and material density; γ – specific weight; g – acceleration of gravity.

The equations of the fourth order, including (1), are generally difficult to solve in an analytical form. Solving a problem is much easier if such an equation could be represented in a symbolic form

$$\left(A\frac{d^2}{dx^2}+B\frac{d}{dx}+C\right)^2 W-D^4W=0, \tag{3}$$

where A, B, C are some functions; $D^4=k^4+\chi^2$; χ^2 is a constant. If the representation (3) is feasible in accordance with the factoring method, then, instead of the fourth-order equation, it would suffice to consider two equations of the second order

$$AW_i''+BW_i'+CW_i\pm D^2W_i=0.\quad (i=1,2). \tag{4}$$

In this case, a solution to the original equation is to be found in the form of the sum of the solutions to two equations (4), that is, $W=W_1+W_2$, where W_1 is the solution to equation (4) at the "plus" sign before D^2, and W_2 – at the "minus" sign. A direct substitution could be used to make sure that equation (1) is identical to equation (3) at $H=a^2x^2+1$, if and only $A=H$; $B=2H'$; $C=-H''$. Once the functions A, B, C are introduced in (4), we obtain $\chi^2=4a^4$; and $D^4=k^4+\chi^2=k^4+4a^4$; $H''=2a^2$, as well as

$$\left(a^2x^2+1\right)W_i''+4a^2xW_i'+\left(\pm D^2-2a^2\right)W_i=0. \qquad (5)$$

It is difficult to obtain closed solutions to equations (5); and it seems impossible at all, which is why we shall use the symmetry method for the equations of the second order [22] to build the required solutions.

5. 2. The scheme of a symmetry method for resolving equations

Replacing the variable $x=x(\varphi)$ converts (5) to the following form

$$W_{i\phi\phi}+\frac{F_\phi}{F}W_{i\phi}+\lambda_i^2W_i=0, \qquad (6)$$

where

$$W_{i\phi\phi}=\frac{d^2W_i}{d\phi^2};\quad W_{i\phi}=\frac{dW_i}{d\phi};$$

$$\lambda_i^2=\begin{cases}+\lambda_1^2=D^2-2a^2=\sqrt{k^4+4a^4}-2a^2;\\ -\lambda_2^2=-\left(D^2+2a^2\right)=-\sqrt{k^4+4a^4}-2a^2.\end{cases}$$

Equation (5), expressed through the new variable $\varphi(x)$, is convenient to write in the following form

$$H\phi_x^2\left[W_{i\phi\phi}+\frac{\left(\phi_xH^2\right)_\phi}{\phi_xH^2}W_{i\phi}\right]+\lambda_i^2W_i=0. \qquad (7)$$

In order for this equation to be represented in form (6), one needs to adopt $H\phi_x^2=1$, by denoting $\varphi_xH^2=F$. Based on this, one can write

$$\phi_x=\frac{1}{\sqrt{H}};\quad \phi=\int\frac{dx}{\sqrt{a^2x^2+1}};\quad F=\frac{H^2}{\sqrt{H}}=H^{3/2}.$$

By completing the integration and appropriate transformations, we obtain

$$\left.\begin{aligned}t=a\phi+\ln a=\ln\left(ax+\sqrt{a^2x^2+1}\right);\\ ax=\frac{1}{2}\left(ae^{a\phi}-\frac{1}{ae^{a\phi}}\right)=\sinh t;\\ H=a^2x^2+1=\cosh^2 t;\\ F=\cosh^3 t.\end{aligned}\right\} \qquad (8)$$

A solution to equation (6) at $F=\cosh^3 t$ is not known but this issue is eliminated by replacing $F(t)$ with the approximating function $F_1(t)$, at which this equation has a precise solution. Choosing the $F_1(t)$ function is feasible once using the symmetry method, the essence and purpose of which is to develop ways to obtain accurate solutions to differential equations with variable coefficients. These coefficients are selected by a special algorithm. Thus, relative to equation (6), expressed through the $t=a\varphi+\ln a$ variable and taking the following form

$$W_{itt}+\frac{F_t}{F}W_{it}+b^2W_i=0\quad\left(b^2=\lambda_i^2/a^2\right), \qquad (9)$$

the symmetry method produces the following results. If, for example, one assumes $F=e^{2t}$, the solution is a closed one because the equation itself does not contain variable coefficients. By using, according to the proposed method, the algorithm of transition from the equation of form (6) at the parameters W and F, to its symmetry – an equation with parameters W_1 and F_1, one can write [22]:

$$\left.\begin{aligned}F_1=\cosh^2 t;\\ W_1=W/\cosh t,\end{aligned}\right\}$$

where W is the solution to an equation in form (9) at $F=e^{2t}$. Thus, to solve the set problem, we choose the following expression as an approximating function

$$F_1=\cosh^2 mt, \qquad (10)$$

where $m=1.26$ is the constant found during approximation. Fig. 2 shows the charts of functions $F=\cosh^3 t$ and $F_1=\cosh^2 mt$ at $t=0\div1.1$.

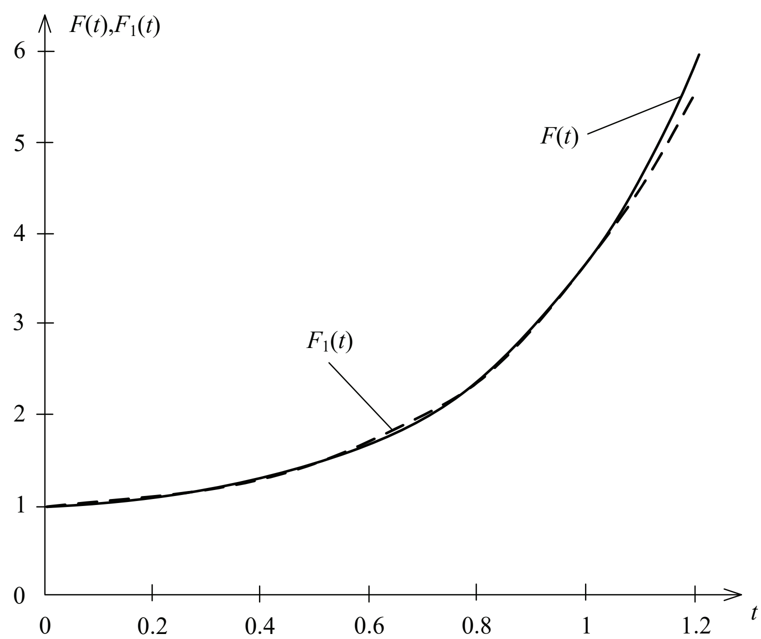


Fig. 2. Graphic interpretation of functions F and F_1

One can see from Fig. 2 and Table 1, which gives the values of F and F_1 in the marked interval, that the match of the compared functions is quite satisfactory. The average value of parameter $\delta=(1-F/F_1)\times100$ % is about 0.5 %.

After introducing (10) in (9), we obtain the following equation

$$W_{itt}+2m\frac{\sinh mt}{\cosh mt}W_{it}+b^2W_i=0\quad\left(b^2=\lambda_i^2/a^2\right).$$

For this equation, after the next replacement of the variable $mt=z$, we move to the following equation

$$W_{izz}+2\frac{\sinh z}{\cosh z}W_{iz}+\frac{\lambda_i^2}{m^2a^2}W_i=0, \tag{11}$$

which has a precise solution built on the symmetry method. According to (11), the problem's resolving equations are written in the following form

$$\left.\begin{aligned}&W_{1zz}+2\tanh zW_{1z}+\frac{D^2-2a^2}{m^2a^2}W_1=0;\\&W_{2zz}+2\tanh zW_{2z}-\frac{D^2+2a^2}{m^2a^2}W_2=0,\end{aligned}\right\}$$

hence

$$W_1=\frac{1}{\cosh z}\left(\begin{aligned}&A_1\sin\left(\left(\sqrt{\frac{D^2-2a^2}{m^2a^2}-1}\right)z\right)+\\&+B_1\cos\left(\left(\sqrt{\frac{D^2-2a^2}{m^2a^2}-1}\right)z\right)\end{aligned}\right);$$

$$W_2=\frac{1}{\cosh z}\left(\begin{aligned}&A_2\sinh\left(\left(\sqrt{\frac{D^2+2a^2}{m^2a^2}-1}\right)z\right)+\\&+B_2\cosh\left(\left(\sqrt{\frac{D^2+2a^2}{m^2a^2}-1}\right)z\right)\end{aligned}\right).$$

Table 1

Values of functions F and F_1 in the interval t=0÷1.2

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.05	1.1	1.2
F	1	0.015	0.0061	1.142	1.263	1.431	1.666	1.977	2.392	2.943	3.67	4.125	4.645	5.936
F_1	1	0.016	0.0065	1.15	1.276	1.452	1.689	2.002	2.41	2.941	3.62	4.042	4.513	5.656
δ, %	0	0.086	0.323	0.656	1.003	1.27	1.368	1.215	0.749	−0.078	−1.295	−2.056	−2.9	−4.96

The overall solution to equation (1) at the predefined beam thickness of $H=a^2x^2+1$ is to be recorded as the sum $W=W_1+W_2$ and, after the transition to the variable $t=z/m$, takes the following form

$$W=\frac{1}{\cosh mt}\left(\begin{aligned}&A_1\sin pt+B_1\cos pt+\\&+A_2\sinh qt+B_2\cosh qt\end{aligned}\right), \tag{12}$$

where

$$p=\sqrt{\frac{D^2}{a^2}-\left(2+m^2\right)};\quad q=\sqrt{\frac{D^2}{a^2}+\left(2-m^2\right)};$$

A_i, B_i (i=1,2) are the constants that are dependent on boundary conditions.

We have considered a solution to the problem of the oscillations of a beam with rigidly fixed ends.

5. 3. Solving a problem about the oscillations of a beam with rigidly fixed ends

For the case of the rigid fixing of the ends' sections of the beam, the deflections and rotation angles at $x=\pm1$ are zero. The boundary conditions of the problem are written in the following form

$$\left(W\right)_{x=\pm1}=0;\ \left(W_x\right)_{x=\pm1}=0. \tag{13}$$

By meeting these conditions after introducing a general solution (12) to them, we obtain the required expressions to determine the natural oscillation frequencies and amplitude coefficients related through A_1, A_2, B_1, B_2. Since the solution to (12) is recorded through a variable t, the conditions for (13) are also appropriate to represent through $t=\ln(ax+$ $+(a^2x^2+1)^{1/2})=\text{arcsinh}(ax)$ (8). The second of the conditions (13) should be written through the W_t derivative according to the ratio $W_x=t_xW_t=a/\cosh(t)W_t$. Hence, instead of (13), the boundary conditions take the form

$$\left(W\right)_{t=\pm t_1}=0;\ \left(W_t\right)_{t=\pm t_1}=0. \tag{14}$$

The limit values of $t_1=\ln(a+(a^2+1)^{1/2})=\text{arcsinh}a$ can be expressed through the ratio of beam thicknesses at x=1 and x=0, which is equal to $\eta=H(1)/H(0)=a^2+1=(\sinh t_1)^2+1=(\cosh t_1)^2$. Table 2 gives the values for the parameters η; a; t_1, valid for the interval of t=0÷1.1.

Table 2

Estimation parameters η; a; t_1

t_1	0	0.481143	0.65841	0.78339	0.88136	0.96242	1.0316	1.077	1.14
a	0	0.5	0.7071	0.866	1	1.118	1.2247	1.322	1.4142
η	1	1.25	1.5	1.75	2	2.25	2.5	2.75	3

Because the initial function F approximation was taken in the interval t=0÷1.1 (Table 1), then the limit values for t_1 in Table 2 were selected from this interval. This means that a satisfactory solution to the problem, in our case, could only be obtained if η≤3. For practical purposes, the results of calculating the oscillations at η=1÷3 are likely to be sufficient, especially as a demonstration of the effectiveness of the method used.

Before substituting a general solution (12) in boundary conditions (14), for convenience, we shall rewrite this solution in the following form

$$W=\frac{1}{\cosh mt}\left(\begin{aligned}&A_1^*\sin p\left(t+t_1\right)+B_1^*\cos p\left(t+t_1\right)+\\&+A_2^*\sinh q\left(t+t_1\right)+B_2^*\cosh q\left(t+t_1\right)\end{aligned}\right). \tag{15}$$

After introducing (15) to boundary conditions (14), we obtain a system of the following ratios

$$B_1^*+B_2^*=0;\quad pA_1^*+qA_2^*=0;$$

$$A_1^*\left(\sin 2pt_1-\frac{p}{q}\sinh 2qt_1\right)+$$
$$+B_1^*\left(\cos 2pt_1-\cosh 2qt_1\right)=0;$$

$$A_1^*\left(\cos 2pt_1-\cosh 2qt_1\right)-$$
$$-B_1^*\left(\sin 2pt_1+\frac{p}{q}\sinh 2qt_1\right)=0.$$

Hence,

$$\left.\begin{aligned} &A_2^* = -\frac{p}{q}A_1^*; \quad B_2^* = -B_1^*; \\ &B_1^* = -A_1^*\frac{\sin 2pt_1 - \frac{p}{q}\sinh 2qt_1}{\cos 2pt_1 - \cosh 2qt_1} = \\ &= A_1^*\frac{\cos 2pt_1 - \cosh 2qt_1}{\sin 2pt_1 + \frac{p}{q}\sinh 2qt_1}. \end{aligned}\right\} \quad (16)$$

The following frequency equation follows from (16):

$$\frac{\sin 2pt_1 - \frac{p}{q}\sinh 2qt_1}{\cos 2pt_1 - \cosh 2qt_1} + \frac{\cos 2pt_1 - \cosh 2qt_1}{\sin 2pt_1 + \frac{p}{q}\sinh 2qt_1} = 0. \quad (17)$$

Because $p=((D/a)^2-m^2-2)^{1/2}$; $q=((D/a)^2-m^2+2)^{1/2}$, then $q^2-p^2=4$, and then equation (17) could be reduced to an expression containing only one desired value, for example, p. Thus, upon substituting $q=(p^2+4)^{1/2}$ in (17), we obtain

$$p\sqrt{p^2+4}\left(1-\cos(2t_1p)\cosh\left(2t_1\sqrt{p^2+4}\right)\right)+$$
$$+2\sin(2t_1p)\sinh\left(2t_1\sqrt{p^2+4}\right)=0. \quad (18)$$

By setting the values for t_1, based, for example, on the chosen concaveness of the beam, defined by the parameter $\eta=H(1)/H(0)=(\cosh t_1)^2$, one can determine, from (18), the values of p_j and q_j (j=1,2,3,...), which meet the natural numbers of problem k_j, which are to be found from the following ratios:

$$k_j = \sqrt[4]{D_j^4 - 4a^4}; \quad D_j = a\sqrt{p_j^2 + m^2 + 2}.$$

Table 3 gives the values of the specified values at j=1,2,3 for the interval η=1÷3. For the case η=1 that corresponds to the beam of a permanent cross-section, the natural numbers k_j can be found from a known equation $\cos(2k)\cosh(2k)-1=0$.

Fig. 3 shows the graphical dependences of values for the first three natural frequencies (k_j number) on the values of $\eta=H(x=1)/H(x=0)$, corresponding to the data in Table 3.

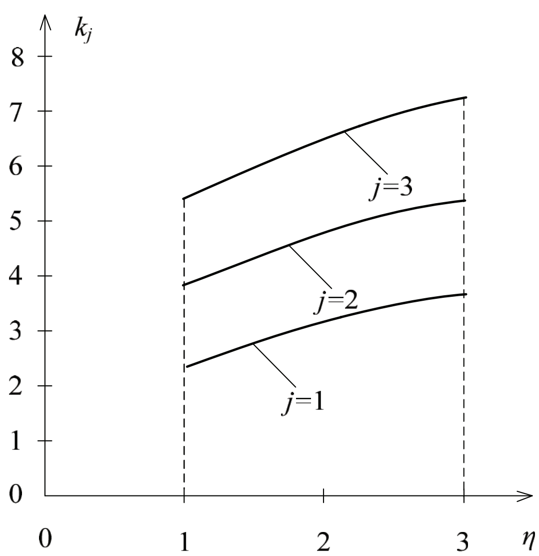


Fig. 3. Dependence of k_j on the width parameter η

Table 3

Frequency parameters of beam natural oscillations

η	j	p_j	q_j	D_j	x	t	k_j
1	1	–	–	–	–	–	2.365
	2						3.9295
	3						5.498
1.01	1	23.678647069	23.76296	2.37543	0.25	0.025	2.37542
	2	39.33828025045	39.38909	3.93839	0.5	0.04998	3.93838
	3	55.084914487228	55.12121	5.51175	0.75	0.07493	5.51175
					–	–	
1.25	1	4.828453144	5.22628	2.59334	0.25	0.12468	2.58974
	2	8.1306165549	8.37299	4.17416	0.5	0.24747	4.1733
	3	11.410822471068	11.58477	5.78348	0.75	0.36672	5.78315
					1	0.48121	
1.5	1	3.47814848	4.01217	2.78063	0.25	0.17464	2.76926
	2	5.922972923	6.25153	4.36598	0.5	0.34421	4.36306
	3	8.328832090661	8.5656	5.99698	0.75	0.50485	5.99585
					1	0.65439	
1.75	1	2.888601327	3.51341	2.99135	0.25	0.21484	2.97012
	2	4.9645775903	5.35229	4.6016	0.5	0.4205	4.59582
	3	6.9928733619383	7.27326	6.27404	0.75	0.61081	6.27176
					1	0.78338	
2	1	2.54165389	3.23419	3.1698	0.25	0.24747	3.13792
	2	4.4022317052	4.83525	4.79242	0.5	0.48121	4.7833
	3	6.209873337915	6.524	6.49231	0.75	0.69315	6.48866
					1	0.88137	
2.25	1	2.30729921	3.05346	3.33742	0.25	0.27598	3.29457
	2	4.022890609	4.49262	4.97117	0.5	0.53335	4.95841
	3	5.6821493	6.02385	6.69629	0.75	0.76245	6.69108
					1	0.9624	
2.5	1	2.13613824	2.92628	3.49645	0.25	0.30158	3.44258
	2	3.74592575264	4.24641	5.14076	0.5	0.57939	5.12412
	3	5.2971148	5.6621	6.88963	0.75	0.82257	6.88274
					1	1.03169	
2.75	1	2.035690765	2.85378	3.67593	0.25	0.32476	3.61283
	2	3.5833404016	4.1037	5.35825	0.5	0.62042	5.33828
	3	5.07120215	5.45134	7.15649	0.75	0.87535	7.14814
					1	1.09179	
3	1	1.898243362	2.75741	3.79231	0.25	0.34657	3.71674
	2	3.3606956996	3.91079	5.45556	0.5	0.65847	5.43076
	3	4.768615348	5.17104	7.25628	0.75	0.92362	7.24579
					1	1.14621	

Note that the trend of curves $k_j(\eta)$ in Fig. 3 confirms the known provision that as the system's rigidity increases, its natural oscillation frequencies should increase.

The shapes of natural oscillations are built using expression (15) after it is supplemented with the values of p_j, q_j and expressions for A_i^*, B_i^*, according to (16). As a result, (15) can be rewritten as

$$W_j = \frac{A_{1j}^*}{\cosh mt}\left\{\begin{aligned} &\sin\left(p_j(t+t_1)\right) - \left(\frac{B_1^*}{A_1^*}\right)_j \cos\left(p_j(t+t_1)\right) + \\ &+\frac{p_j}{q_j}\left[\begin{aligned} &\cosh\left(q_j(t+t_1)\right) - \\ &-\sinh\left(q_j(t+t_1)\right) \end{aligned}\right] \end{aligned}\right\}, \quad (19)$$

where

$$\left(\frac{B_1^*}{A_1^*}\right)_j = \frac{\sin\left(2t_1 p_j\right) - \frac{p_j}{q_j}\sin\left(2t_1 q_j\right)}{\cos\left(2t_1 p_j\right) - \cosh\left(2t_1 q_j\right)}. \qquad (20)$$

After replacing t with x using a ratio in (8) t=arcsinhax, expression (19) takes the form that is convenient for the graphic representation of oscillation shapes W_j (k_j) in the interval $-1 \leq x \leq 1$. The approximate trendline of charts W_j (j=1, 2, 3) for η=1.01; 2; 3 using data in Table 3 is shown in Fig. 4.

The η=1.01 parameter is little different from η=1 so that when one proceeds from η=1 1 to η=1.01, the natural frequencies (Table 3) also differ little. Given this, in order to construct and compare the shapes of oscillations built using a single expression (19), the initial variant η=1.01 was chosen instead of natural η=1.

It is common knowledge that for η=1 (a beam of constant cross-section) the shapes of oscillations can be built in the interval $-1 \leq x \leq 1$ using the following expression

$$W_{j=1,2,3,...} = \\ = A\left\{\begin{array}{l}\sin\left(k_j(x+1)\right) - \sinh\left(k_j(x+1)\right) - \\ -\dfrac{\sin 2k_j - \sinh 2k_j}{\cos 2k_j - \cosh 2k_j}\left[\begin{array}{l}\cos\left(k_j(x+1)\right) - \\ -\cosh\left(k_j(x+1)\right)\end{array}\right]\end{array}\right\}. \qquad (21)$$

Table 4 gives data for the quantitative analysis of the shapes of oscillations shown in Fig. 4.

Table 4

Coefficients for building oscillation shapes

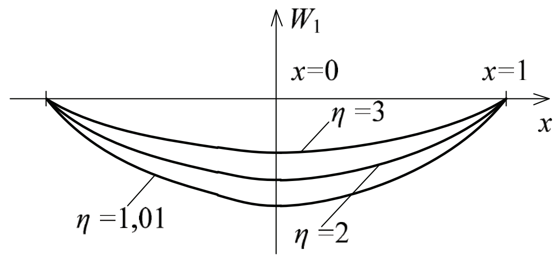
η	j	p_j/q_j	$\left(B_1^*/A_1^*\right)_j$
1.01	1	0.99645	1.01397
	2	0.99871	0.99794
	3	0.99934	0.99937
2	1	0.78587	0.79114
	2	0.91045	0.91008
	3	0.95185	0.95187
3	1	0.68841	0.6909
	2	0.85934	0.85912
	3	0.92218	0.92219

Note that in Table 4 the p_j/q_j values are calculated according to the data in Table 3, and the $\left(B_1^*/A_1^*\right)_j$ values – from formula (20).

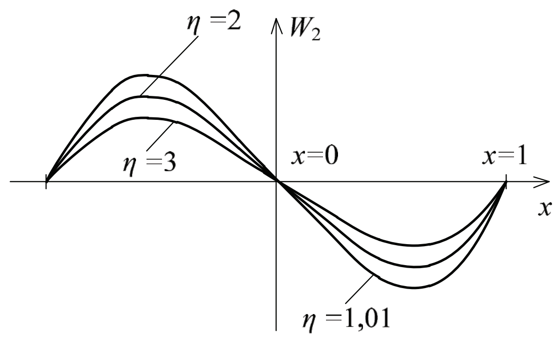
As an example related to the quantitative analysis, we give the results from calculating the deflections $W_{j=1}$ at x=0 (t=0) for η given in Table 4. The following expression

$$W_1 = \sin p_1 t_1 - \left(\frac{B_1^*}{A_1^*}\right)_1 \cos p_1 t_1 + \\ + \frac{p_1}{q_1}\left(\cosh q_1 t_1 - \sinh q_1 t_1\right),$$

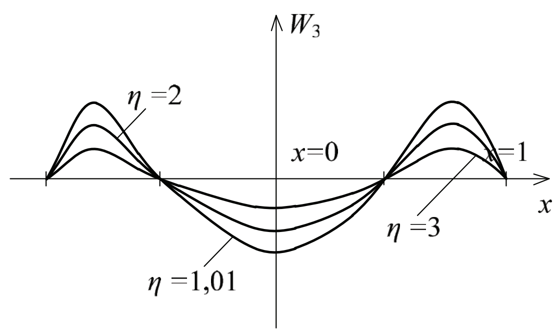
which follows from (19) at x=0; j=1; $A_1^* = 1$ produces, for η=1,01;2;3, W_1=1.517; 1.32;1.244, respectively. By attaching W_1=1.616 to this sequence as a result of the calculation, according to (21), of a deflection in the center of the beam (x=0), we shall obtain the dependence of maximum deflections W_1 on the η parameter, which characterizes the rigidity of the beam. This dependence leads to the obvious conclusion that the deflections decrease with an increase in the rigidity η (an increase a).



a



b



c

Fig. 4. The first three shapes of the beam's natural oscillations for three values of the thickness parameter η: *a* – first shape; *b* – second shape; *c* – third shape

The problem considered could form a basis for the rational design of highly loaded beam elements of the structure, for which the nature of the distribution of cyclical operational stresses is very important. It is obvious that, by changing the η parameter, one could achieve the desired change in the stressed state of the beam – for example, to reduce normal stresses in the fixation and increase them in the conditioned cross-section, far from the place of anchoring.

6. Discussion of results of exploring the problem about the oscillations of a beam

Based on the concept of a factorization method (FM), it can be concluded that the representation of the fourth-order equation in form (3) requires the presence of a constant coefficient in the form $\pm k^4$ in the function W_i. If we follow this rule, we shall obtain the FM implementation criterion in the form of condition $H''=0$. This leads to the simplest and only case $H=bx+c$, which was used by Kirchhoff in a known problem about wedge oscillations. Our modernization of FM implies the introduction of an arbitrary constant in the coefficient in W_i, which is denoted in (3) as $\pm(D^4)^{1/2}=\pm D^2=$ $=\pm(k^4+\chi^2)^{1/2}=\pm(k^4+(2a^2)^2)^{1/2}$. As a result, the FM implementation criterion rank is increased and takes the form of $H'''=0$. Hence the configuration of the beam, defined by the law $H=a^2x^2+1$, considered in this paper. It is also obvious that it follows from this criterion that there is a more general expression $H=a^2x^2+bx+c$, which implements FM. In the latter case, the choice of constants a, b, c would significantly diversify the types of beams for which the problem about oscillations could be solved analytically. The limitation of a factoring method is due to its constraints dictated by the aforementioned criterion $H'''=0$.

The method of symmetry, as shown in the current work, makes it possible to solve problems about elastic body oscillations correctly enough, even when the solutions to equations obtained on the basis of FM are not known.

The caveats of our study relate to the restriction of the parameter $\eta\leq 3$, as the approximation function F_1 is set within a limited interval of $t=0\div(\pm 1.1)$. Increasing the approximation interval with the selected F_1 leads to a margin of error that could be reduced by selecting a different function as F_1, using the symmetry method scheme outlined in this work.

The example of solving the problem about natural oscillations of the parabolic beam fixed on the end sections illustrates the effectiveness of the methods used in their combination. In a very similar way, the oscillations of the beams of this group could be studied, for example, the case of a convex beam whose thickness is $H=1-a^2x^2$ at different ways of fixing it, not excluding the case of a completely free beam.

7. Conclusions

1. The original fourth-order equation with variable coefficients has been decomposed by using the factorization method into two self-adjoint second-order equations. This decomposition has made it possible to replace the immediate search for a solution to the more complex fourth-order equation with the search for a solution to the simpler self-adjoint equation of the second order. It has been shown that when the beam thickness is $H=a^2x^2+1$, where a is the coefficient that determines the degree of concavity, such a decomposition is easy enough.

2. In order to find a solution to the resulting second-order equations, the symmetry method and the scheme of its application were used to find an approximating function, with which we derived a precise solution to these equations. In this case, the function $F=e^{2t}$ is approximated by the function $F_1=\cosh^2 mt$, where $t=\ln(ax+(a^2x^2+1)^{0,5})$; m is the constant, established during approximation.

3. The problem about the oscillations of a symmetrical concave beam, rigidly fixed at both ends, has been solved. The frequency equation has been derived; the frequency numbers k_j ($j=1\div 3$) have been found; graphic illustrations have been constructed for k_j and a series of oscillation shapes for $n=1\div 3$ depending on the beam's degree of concavity determined by the thickness parameter $\eta=H(1)/H(0)$. It has been shown that as η increases, the natural frequencies (k_j numbers) increase regardless of the frequency number j. For example, for $\eta=1;2;3$, we received, respectively, $k_1=2.365$; 3.137; 3.716; $k_2=3.929$; 4.783; 5.4307; $k_3=5.498$; 6.488; 7.245. There is a real possibility to directly apply the proposed approaches and the algorithm in general to study the oscillations of other beams of the parabolic group, including different ways of fixing them, or free.

References

1. Saurin, V. V. (2019). Analysis of Dynamic Behavior of Beams with Variable Cross-section. Lobachevskii Journal of Mathematics, 40 (3), 364–374. doi: https://doi.org/10.1134/s1995080219030168
2. Safronov, V. S., Antipov, A. V. (2020). Evaluation of dynamic qualities of a metal road bridge according to data of natural tests and test calculations. Stroitel'naya mekhanika i konstruktsii, 1 (24), 39–53. Available at: https://cchgeu.ru/science/nauchnye-izdaniya/stroitelnaya-mekhanika-i-konstruktsii/spisok-vypuskov/24.pdf
3. Kaynardag, K., Battaglia, G., Ebrahimkhanlou, A., Pirrotta, A., Salamone, S. (2020). Identification of Bending Modes of Vibration in Rails by a Laser Doppler Vibrometer on a Moving Platform. Experimental Techniques, 45 (1), 13–24. doi: https://doi.org/10.1007/s40799-020-00401-9
4. Mahnenko, O. V., Saprykina, G. Yu., Mirzov, I. V., Pustovoy, A. D. (2014). Prospects for manufacture of welded structures of load-carrying elements of freight wagon carriage. Avtomaticheskaya svarka, 3 (730), 36–42. Available at: http://dspace.nbuv.gov.ua/handle/123456789/103390
5. Cao, D., Gao, Y. (2018). Free vibration of non-uniform axially functionally graded beams using the asymptotic development method. Applied Mathematics and Mechanics, 40 (1), 85–96. doi: https://doi.org/10.1007/s10483-019-2402-9
6. Cao, D. X., Wang, J. J., Gao, Y. H., Zhang, W. (2019). Free Vibration of Variable Width Beam: Asymptotic Analysis with FEM Simulation and Experiment Confirmation. Journal of Vibration Engineering & Technologies, 7 (3), 235–240. doi: https://doi.org/10.1007/s42417-019-00116-1
7. Mylenko, A., Rozhkovsksaya, E., Garanenko, T. (2014). Increasing endurance limit of gas-turbine engines wide chord fan in ultrasonic field. Vestnik dvigatelestroeniya, 1, 67–70. Available at: http://vd.zntu.edu.ua/article/view/97952/93278
8. Sayyad, A. S., Ghugal, Y. M. (2018). Analytical solutions for bending, buckling, and vibration analyses of exponential functionally graded higher order beams. Asian Journal of Civil Engineering, 19 (5), 607–623. doi: https://doi.org/10.1007/s42107-018-0046-z
9. Pham, P.-T., Hong, K.-S. (2020). Dynamic models of axially moving systems: A review. Nonlinear Dynamics, 100 (1), 315–349. doi: https://doi.org/10.1007/s11071-020-05491-z
10. Adikov, S. G. (2006) Design features of ultrasonic sawbow. Derevoobrabotka: tekhnologii, oborudovanie, menedzhment XXI veka: trudy mezhdunarodnogo evraziyskogo simpoziuma. Ekaterinburg, 125–132. Available at: https://elar.usfeu.ru/bitstream/123456789/4213/1/Adikov.pdf

11. Tomas, K. I., Il'yaschenko, D. P. (2011). Tekhnologiya svarochnogo proizvodstva. Tomsk: Izd-vo Tomskogo politekhnicheskogo universiteta, 247.
12. Tarasov, S. V., Mischanin, L. V. (2008). Metodika rascheta sobstvennyh chastot uprugoy lopasti vertikal'no-osevoy vetroenergeticheskoy ustanovki. Vestnik Dnepropetrovskogo universiteta. Seriya: "Fizika. Radioelektronika", 16 (15 (2)), 179–183. Available at: http://www.vdnu.narod.ru/v15t2/pdf/s28_15t2.pdf
13. Korobenko, A., Bazilevs, Y., Takizawa, K., Tezduyar, T. E. (2018). Computer Modeling of Wind Turbines: 1. ALE-VMS and ST-VMS Aerodynamic and FSI Analysis. Archives of Computational Methods in Engineering, 26 (4), 1059–1099. doi: https://doi.org/10.1007/s11831-018-9292-1
14. Sazonov, I. A. (1978). Kontsentratory izgibnyh voln. Akusticheskiy zhurnal, 24 (6), 925–931. Available at: http://www.akzh.ru/pdf/1978_6_925-931.pdf
15. Trapezon, A. G. (1983). Raschet uprugih elementov pri rezonansnyh ustalostnyh ispytaniyah. Kyiv: Naukova dumka, 96.
16. Ghazaryan, D., Burlayenko, V. N., Avetisyan, A., Bhaskar, A. (2017). Free vibration analysis of functionally graded beams with non-uniform cross-section using the differential transform method. Journal of Engineering Mathematics, 110 (1), 97–121. doi: https://doi.org/10.1007/s10665-017-9937-3
17. Moreno-García, P., dos Santos, J. V. A., Lopes, H. (2017). A Review and Study on Ritz Method Admissible Functions with Emphasis on Buckling and Free Vibration of Isotropic and Anisotropic Beams and Plates. Archives of Computational Methods in Engineering, 25 (3), 785–815. doi: https://doi.org/10.1007/s11831-017-9214-7
18. Li, Z., Xu, Y., Huang, D., Zhao, Y. (2019). Two-dimensional elasticity solution for free vibration of simple-supported beams with arbitrarily and continuously varying thickness. Archive of Applied Mechanics, 90 (2), 275–289. doi: https://doi.org/10.1007/s00419-019-01608-y
19. Pavliuchenkov, M. V. (2014). Structure rationalization of tank cars support devices for fluids. Science and Transport Progress. Bulletin of Dnipropetrovsk National University of Railway Transport, 1 (49), 151–159. doi: https://doi.org/10.15802/stp2014/22681
20. Gusev, B. V., Saurin, V. V. (2019). On Free Bending Vibrations of Concrete Beams With Variable Cross Section. Industrial and Civil Engineering, 8, 93–98. doi: https://doi.org/10.33622/0869-7019.2019.08.93-98
21. Tymoshenko, S. (1928). Vibration problems in engineering. New York: D. Van Nostrand Co Inc, Constable and Co., 351.
22. Trapezon, K. A. (2006). The symmetry method in calculating and designing of acoustic thickeners. Akusticheskiy vestnik, 9 (4), 50–55. Available at: http://hydromech.org.ua/content/pdf/av/av-09-4(50-55).pdf
23. Chang, P., Zhao, X. (2020). Exact solution of vibrations of beams with arbitrary translational supports using shape function method. Asian Journal of Civil Engineering, 21 (7), 1269–1286. doi: https://doi.org/10.1007/s42107-020-00275-7