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INTEGRATING LINEAR ORDINARY FOURTH-ORDER DIFFERENTIAL EQUATIONS IN THE MAPLE PROGRAMMING ENVIRONMENT

Irina Belyaeva
PhD, Associate Professor
Department of Computer Science,
Natural Sciences and Teaching Methods
Belgorod State University
Pobedy str., 85, Belgorod, Russian Federation, 308015
Igor Kirichenko
Corresponding author
Doctor of Physic-Mathematical Sciences, Professor*
E-mail: ikir238@gmail.com
Oleh Ptashnyi
PhD, Associate Professor*
Natalia Chekanova
PhD, Associate Professor*
Tetiana Yarkho
Doctor of Pedagogical Sciences, Professor*
*Department of Higher Mathematics
Kharkiv National Automobile and Highway University
Yaroslava Mudroho str., 25, Kharkiv, Ukraine, 61002

This paper reports a method to solve ordinary fourth-order differential equations in the form of ordinary power series and, for the case of regular special points, in the form of generalized power series. An algorithm has been constructed and a program has been developed in the MAPLE environment (Waterloo, Ontario, Canada) in order to solve the fourth-order differential equations. All types of solutions depending on the roots of the governing equation have been considered. The examples of solutions to the fourth-order differential equations are given; they have been compared with the results available in the literature that demonstrate excellent agreement with the calculations reported here, which confirms the effectiveness of the developed programs. A special feature of this work is that the accuracy of the results is controlled by the number of terms in the power series and the number of symbols (up to 20) in decimal mantissa in numerical calculations. Therefore, almost any accuracy allowed for a given electronic computing machine or computer is achievable. The proposed symbolic-numerical method and the work program could be successfully used for solving eigenvalue problems, in which controlled accuracy is very important as the eigenfunctions are extremely (exponentially) sensitive to the accuracy of eigenvalues found. The developed algorithm could be implemented in other known computer algebra packages such as REDUCE (Santa Monica, CA), MATHEMATICA (USA), MAXIMA (USA), and others. The program for solving ordinary fourth-order differential equations could be used to construct Green's functions of boundary problems, to solve differential equations with private derivatives, a system of Hamilton's differential equations, and other problems related to mathematical physics

Keywords: computer simulation, ordinary fourth-order differential equations, generalized power series, regular special points

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1. Introduction

Most differential equations and systems of differential equations found in mathematics, physics, and other natural sciences are not integrated in an explicit form [1]. Typically, there are no universal numerical methods for finding solutions to them [2, 3]. These equations include, specifically, Schrodinger's stationary equation (when solving its eigenvalue problem), the system of differential equations of Hamilton.

An important role for the mathematical models in different fields of physics, in particular, in classical and quantum mechanics, in applied mathematics, in technology belongs to eigenvalue problems [4]. These problems arise when investigating stability, as well as the critical and bifurcation regimes of complex physical systems [5].

It should be noted that the relevance of the theory of solving the fourth-order differential equations is due to that there are a number of physical states whose consideration involves the fourth-order differential equations. Those are the plate bending theory, elasticity theory, torsion theory of anisotropic rods, shell theory, and so on. These physical states were analyzed in work [1]. The relevance of solving, specifically, ordinary fourth-order differential equations was discussed in detail in paper [6]. The relevance of solving ordinary differential equations of any order, the relevance of their research methods, as well as the area of their applicability in various fields of technical sciences were discussed in detail in work [7]. The relevance of exploring ordinary differential equations is confirmed by the fact that every year there are international conferences on differential equations in the world.

2. Literature review and problem statement

Ordinary differential equations, both linear and non-linear, are widely used in various fields of science and technology.

However, most equations do not produce solutions in an explicit analytical way. For example, the solutions to such simple-looking equations as $y'(x)=y^2-x$, $y'(x)=x^2+y^2$ are not expressed as the ultimate combination of elementary and algebraic functions and their integrals. In addition, the cumbersome formula that yields a solution in the explicit form is often not useful at all as it does not provide a constructive technique to derive a solution, which was very well analyzed in works [1, 6, 7]. Therefore, in the theory of differential equations, various approximate analytical methods have been developed and advanced, in particular the methods of perturbation theory [8, 9], averaging methods, and direct numerical methods [2, 3, 8–11] to solve them.

As regards direct numerical methods for solving ordinary differential equations, there are a large number of machine programs that reliably find solutions to such equations with great accuracy, and most of these programs employ an algorithmic language.

In recent decades, the so-called hybrid methods involving different computer algebra systems have shown promises for modern areas related to solving problems in mathematical physics. When the preliminary algebraic (analytical) transformations are combined with the derivation of basic formulae in the form of explicit expressions, subsequent numerical calculations are performed if necessary.

An important class of equations for which the algorithmic solution procedure has been developed is composed of linear homogeneous differential equations of the second order with rational coefficients-functions. Such an algorithm resembles an algorithm for calculating uncertain integrals with a solution built by using rational operations, algebraic functions, defined by the polynomial equations of demonstrative functions and integrals. It should be noted that until now no algorithmic procedure for solving differential equations above the second order has been developed.

Solutions to the differential equation by using computer algebra systems were also analyzed in work [12]. An Euler-type equation was investigated and several algorithms were developed to obtain approximate solutions to such equations, with an assessment of the accuracy of approximate solutions. It is worth noting that Euler's equations, through a known simple replacement, are reduced to linear differential equations with constant coefficients and make it possible to derive a general solution in an analytical form.

Study [12] presented two algorithms for solving ordinary linear differential equations of the second order. One algorithm converts the assigned differential equation into an equation with constant coefficients, and the second algorithm represents the original differential equation in the form of a product of two simple differential expressions (a factorization method).

Paper [13] also reported an algorithm for solving linear differential equations of arbitrary orders with polynomial coefficients (that is, differential equations of a certain class), for which the task of constructing all rational solutions is solved.

Ordinary differential equations arise to find solutions to differential equations with partial derivatives when using a Fourier method. In this case, the so-called linearization procedure is often applied, first considered in work [14], which leads to ordinary differential equations.

Subsequent studies [15–20] developed the above heuristic methods for exploring differential equations by using computer algebra. Algebraic substitutions are employed to solve a differential equation in [15]. Functional substitutions are used to solve a differential equation and the original differential equations of a special kind are considered in [16]. Cylindrical coordinates are applied to solve a differential equation by using algebraic substitutions in [17].

The authors of work [18] developed an algorithm for constructing asymptotic approximations (of arbitrary order) by building the asymptotes of Lagrange multipliers in the form of expansion into the integer powers of a small parameter. The computational procedure of the algorithm involves solving the linear problem of optimal management, integrating the systems of linear differential equations, as well as finding solutions to nondegenerate linear algebraic systems.

Paper [19] proposed methods for solving an external problem for the Laplace equation. The primary method is to set an artificial integral boundary condition with iterative refinement. It was shown that the iterative methods converge at the speed of geometric progression. The applicability of the methods for solving external problems was confirmed by computational experiments for the two-dimensional and three-dimensional cases.

The seven-stage Runge-Kutta methods of the sixth-order order were used to solve a differential equation in [21, 22].

In work [23], the Runge-Kutta method is used to examine the solutions to the initial problems for the ordinary fourth-order differential equations.

In works [21–23], when solving differential equations, the first attempts were made to represent solutions analytically in the form of power polynomials based on the numerical solution to a differential equation. Particularly noteworthy is paper [21] in which a three-stage method, which is designated as RKFD of the fourth order, and the three-stage RKFD method of the fifth order were built on the basis of the conditions of order. The authors reported numerical results to demonstrate the effectiveness of the new RKFD methods by comparing them with other methods described in the scientific literature, such as Runge-Kutta-Nystrom (RKN) and Runge-Kutta (RK), after converting the problem into a system of ordinary linear differentials of the second order.

In work [20], numerical calculations were performed using the method of targeting in solving the problems of optimal control with switching. An algorithm has been worked out to ensure that Newton's method is converged in problems of this kind. An analysis of the accuracy of the calculations has been carried out.

It should be noted that in all recent works [24, 25] the solutions to differential equations by approximate numerical methods were obtained without their specific analytical representation; paper [25] even proposed an algorithm for numerical solving of ordinary differential equations of the highest order.

Our review reveals that one of the effective and constructive methods for solving linear ordinary differential equations is to derive them in the form of power series [26]. There may be cases where function coefficients have singularities at some points (so-called special points) whose presence sometimes leads to features in the solutions themselves [1].

3. The aim and objectives of the study

The aim of this work is to find a solution to the linear ordinary fourth-order differential equations in an analytical form.

To accomplish the aim, the following tasks have been set:

– to construct a method for solving a linear ordinary differential equation of the fourth order in the form of a power series;

– to consider all types of solutions to the fourth-order differential equations depending on the roots of the governing equation;

– to develop a computer program algorithm in the MAPLE environment to solve the fourth-order differential equations;

– to provide examples of the solutions that would make it possible to evaluate the possibilities of the method for the analytical representation of solutions to the differential equation in the idea of power series.

4. The study materials and methods

The starting material of this study is the power series. Underlying the construction of a given method for solving a differential equation is the representation of the functional dependence of the function desired from the equation in the form of a power series. That would make it possible to find a solution to the differential equation analytically. It should be noted, however, that no solution has yet been obtained for the linear ordinary differential equations in an analytical form. By developing a software algorithm in the MAPLE analytical computing environment, one could obtain the result of computation depending on any number of terms in the power series. That makes it possible to obtain a solution [26] and be able to control it with any accuracy of computation in numerical calculations [27, 28]. The accuracy of the calculations would depend only on a given electronic computing machine.

It should be noted that there may be cases where function coefficients have singularities at some points (so-called special points) whose presence sometimes leads to features in the solutions themselves [1]. In the presence of special points, a given method employs generalized power series. However, each particular differential equation has to be theoretically investigated.

5. The results of studying the integration of the linear fourth-order differential equations

5. 1. A general scheme of the method for solving a linear differential equation of the fourth order with the help of power series

We shall describe the basic provisions of the theory for finding linearly independent solutions to a linear ordinary differential equation of the fourth order, which takes the following form:

$$y^{(IV)}(x)+P_3(x)\cdot y'''(x)+P_2(x)\cdot y''(x)+$$
$$+P_1(x)y'(x)+P_0(x)\cdot y(x)=0. \quad (1)$$

If the functions-coefficients $P_3(x)$, $P_2(x)$, $P_1(x)$, $P_0(x)$ do not contain regular special points and are the holomorphic functions in the vicinity of point $x=x_0$, that is, represented in the form of converging series:

$$P_0(x)=\sum_{k=0}^{\infty} p_k^{(0)}(x-x_0)^k,$$

$$P_1(x)=\sum_{k=0}^{\infty} p_k^{(1)}(x-x_0)^k,$$

$$P_2(x)=\sum_{k=0}^{\infty} p_k^{(2)}(x-x_0)^k,$$

$$P_3(x)=\sum_{k=0}^{\infty} p_k^{(3)}(x-x_0)^k,$$

then four linearly independent solutions y_1, y_2, y_3 and y_4 can be represented by the following power series:

$$y_1(x)=1+\sum_{k=2}^{\infty} c_k^{(1)}(x-x_0)^k,$$

$$y_2(x)=(x-x_0)+\sum_{k=2}^{\infty} c_k^{(2)}(x-x_0)^k,$$

$$y_3(x)=(x-x_0)^2/2+\sum_{k=2}^{\infty} c_k^{(3)}(x-x_0)^k,$$

$$y_4(x)=(x-x_0)^3/3+\sum_{k=2}^{\infty} c_k^{(4)}(x-x_0)^k. \quad (2)$$

The coefficients $c_k^{(1)}$, $c_k^{(2)}$, $c_k^{(3)}$, $c_k^{(4)}$ are determined in the only way by the substitution of series (2) into equation (1) and by equating to zero the coefficients at different powers of an independent variable in the left-hand part of the derived equality (the method of uncertain coefficients).

If there are poles no higher than the fourth order at point $x=x_0$, the form of solutions to (2) would be different depending on the roots of the governing equation [1]. It is known from the theory of ordinary differential equations [1] that in order for the equation in form (1) to have at least one particular solution in the vicinity of the special point $x=x_0$ in the form of a generalized power series, for example:

$$y_4(x)=(x-x_0)^{\rho}\sum_{k=2}^{\infty} c_k(x-x_0)^k,\ (c_0\neq 0), \quad (3)$$

where ρ is some constant number, it is required that this equation should take the following form:

$$y^{(IV)}(x)+\frac{\sum_{k=0}^{\infty} p_k^{(3)}}{(x-x_0)}y^{(III)}(x)+\frac{\sum_{k=0}^{\infty} p_k^{(2)}}{(x-x_0)^2}y^{(II)}(x)+$$
$$+\frac{\sum_{k=0}^{\infty} p_k^{(1)}}{(x-x_0)^3}y'(x)+\frac{\sum_{k=0}^{\infty} p_k^{(0)}}{(x-x_0)^4}y(x)=0, \quad (4)$$

that is, its pole should not exceed the fourth order. The indicator ρ is determined from the so-called governing equation:

$$\rho(\rho-2)(\rho-3)+\rho(\rho-1)(\rho-2)p_0^{(3)}+$$
$$+\rho(\rho-1)p_0^{(2)}+\rho p_0^{(1)}+p_0^{(0)}=0. \quad (5)$$

Let ρ_1, ρ_2, ρ_3, and ρ_4 be the roots to equation (5). Then, if the roots of the governing equation ρ_1, ρ_2, ρ_3, and ρ_4 are different, and no two of them differ by an integer number, then each number ρ corresponds to a certain sequence of coefficients $c_k^{(\alpha)}$, $\alpha=1, 2, 3, 4$, and there are four independent solutions that form the fundamental system:

$$y_1(x)=(x-x_0)^{\rho_1}\sum_{k=0}^{\infty} c_k^{(1)}(-x_0)^k,\ (c_0^{(1)}\neq 0),$$

$$y_2(x)=(x-x_0)^{\rho_2}\sum_{k=0}^{\infty} c_k^{(2)}(-x_0)^k,\ (c_0^{(2)}\neq 0),$$

$$y_3(x)=(x-x_0)^{\rho_3}\sum_{k=0}^{\infty} c_k^{(3)}(-x_0)^k,\ (c_0^{(3)}\neq 0),$$

$$y_4(x)=(x-x_0)^{\rho_4}\sum_{k=0}^{\infty} c_k^{(4)}(-x_0)^k,\ (c_0^{(4)}\neq 0). \quad (6)$$

All coefficients $c_k^{(\alpha)}$, $\alpha = 1, 2, 3, 4$ are found by the substitution of series (6) into a differential equation (4) and by equating the coefficients at the same power of variable x (the method of uncertain coefficients). Coefficients $c_k^{(\alpha)}$, $\alpha = 1, 2, 3, 4$ are assumed to equal unity.

5. 2. Solutions to the fourth-order differential equations depending on the roots of the governing equation

If of the four ρ values found, two or several roots differ by an integer number, they can be arranged as independent sequences:

$$\rho_1,\ \rho_2, \ldots, \rho_{\alpha-1}, \rho_\alpha, \rho_{\alpha+1}, \ldots, \rho_{\beta-1},$$

in this case, the values in each sequence differed only by integer numbers, and their physical parts would form a non-ascending sequence. Then only the first term of each sequence gives a solution in the form (3).

If the difference $\rho_1 - \rho_2$ of any roots is equal to an integer positive number or zero, then one of the solutions takes the following form:

$$y_1(x) = (x - x_0)^{\rho_1},$$

and is found according to (6), but the second solution takes the following form [1]:

$$y_2(x) = A \cdot y_1(x) \cdot \ln(x - x_0) + (x - x_0)^{\rho_2} \sum_{k=0}^{\infty} c_k (x - x_0)^k. \quad (7)$$

A linearly independent solution $y_2(x)$ can be found either from a known Liouville-Ostrogradsky formula or from expression (7) by the method of uncertain coefficients. At the same time, for the case of $\rho_1 - \rho_2 = 0$, the logarithmic term would be contained ($A = 0$) in the solution, and if the difference $\rho_1 - \rho_2$ is a positive integer, the logarithmic term may be missing.

A general solution to differential equation (1) is determined from the following expression:

$$y(x) = C_1 \cdot y_1(x) + C_2 \cdot y_2(x) + C_3 \cdot y_3(x) + C_4 \cdot y_4(x). \quad (8)$$

According to the above formulae (6), a program to calculate four linearly independent solutions to a linear ordinary differential equation of the fourth order (1) and its general solution (8) has been developed, using the computer system of algebraic computation MAPLE: the algorithm is represented below.

5. 3. An algorithm for finding solutions to a linear ordinary differential equation of the fourth order in an analytical form

Assume $P_k(x)$, where $k = 0, 1, 2, 3$ are the coefficients-functions of the assigned differential equation (1); k is the maximum degree of power of the power series used; $x_0 0$ is the special point in an equation, if any.

The algorithm steps are as follows:

1. Introduce four coefficients-functions $P_3(x)$, $P_2(x)$, $P_1(x)$, $P_0(x)$, which determine the differential equation itself, as well as the desired maximum order of the power series n.

2. Designate the *potencial* parameter-flag:

– if *potencial*=1, then the coefficients-functions of equation (1) do not contain features at point x_0;

– if *potencial*=0, then the equation has a pole no higher than the fourth order at point x_0.

3. Find four linearly independent solutions y_1, y_2, y_3, and y_4 from formulae (3) if the coefficients-functions $P_3(x)$, $P_2(x)$, $P_1(x)$, $P_0(x)$ do not contain regular special points and are the holomorphic functions in the vicinity of point $x = x_0$.

4. If there are poles not higher than the fourth, the equation takes the form (4).

5. Find the roots ρ_1, ρ_2, ρ_3 and ρ_4 of governing equation (5).

6. If the roots of the governing equation ρ_1, ρ_2, ρ_3, and ρ_4 are different, and no two of them differ by an integer or zero, then the coefficients $c_k^{(\alpha)}$, $\alpha = 1, 2, 3, 4$ are determined by the substitution of series (6) into equation (1).

7. If the four ρ values found are such that two or more differ by an integer, the linearly independent solutions are found from formula (7).

8. Find a general solution to equation (1) according to expression (7).

5. 4. Examples of finding solutions to the linear fourth-order differential equations in the form of power series

The method built and the algorithm developed allow us to find solutions to the fourth-order differential equations in the form of power series, in general, of an arbitrary power n, limited, however, by the capabilities of a particular computer. A given program was used to derive solutions to some differential equations.

Example 1.

The following equation:

$$y^{(IV)} + 2y''' + 10y'' + 18y' + 9y = 0, \quad (9)$$

is a homogeneous linear differential equation with constant coefficients. Its linearly independent solutions are equal to:

$$y_1(x) = e^{-x},\ y_2(x) = x \cdot e^{-x},$$
$$y_3(x) = \sin 3x,\ y_4(x) = \cos 3x, \quad (10)$$

because its characteristic equation:

$$\lambda^4 + 2\lambda^3 + 10\lambda^2 + 18\lambda + 9 = 0,$$

has the roots $\lambda_k = \{\pm 3i, -1, -1\}$.

Equation (9) has no special points. The following four linearly independent solutions to this equation were obtained using the developed program:

$$y_1(x) = 1 - \frac{3}{8}x^4 + \frac{3}{20}x^5 + \frac{3}{40}x^6 - $$
$$-\frac{1}{40}x^7 - \frac{59}{4480}x^8 + \frac{11}{3360}x^9 + \ldots,$$

$$y_2(x) = x - \frac{3}{4}x^4 + \frac{9}{40}x^5 + \frac{7}{40}x^6 -$$
$$-\frac{11}{280}x^7 - \frac{33}{1120}x^8 + \frac{41}{8064}x^9 + \ldots,$$

$$y_3(x) = \frac{1}{2}x^2 - \frac{5}{12}x^4 + \frac{1}{60}x^5 + \frac{29}{240}x^6 -$$
$$-\frac{1}{360}x^7 - \frac{197}{10080}x^8 + \frac{11}{30240}x^9 + \ldots,$$

$$y_4(x) = \frac{1}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{10}x^5 + \frac{7}{180}x^6 +$$
$$+\frac{59}{2520}x^7 - \frac{11}{1680}x^8 + \frac{131}{45360}x^9 + \ldots \quad . \quad (11)$$

The resulting solutions (11) are the linear combinations of precise solutions (10).

Example 2.

Solve an equation in which the curved axis of the beam $y(x)$, loaded by a continuously distributed reaction from the base with an intensity equal to $k \cdot y(x)$ (for simplicity, taken to be $k=4$), is determined by the following differential equation of the fourth order:

$$y^{(IV)}(x)+4y(x)=0.$$

In this equation, the coefficients-functions, equal to the function $P_3(x)=0$, $P_2(x)=0$, $P_1(x)=0$, $P_0(x)=4$, do not have special points. This equation also has no special points and also is a homogeneous linear equation with constant coefficients, so it has precise analytical solutions:

$$y_1(x)=e^x\sin x,\ \ y_2(x)=e^x\cos x,$$

$$y_3(x)=e^{-x}\sin x,\ \ y_4(x)=e^{-x}\cos x,$$

because its characteristic equation $\lambda^4+4=0$ has the roots $\lambda_k=\{1\pm i,\ -1\pm i\}$. Our program has produced th following four linearly independent solutions:

$$y_1(x)=1-\frac{1}{6}x^4+\frac{1}{2520}x^8-\frac{1}{7484400}x^{12}+$$
$$+\frac{1}{81729648000}x^{16}-\frac{1}{2375880867360000}x^{20},$$

$$y_2(x)=x-\frac{1}{30}x^5+\frac{1}{22680}x^9-$$
$$-\frac{1}{97297200}x^{13}+\frac{1}{1389404016000}x^{17},$$

$$y_3(x)=\frac{1}{2}x^2-\frac{1}{180}x^6+\frac{1}{226800}x^{10}-$$
$$-\frac{1}{1362160800}x^{14}+\frac{1}{2500972288000}x^{18},$$

$$y_4(x)=\frac{1}{3}x^3-\frac{1}{630}x^7+\frac{1}{1247400}x^{11}-$$
$$-\frac{1}{10216206000}x^{15}+\frac{1}{237588086736000}x^{19},$$

that exactly coincide with its analytical solution. The resulting solutions are also the linear combinations of precise solutions $y_k(x)$, $(k=1, 2, 3, 4)$.

Example 3.

Consider the following equation:

$$x^4y^{(IV)}+x^2y''-xy'=0.$$

This equation has a special point $x=x_0=0$, governing equation (5) has equal roots $\rho_1=\rho_2=\rho_3=2$ and a simple root $\rho_4=0$. The following linearly independent solutions have been obtained using the program:

$$y_1(x)=x^2,$$

$$y_2(x)=x^2+x^2\ln x,$$

$$y_3(x)=x^2\ln^2 x+2x^2\ln x+x^2,$$

$$y_4(x)=1.$$

The resulting solutions are precise solutions to a given differential equation.

Example 4.

Consider the following equation:

$$x^4y^{(IV)}+2x^3y'''+x^2y''-xy'+y=0.$$

A special point at $x=x_0=0$. The governing equation has the roots $\rho_1=\rho_2=\rho_3=\rho_4=1$. The following solutions were derived:

$$y_1(x)=x,$$

$$y_2(x)=x\ln x+x,$$

$$y_3(x)=x\ln^2 x+2x\ln x+x,$$

$$y_4(x)=x\ln^3 x+3x\ln^2 x+2x\ln x+x.$$

The resulting solutions are the precise solutions to a given differential equation.

Example 5.

Consider the following equation:

$$xy^{(IV)}+5y'''=0.$$

A special point at $x=x_0=0$. Governing equation (5) has the roots $\rho_1=2$, $\rho_2=1$, $\rho_3=0$, $\rho_4=-2$. Below are the derived linearly independent solutions:

$$y_1(x)=x^2,\ \ y_2(x)=x,\ \ y_3=1,\ \ y_4(x)=x^{-2},$$

that coincide with the solutions reported in work [11].

Example 6.

Consider the following equation:

$$x^2y^{(IV)}(x)+4xy'''(x)+2y''(x)=0.$$

This equation has a special point $x=x_0=0$. In this case, the governing equation has the roots equal to $\rho_1=\rho_2=1$, $\rho_3=\rho_4=0$. Four linearly independent solutions have been obtained using the developed program:

$$y_1(x)=x,$$

$$y_2(x)=x\ln x+x,$$

$$y_3(x)=1,$$

$$y_4(x)=\ln x+1,$$

that are exactly the same as its analytical solution in [11].

6. Discussion of results of solving the linear ordinary fourth-order differential equations

This paper reports the developed method that makes it possible to find solutions to the linear ordinary fourth-order differential equations in the form of ordinary power series. Thus, four independent solutions have been obtained that constitute a fundamental system of solutions (6). In the case of regular special points, the solutions to differential equations are represented in the form of generalized power series. In the latest scientific research, numerical [6, 13, 20] methods, as well as approximate methods [14, 22–24], are used to find solutions to differential equations. A special feature of the method for solving the linear ordinary fourth-order differential equations, proposed in this work, is to obtain a solution analytically in the form of power series. That makes it possible to control the accuracy of the solution by the necessary number of terms in the power series derived.

An algorithm has been constructed taking into consideration the number of roots of the governing equation, which makes it possible to find solutions to the linear ordinary fourth-order differential equations in the form of ordinary power series.

According to the proposed method and the above algorithm, a program has been developed for computing four linearly independent solutions to a linear ordinary differential equation of the fourth order using the computer system of algebraic calculations Maple. A given program eliminates all possible limitations in this area of research and makes it possible, at the numerical representation of solutions to an equation, to use any number of decimals in the solution, depending on the required accuracy of research.

The derived solutions to ordinary linear differential equations make it possible to establish the functional dependence of a solution to nonlinear differential equations and have the widest application potential for solving technical tasks. Depending on the functional type of solution, our results could be used in the theory of fluctuations, stability, the rigidity of technical devices and their structures, etc.

The advantage of a given method for obtaining solutions to ordinary linear differential equations, compared to the currently known numerical and approximate methods, is the accuracy of exploring a particular technical process, which is very important for the modern development of equipment.

In the future, based on the developed algorithm and the devised program, it is planned to build a Green's function for differential equations of the fourth order, which contain regular special points.

In conclusion, we note that the study reported in this paper has been conducted for those linear ordinary fourth-order differential equations that may contain the proper special points.

7. Conclusions

1. A method has been constructed for solving a linear ordinary differential equation of the fourth order in an analytical form. The functional dependence of the solution is represented in the form of power series.

2. We have considered all types of solutions to the fourth-order differential equations, depending on the roots of the governing equation derived by the proposed solving method. The theoretical justification for each solution has been, depending on the heterogeneity of the differential equation.

3. An algorithm of computer software has been developed in the MAPLE environment for solving a linear ordinary differential equation of the fourth order whose functional dependence is represented in the form of a polynomial from power functions. The resulting solution provides for any accuracy of numerical calculations depending on the number of terms of the power series taken in a given solution.

4. The given examples of the found solutions for heterogeneous differential equations depending on the functional form of their right-hand part have shown that in all the cases examined the solutions were in excellent agreement with those reported in the literature.

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