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GENERALIZING THE SAMPLING THEOREM FOR A FREQUENCY-TIME DOMAIN TO SAMPLE SIGNALS UNDER THE CONDITIONS OF A PRIORI UNCERTAINTY

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The radio monitoring of radiation and interference with electronic means is characterized by the issue related to the structural-parametric a priori uncertainty about the type and parameters of the ensemble of signals by radio-emitting sources. Given this, it is a relevant task to devise a technique for the mathematical notation of signals in order to implement their processing, overcoming their a priori uncertainty in terms of form and parameters.

A given problem has been solved by the method of generalization and proof for the finite signals of the Whittaker-Kotelnikov-Shannon sampling theorem (WKS) in the frequency-time domain. The result of proving it is a new discrete frequency-temporal description of an arbitrary finite signal in the form of expansion into a double series on the orthogonal functions such as sinx/x, or rectangular Woodward strobe functions, with an explicit form of the phase-frequency-temporal modulation function. The properties of the sampling theorem in the frequency-time domain have been substantiated. These properties establish that the basis of the frequency-time representation is orthogonal, the accuracy of approximation by the basic functions sinx/x and rectangular Woodward strobe functions are the same, and correspond to the accuracy of the UCS theorem approximation, while the number of reference points of an arbitrary, limited in the width of the spectrum and duration, signal, now taken by frequency and time, is determined by the signal base.

The devised description of signals in the frequency-time domain has been experimentally investigated using the detection-recovery of continuous, simple pulse, and linear-frequency-modulated (LFM) radio signals. The constructive nature of the resulting description has been confirmed, which is important and useful when devising methods, procedures, and algorithms for processing signals under the conditions of structural-parametric a priori uncertainty

Keywords: radio monitoring, a priori uncertainty, sampling theorem, frequency-time domain, signal detection-recovery, Fourier processor

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1. Introduction

The current radio-electronic environment (REE) is characterized by significant a priori uncertainty, which is represented by the following:

– the wide range and bands of work frequencies of the radio-electronic means (REM) radiation;

– a large unfixed ensemble of used signals and types of transmissions;

– readjusting the REM operational modes and the signal-code structure (SCS) parameters in the process of operation;

– the uncertainty of the time of radiation and the short-term REM broadcasting.

The effectiveness of the radio monitoring of such a complex REE and radio suppression of REM can be significantly improved by devising optimal methods for the detection and recovery of signals for the specified conditions of structural-parametric a priori uncertainty. When constructing such methods, absolutely important is to use and implement a technique for the mathematical notation of signals, which could ensure the following:

– to adequately model real processes;

– to overcome the specified a priori uncertainty;

– to optimize the processing procedure;

– to acquire information from measurements not only about the parameters but also about the form (type) of the signal.

Theoretically, any signal with a finite frequency band is unlimited in time and, on the contrary, the signal of finite duration has a spectrum that is unlimitedly long along the frequency axis. In radio engineering, various techniques are used to mathematically describe finite signals in the spectral and temporal domains [1, 2]. Among the mathematical notations of finite signals, their discrete representation in accordance with the Whittaker-Kotelnikov-Shannon (WKS) sampling theorem [3–6] occupies an important place, widely used in radio communications, radar equipment [7, 8], as well as other fields of physics and engineering [9].

However, given the specified a priori uncertainty, using the expansion of signals into a WKS series requires knowledge of either their spectrum width or duration in order to optimize processing. In addition, the WKS series does not explicitly provide information about the evolution of the frequency or phase during the signal needed to determine its form (type). To overcome the structural-parametric a priori uncertainty in terms of the types and parameters of different signals, it is tempting to apply their description in a frequency-time domain. The most appropriate for these conditions could be the frequency-time representation of

Gabor's signals using Gauss functions, Fourier-conjugated, and modulating cosine or sine waves [10, 11]. However, as shown in [12], this expansion has proved inconvenient to use because the Gauss functions are not orthogonal.

In this regard, it is a relevant task to devise, based on the theoretical generalization and by proving the WKS sampling theorem in the frequency-time domain, a new discrete description of signals, which would make it possible to overcome the existing a priori uncertainty during radio monitoring. The current study is a continuation of the research whose results were reported at conferences [13, 14].

2. Literature review and problem statement

The WKS sampling theorem underlies the discrete transformation of signals from the analog form, when detecting them, and from the digital form to analog, when recovering (representing) them.

Since the time of stating, proving, physically interpreting, and justifying the cope of application by the founders of the WKS sampling theorem [3–6], many scientists have explored its various historical, scientific, and practical aspects. The most significant analytical review of research into the sampling of finite signals is work [15]. It reports four groups of studies. First, the essence and detailed analysis of the WKS sampling theorem are presented. Second, its various generalizations are considered: in terms of the *n*-dimensional sampling, sampling for derivatives, sampling of random processes, sampling at uneven counts, sampling of band signals, implicit sampling, and a number of other generalizations. Third, the errors of the discrete representation have been analyzed. Finally, the application of the sampling theorem in physics, optics, image recognition, etc. is described.

Several subsequent review publications tackled further research and the status of the WKS sampling theorem for its certain anniversaries [16–21]. Thus, in work [16,] it is demonstrated how practitioners, theorists, and mathematicians have discovered the value of the sampling theorem almost independently of each other. Article [17] gives a report on the current state of the sampling theorem. The cited article suggests a modern formulation of the sampling theorem in the Gilbert space, with an emphasis on regular sampling, and the interpretation of the sampling procedure as an orthogonal projection onto the subspace of functions limited by zones. It then expands the standard sampling paradigm to represent functions in a more general class of functional spaces, including splines and wavelets. Paper [18] reports some aspects of sampling with a particular emphasis on the band-unlimited signals, point stability of recovery (reproduction), and recovery from heterogeneous samples. Applications in multi-resolution computations and digital spline interpolation are also considered. Stating and proving the sampling theorem at the level of rigor of applied mathematics for the class of ergodic stationary random signals with limited power, widely used in radio electronics, are considered in [19].

Study [20] is a report on the current state of the sampling theorem, focusing on some of the new sampling trends in the first decade of the 21st century. First, the issue of expanding the vector sample is considered. Then the reconstruction of the signal from the local means is shown. And, further, the issue related to the sampling theorem in the wavelet subspaces is investigated with some results represented. The authors analyzed and summarized the application of sample extension and compression for image processing. However, the frequency-temporal sampling under the conditions of a priori uncertainty was not considered. New representations of one-dimensional and two-dimensional generalized Kravchenko-Kotelnikov sampling theorems based on the atomic function up(t) were proposed and substantiated in [21]. The advantage of the new series is shown in comparison with the classic WKS series. However, the issues related to the frequency-time discreteness of signals under the conditions of a priori uncertainty remained unresolved. Apparently, the reason for this lies in the fact that the transition to frequency-time representation is not in demand in the proposed fields. Review [22] describes a mathematical apparatus and a method of generalized consideration of real band signals with the finite (imperfect) steep slope of the spectrum outside its work band in accordance with the WKS sampling theorem. The procedure under consideration implies a clear knowledge of the amplitude-frequency spectrums of processed signals and limits the application of the proposed model under the conditions of structural-parametric a priori uncertainty. Article [23] reports a sampling theorem for subspaces invariant relative to the shift in the mixed Lebesgue spaces $L^{p,q}$ ($\mathbb{R}^{d+1}$). When the sampling density is large enough, this sampling theorem can reconstruct (reproduce) exactly the signals in subspaces that are not dependent on the shift. Despite this, the issues of frequency-time sampling of signals within invariant and non-invariant subspaces with respect to the shift were not considered.

In a general case, the signal processing under the conditions of a priori certainty in terms of their spectrum width or duration does not require two-dimensional frequency-temporal sampling. Therefore, at present, most sampling studies mainly focus on the one-dimensional methods for improving the accuracy of the signal approximation, reducing sampling losses, and practical applications.

Thus, paper [24] proved a sampling theorem for the finite polyharmonic processes involving the derivation of analytical formulas for the boundaries of the discrete spectrum of harmonics frequencies depending on the frequency band index and the number of interpolated sampling. However, the issue of a priori uncertainty was not considered.

The results of studying the accuracy of the recovery of periodic discrete signals (DSs) of finite duration using the interpolation Kotelnikov basis are reported in [25]. The ratio of signal power to the power of recovery error was used to assess the accuracy of the signal recovery. The ratio between the frequency of the periodic signal, the frequency of sampling, and the number of DS samples were found, which provide for the lowest errors in the recovery of a given signal. However, the results reported are only applicable to DSs with known parameters.

It is shown in [26] that the moment of the first information sample is necessary to accurately recover an unknown function by its even samples with an interval determined by the upper frequency in the $2f_v$ spectrum. However, it is impossible to establish the moment of the first informational sample under the conditions of a priori uncertainty.

Paper [27] reports a study into choosing the optimal frequency difference between the channels of the cross-shaped interferometer receiving device with frequency scanning based on the generalized two-dimensional WKS theorem, taking into consideration the direction of the scan. The dependence of the boundaries of frequency scattering on the clock angle and inclination has been obtained. The radio telescope's focus chart is seen as a spatial frequency filter. It is emphasized that the transformation of a two-dimensional signal is implemented in a two-stage procedure: sampling and elemental quantization.

However, both with the two-dimensional [21, 27] and multidimensional [10] sampling, the independence of the samples for different coordinates was tacitly assumed. In the case of sampling the signals in the frequency-time domain, there is a dialectic relation between the frequency and time, and the two-dimensional sampling has advantages under the conditions of structural-parametric a priori uncertainty.

All of this suggests that it was appropriate to conduct research into the frequency-time sampling of signals under the conditions of a priori uncertainty.

3. The aim and objectives of the study

The purpose of this study is to devise, based on the theoretical generalization and by proving a sampling theorem in the frequency-time domain, a new discrete description of signals, which would make it possible to overcome the existing structural-parametric a priori uncertainty in the radio monitoring of radiation.

To achieve the set aim, the following tasks have been solved:

– to theoretically generalize and prove the sampling theorem in the frequency-time domain;

– to analyze the basic properties and merits of the resulting generalization of the discrete frequency-time representation of signals;

– to experimentally confirm the practical significance of the resulting discrete frequency-time description of signals during the detection-recovery of signals of a priori uncertain kind and parameters.

4. The study materials and methods

To solve research problems, a frequency-time approach was used, involving the methods of signal sampling in the frequency and temporal domains, the methods of spectral and temporal signal analysis, and a Fourier transform. The source material for the theoretical generalization of the sampling theorem for the frequency-time domain was a WKS series in the temporal domain. The procedure for proving the sampling theorem in the frequency-time domain is based on the even grouping of time samples at fixed time intervals ΔT, transferring them from time to frequency domain using a Fourier transform, and justifying the fact that the averages of the complex amplitude of grouped samples are the values of the function amplitude itself, taken at the k-th point of time at the ℓ-th frequency. The basic properties and advantages of the resulting generalization of the discrete frequency-time representation of signals are substantiated by the system-theoretical method. Standard measuring devices, as well as devices, designed under the guidance of this paper's author, for signal detection and signal detection-recovery, based on the dispersion Fourier processors with pulse compression, were used for the experimental research.

5. The results of studying the sampling of signals in a frequency-time domain

5. 1. Generalizing the sampling theorem for a frequency-time domain

Theorem. An arbitrary narrow-band signal $s(t)$ with a limited spectrum, confined to the frequency band $\pm\Delta f_c/2$, is fully defined by its values taken for frequency at intervals $\Delta F_\ell=\Delta f_c/\ell$ and the time $\Delta T_k=1/\Delta F_\ell$ at any integer value $\ell=1, 2, ..., L$ and $k=1, 2, ..., \infty$.

Proof. Given the narrow band of the $s(t)$ signal, use its representation through a complex envelope $\dot{S}(t)$

$$s(t)=\mathrm{Re}\left\{\dot{S}(t)e^{j2\pi f_0 t}\right\}. \quad (1)$$

In turn, the complex envelope is related through a Fourier transform

$$\dot{S}(t)=\int_{-\infty}^{\infty}\dot{C}(f)e^{j2\pi ft}\mathrm{d}f \quad (2)$$

to the complex spectrum

$$\dot{C}(f)=\int_{-\infty}^{\infty}\dot{S}(t)e^{-j2\pi ft}\mathrm{d}t.$$

Given the limited nature of spectrum $\dot{C}(f)=0$ at $|f|>\Delta f_c/2$ of the narrowband signal (1), one can record

$$\dot{S}(t)=\int_{-\Delta f_c/2}^{\Delta f_c/2}\int_{-\infty}^{\infty}\dot{S}(\tau)e^{-j2\pi f\tau}\mathrm{d}\tau\cdot e^{j2\pi ft}\mathrm{d}f. \quad (3)$$

In accordance with the WKS theorem [1], a spectrum-limited integrated envelope (2) is represented within a temporal domain by the following series

$$\dot{S}(t)=\sum_{n=-\infty}^{\infty}\dot{S}(n\Delta t)\cdot\frac{\sin\pi\Delta f_c(t-n\Delta t)}{\pi\Delta f_c(t-n\Delta t)}=$$
$$=\sum_{n=-\infty}^{\infty}\dot{S}(n\Delta t)\cdot\phi_n(t), \quad (4)$$

where $\dot{S}(n\Delta t)$ is the sampling of a complex envelope at time $t=n\Delta t$; $\Delta t=1/\Delta f_c$ is the interval between samples.

At the same time, the spectral density of functions $\phi_n(t)$ in the frequency band $|f|=\Delta f_c/2$

$$\dot{C}_n(f)=\begin{cases}1/\Delta f_c\, e^{-j2\pi fn\Delta t}, \text{at } |f|\le\Delta f_c/2,\\ 0, \text{ at } |f|>\Delta f_c/2.\end{cases} \quad (5)$$

In this case, the signal-occupied frequency-time domain is broken down into temporal elements $\Delta t=1/\Delta f$ (Fig. 1, *a*), and, when the signal is represented by the WKS series in the frequency domain, into the frequency elements $\Delta f=1/\tau_c$ (Fig. 1, *b*). With limited spectrum width and signal duration, the number of sampling elements in the frequency-time domain and the total number of complex samples in both cases is $K_0=L_0=\Delta f_c\,\tau_c$, which is equivalent to $N=2K_0=2L_0=\Delta f_c\,\tau_c$ valid samples.

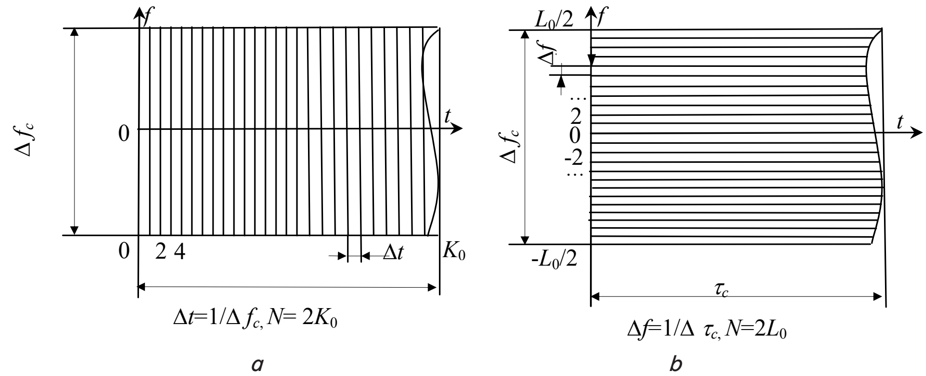


Fig. 1. Sampling a frequency-time plane: *a* – in the temporal domain; *b* – in the frequency domain

Group the complex samples in (4) evenly for $\ell = L'$ in such a way that

$$\dot S(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell'=-(L'-1/2)}^{L'+1/2}\dot S(n_{k\ell'}\Delta t)\cdot\frac{\sin\pi\Delta f_c(t-n_{k\ell'}\Delta t)}{\pi\Delta f_c(t-n_{k\ell'}\Delta t)}, \qquad (6)$$

where $n_{k\ell'}=\ell'+(k-1)L'$ is the variable summing index.

Substituting (6) in (3), by changing the order of summing up and integrating operations, calculating the integral in infinite limits, taking into consideration the spectral density of functions $\phi_n(t)$ (5) in the frequency band $|f|\le\Delta f_c/2$, the following is obtained

$$\dot S(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell'=-(L'-1/2)}^{L'+1/2}\dot S(n_{k\ell'}\Delta t)\times$$
$$\times\frac{1}{\Delta f_c}\int_{-\Delta f_c/2}^{\Delta f_c/2}e^{-j2\pi f(t-n_{k\ell'}\Delta t)}\mathrm{d}f.$$

The integral in the last expression can be represented as the sum of L integrals so that

$$\dot S(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell'=-(L'-1/2)}^{L'+1/2}\dot S(n_{k\ell'}\Delta t)\times$$
$$\times\sum_{\ell=-(L-1/2)}^{L+1/2}\frac{1}{\Delta F}\int_{(\ell-1)/2\Delta F}^{(\ell+1)/2\Delta F}e^{-j2\pi f(t-n_{k\ell'}\Delta t)}\mathrm{d}f, \qquad (7)$$

where $\Delta F=\Delta f_c/L$.

By introducing the designation $\Delta T=L'\times\Delta t$ and replacing the variable

$$f=F+\ell\cdot\Delta F, \qquad (8)$$

transform (6) to the following form

$$\dot S(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell'=-(L'-1/2)}^{L'+1/2}\dot S\left[\ell'\Delta t+(k-1)\Delta T\right]\times$$
$$\times\sum_{\ell=-(L-1/2)}^{L+1/2}\frac{1}{\Delta F}\int_{-\Delta F/2}^{\Delta F/2}e^{j2\pi(F+\ell\Delta F)[t-\ell'\Delta t-(k-1)\Delta T]}dF. \qquad (9)$$

Taking out from under the integral in (9) those terms that are not dependent on F, by conducting the integration considering (5), and summing up ℓ', the following representation of the complex envelope is obtained

$$\dot S(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell=-(L-1/2)}^{L+1/2}\bar{\dot S}_\ell(k\Delta T)\dot\psi_{k\ell}(t),$$

where

$$\bar{\dot S}_\ell(k\Delta T)=\sum_{\ell'=-(L'-1/2)}^{L'+1/2}\dot S\left[\ell'\Delta t+(k-1)\Delta T\right]/L,$$

$$\dot\psi_{k\ell}(t)=\frac{\sin\pi\Delta F(t-k\Delta T)}{\pi\Delta F(t-k\Delta T)}e^{j2\pi\ell\Delta F(t-k\Delta T)}. \qquad (10)$$

The complex amplitude in (10) at the interval ΔT is the average function value over L' samples. Show that these averages represent the function values taken at the ℓ-th frequency at the k-th point in time. The Fourier transform's limited-spectrum function (1) can be represented as the sum of Fourier's partial transformations

$$\dot S(t)=\int_{-\Delta f_c/2}^{\Delta f_c/2}\dot C(f)e^{j2\pi ft}\mathrm{d}f=\sum_{\ell=-(L-1/2)}^{L+1/2}\dot S_\ell(t), \qquad (11)$$

where, considering (8),

$$\dot S_\ell(t)=\int_{-\Delta F/2}^{\Delta F/2}\dot C(F+\ell\Delta F)e^{j2\pi(F+\ell\Delta F)t}\mathrm{d}F. \qquad (12)$$

By following [1, 9], the complex function $\dot C(F+\ell\Delta F)$ at the interval $(-\Delta F/2,\ \Delta F/2)$ is to be expanded into a Fourier series as follows

$$\dot C(F+\ell\Delta F)=\sum_{k=-\infty}^{\infty}\dot C_k(\ell\Delta F)e^{-j2\pi(F+\ell\Delta F)k/\Delta F}, \qquad (13)$$

where considering (8),

$$\dot C_k(\ell\Delta F)=\frac{1}{\Delta F}\int_{-\Delta F/2}^{\Delta F/2}\dot C(F+\ell\Delta F)e^{j2\pi(F+\ell\Delta F)k/\Delta F}\mathrm{d}F,$$

$1/\Delta F=\Delta T$ is the sampling period.

Substituting (13) and (12) in (11) produces the following

$$\dot S(t)=\sum_{\ell=-(L-1/2)}^{L+1/2}\int_{-\Delta F/2}^{\Delta F/2}\sum_{k=-\infty}^{\infty}\dot C_k(\ell\Delta F)e^{j2\pi(F+\ell\Delta F)(t-k/\Delta F)}\mathrm{d}F. \qquad (14)$$

By introducing in (14) the integral under the sign of the sum for k and, integrating for F, the following is obtained

$$\dot{S}(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell=-(L-1/2)}^{L+1/2}\dot{C}_k(\ell\Delta F)\frac{\sin\pi\Delta F(t+k/\Delta F)}{\pi\Delta F(t+k/\Delta F)}e^{j2\pi\ell\Delta F(t-k/\Delta F)}. \quad (15)$$

Comparing (13) to (11) determines the following

$$\dot{C}_k(\ell\Delta F)=\frac{1}{\Delta F}\dot{S}_\ell(k\Delta T). \quad (16)$$

Substituting (16) in (15), considering the sampling period ΔT, produces

$$\dot{S}(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell=-(L-1/2)}^{L+1/2}\dot{S}_\ell(k\Delta T)\frac{\sin\pi\Delta F(t-k\Delta T)}{\pi\Delta F(t-k\Delta T)}e^{j2\pi\ell\Delta F(t-k\Delta T)}. \quad (17)$$

It follows from the comparison of (9) to (17) that $\bar{\dot{S}}_\ell(k\Delta T)=\dot{S}_\ell(k\Delta T)$ and, therefore, (9) can be written in the following form

$$\dot{S}(t)=\sum_{k=-\infty}^{\infty}\sum_{\ell=-(L-1/2)}^{L+1/2}\begin{array}{c}\dot{S}_\ell(k\Delta T)\times\\ \times\dfrac{\sin\pi\Delta F(t-k\Delta T)}{\pi\Delta F(t-k\Delta T)}e^{j2\pi\ell\Delta F(t-k\Delta T)}\end{array}. \quad (18)$$

Expression (18) describes a complex envelope of the narrow-band signal in the form of a two-coordination (matrix) expansion for time and frequency. At the same time, the frequency-time plane occupied by a signal is sampled into the frequency-time elements with the frequency band $\Delta F=\Delta f_c/L$ and the duration $\Delta T=1/\Delta F$ (Fig. 2). The basis functions of the expansion are functions such as sin x/x with frequency filling (except for $\ell=0$), shifted by time at ΔT and, by frequency, at ΔF.

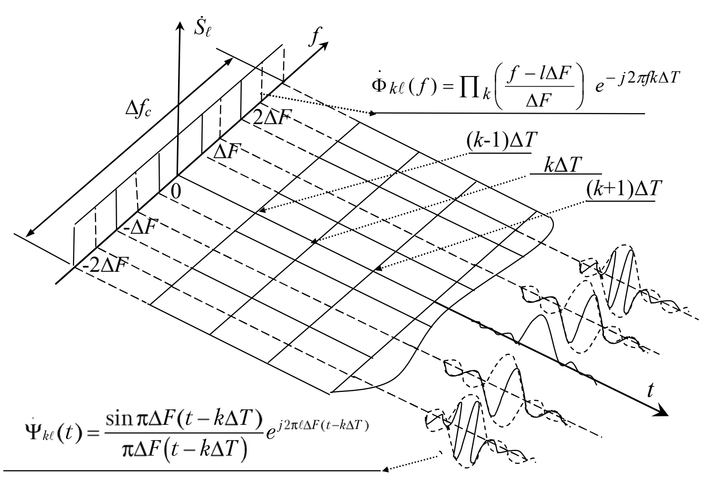


Fig. 2. A two-coordinate sampling of the signal complex envelope for time (t) and frequency (f)

At $\Delta F\,\Delta T=1$, the basis expansion functions are the superposition of the sin x/x function and the harmonics of the Fourier series.

$$\Psi_{k\ell}(t)=\frac{\sin\left[\pi(\Delta Ft-k)\right]}{\pi(\Delta Ft-k)}e^{j2\pi\ell(\Delta Ft-k)}=$$
$$=\frac{(-1)^k\sin(\pi\Delta Ft)}{\pi(\Delta Ft-k)}e^{j2\pi\ell\Delta Ft}. \quad (19)$$

Given the equalities $\Delta t=1/\Delta f_c$, $\Delta T=L'\times\Delta t$, $\Delta T=\Delta L'$, $\Delta F=\Delta f_c/L$, the number of L-frequency samples at each time sampling interval $k\Delta T$ is equal to the number of the grouped samples over time L'. When deriving (18), there were no restrictions on the number of grouped samples. Therefore, (18) holds for any integer value $\ell'=\ell=1, 2,\ldots, L$. The magnitude of sampling intervals by the frequency $\Delta F=\Delta F_\ell$ and the time $\Delta T=\Delta T_k$ depends on the specific value of ℓ. Thus, the theorem is proven.

The collateral follows from it for the conditions of the structural-parametric a priori uncertainty.

Collateral. An arbitrary narrow-band signal $s(t)$ with a priori uncertain finite spectrum $\Delta f_c/2$ is fully determined over the given frequency band $\Pi\geq\Delta f_{c\max}$ by its frequencies taken for frequency at intervals $\Delta F_\ell=\Pi/\ell$ and time $\Delta T_k=1/\Delta F_\ell$ at any integer value $\ell=1, 2, \ldots, L$ and $k=1,2,\ldots,\infty$.

5. 2. The basic properties of the sampling theorem in a frequency-time domain

Let us analyze the properties and merits of the resulting discrete frequency-time representation of signals (18), which should be utilized in their practical application.

1. Make sure that the basis functions of sinx/x expansion in (18) are orthogonal along the time axis. At fixed ℓ

$$I_1=\int_{-\infty}^{\infty}\dot{\Psi}_{k\ell}(t)\dot{\Psi}_{m\ell}^{*}(t)\mathrm{d}t=$$
$$=\frac{(-1)^{k+m}}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2(\pi\Delta Ft)}{(\Delta Ft-k)(\Delta Ft-m)}\mathrm{d}t. \quad (20)$$

Introducing the substitution of variable $\tau=\Delta F\cdot t$ in (20) produces the following

$$I_1=\frac{(-1)^{k+m}}{\pi\Delta A}\int_{-\infty}^{\infty}\frac{\sin^2\pi\tau}{(\tau-k)(\tau-m)}\mathrm{d}\tau=$$
$$=\begin{cases}\ell\Delta T, \text{ at } k=m,\ \Delta F\cdot\Delta T=i,\ i=1,2,\ldots\\ 0, \quad \text{at } k\neq m.\end{cases} \quad (21)$$

Therefore, for any ℓ, the basis functions of expansion are orthogonal along the entire axis $-\infty<t<\infty$. The orthogonalization of sinx/x functions along the time axis is obtained with a minimal shift of the argument satisfying the condition $\Delta F\,\Delta T=1$. At the same time, the distance between the sinx/x functions along the frequency axis is $\Delta F=1/\Delta T$.

Let us prove the orthogonality of the basis functions of expansion along the frequency axis. To this end, determine spectrum (18) using (2). Introducing to the right-hand part of the Fourier transform the integral for time under the sum signs, and using the F and t interchangeable property for even functions, after computing the integral, the following is obtained

$$\dot{C}(f)=\sum_{\ell=-(L-1/2)}^{L+1/2}\sum_{k=1}^{K}\begin{array}{c}C_K(\ell\Delta F)e^{j\phi_k(\ell\Delta F)}\times\\ \times\Pi_k\left(\dfrac{f-\ell\Delta F}{\Delta F}\right)e^{-j2\pi(f-\ell\Delta F)k\Delta T}\end{array}, \quad (22)$$

where $\dot{C}(\ell\Delta F,k\Delta T)=\Delta T\cdot\dot{S}(\ell\Delta F,k\Delta T)$,

$$\Pi_k(f)=\begin{cases}1,\ at\,|f|\leq 1/2\,\Delta F,\\ 0,\ at\,|f|>1/2\,\Delta F\end{cases}$$

– a rectangular strobe function with a frequency band ΔF and a center at the point $F=\ell\Delta F$, introduced by Woodward [2].

At $\Delta F\cdot\Delta T=1$, the expansion functions are

$$\dot{\Phi}_{kl}(f)=\Pi_k\left(\frac{f-\ell\Delta F}{\Delta F}\right)e^{-j2\pi f k\Delta T},$$

and the condition of orthogonality along the frequency axis at fixed k is

$$I_2=\Delta F\int_{-\Delta f_c/2}^{\Delta f_c/2}\dot{\Phi}_{k\ell}(f)\dot{\Phi}^*_{km}(f)\mathrm{d}f=$$
$$=\int_{-\Delta f_c/2}^{\Delta f_c/2}\Pi_k\left[\frac{f-\ell\Delta F}{\Delta F}\right]\Pi_k\left[\frac{f-m\Delta F}{\Delta F}\right]\mathrm{d}f.$$

By introducing a replacement for variables $f=S\cdot\Delta F$, the following is obtained

$$I_2=\Delta F\int_{-\Delta f_c/2}^{\Delta f_c/2}\Pi_k[s-\ell]\Pi_k[s-m]\mathrm{d}s=$$
$$=\begin{cases}\Delta F, & \text{at } \ell=m,\\ 0, & \text{at } \ell\neq m.\end{cases}\quad(23)$$

Woodward's basis functions do not overlap in frequency and, therefore, satisfy the condition of orthogonality, and, therefore, the functions sin x/x in (18) are orthogonal along the frequency axis. At the same time, due to the orthogonality of expansion functions, as opposed to the Gabor expansion [10, 11], the samples taken for frequency at points $\ell\Delta F$ and for time at points $k\Delta T$ are not correlated.

2. Due to the fact that at $k\to\infty$ the norm of the basis expansion functions, in accordance with [1],

$$\left\|\psi_{k\ell\sin x/x}\right\|^2=\left\|\Phi_{k\ell\Pi}\right\|^2=\Delta T,\quad(24)$$

the accuracy of the approximation of the signals by these functions is the same. This property of the derived discrete frequency-time representation of signals indicates the invariance of approximating functions and the order of receiving samples for the frequency and time coordinates.

3. In practice, the most commonly used are signal models, limited both in the width of the spectrum and in duration. Such models with sufficient precision describe the actually observed signals. Then the complex envelope and the spectrum of a signal of duration τ_c and the width of the spectrum Δf_C can be represented, in accordance with (18) and (22), in the frequency-time plane in the following form

$$\dot{S}(t)=\sum_{k=1}^{K}\sum_{l=-(L-1/2)}^{L+1/2}\frac{S_\ell(k\Delta T)e^{j\phi_\ell(k\Delta T)}\times}{\times\dfrac{\sin\pi\Delta F(t-k\Delta T)}{\pi\Delta F(t-k\Delta T)}e^{j2\pi\ell\Delta F(t-k\Delta T)}},\quad(25)$$

$$\dot{C}(f)=\sum_{\ell=-(L-1/2)}^{L+1/2}\sum_{k=1}^{K}\frac{C_K(\ell\Delta F)e^{j\phi_k(\ell\Delta F)}\times}{\times\prod_K\left(\dfrac{f-\ell\Delta F/2}{\Delta F}\right)e^{j2\pi(f-\ell\Delta F)k\Delta T}}.\quad(26)$$

Proving the theorem similarly to chapter 4. 1, based on the samples in the frequency domain, produces the following result

$$\dot{C}(f)=\sum_{\ell=-(L-1/2)}^{L+1/2}\sum_{k=1}^{K}\frac{C_k(\ell\Delta f)e^{j\phi_k(\ell\Delta F)}\times}{\times\dfrac{\sin\pi\Delta T(f-\ell\Delta F)}{\pi\Delta T(f-\ell\Delta F)}e^{j2\pi(f-\ell\Delta F)k\Delta T}}.\quad(27)$$

$$\dot{S}(t)=\sum_{k=1}^{K}\sum_{k=-(L-1/2)}^{L+1/2}\frac{S_\ell(k\Delta T)e^{j\phi_\ell(k\Delta T)}\times}{\times\prod\left(\dfrac{k-k\Delta T/2}{\Delta T}\right)e^{j2\pi\ell\Delta F(k-k\Delta T)}}.\quad(28)$$

Comparing (25) and (26) with (28) and (27), respectively, one can see that the original function (1) or complex spectrum (2) can be approximated by discrete two-coordination expansions. The approximation can be performed both by the sin x/x-type functions and by the rectangular strobe functions with Fourie-conjugated frequency filling. Such a representation can be interpreted as an approximation of signals in the form of a matrix of discrete elements in a frequency-time plane with appropriate envelopes, amplitudes, and phases at coordinate points ($k\Delta T$, $\ell\Delta F$). These expressions can be used in both a complex and material form to synthesize and analyze the processing and interpretation of the results.

4. For signals with limited spectrum width and duration in the form of (25) to (28), the total number of sample points (sampling elements) is

$$N=K\cdot L=\frac{\tau_C}{\Delta T}\cdot\frac{\Delta f_C}{\Delta F}=\tau_C\cdot\Delta f_C.\quad(29)$$

At the same time, two parameters should be defined for each element at the time points $k\Delta T$ at the frequency points $\ell\Delta F$: amplitude and phase. Consequently, the total number of samples corresponds to the number of samples when the signals are represented by the WKS series and equals $2\tau_c\Delta f_c$. However, in (25) to (28), the samples are taken both for time and frequency. For complex modulated signals, in order to obtain the same number of samples by time and frequency, the number of reference points for each coordinate is appropriate to choose equal to

$$N_t=N_F=\sqrt{\tau_C\cdot\Delta f_C}.\quad(30)$$

5. Expressions (25) and (28) explicitly contain the known expansion of signals into a Kotelnikov series by the functions sin x/x in a temporal domain at the known width of the signal spectrum Δf_c ($\ell=0$). And, expressions (26) and (27) contain the expansion of signals into a Kotelnikov series in the frequency domain based on the rectangular Woodward strobe functions at a known duration of the signal τ_c (ķ=1). This fact indicates that the accuracy of the representation of signals in the form of (25), (26) matches the accuracy of their representation by a Kotelnikov series based on the criterion of a minimal rms error.

6. An important consequence of generalization of the sampling theorem for the frequency-time domain is the presence in the resulting description of the explicit form of the phase-frequency-temporal function $V_{k\ell}(t)=e^{j2\pi\Delta F\ell(t-k\Delta T_\ell)}$. As a result, one can evaluate both the evolution and parameters of the functions of the phase-time or frequency-time modulation of signals, which makes it possible to determine not only their parameters but also the type of signal.

And the main advantage of generalizing the sampling theorem is its constructive nature. The theorem not only substantiates the technique for sampling signals in a frequency-time plane but also determines the way to restore the signals set by their sample values for frequency and time, which is confirmed below by the experimental results.

5. 3. The results of the experimental research into determining and reproducing signals under the conditions of a priori uncertainty

To confirm the practical significance of the results reported above, an experimental study was conducted to detect and reproduce signals using a laboratory setup whose structural scheme is shown in Fig. 3.

Digits in Fig. 3 designate the following:

– 1– frequency meter;

– 2 – a laboratory radio signal simulator;

– 3, 11 – direct dispersion Fourier processors;

– 4, 12 – analog-digital converters of frequency-time samples;

– 5, 13 – REM units;

– 6, 14 – TV-type raster indicators;

–7 – a standard signal generator;

– 8 – reverse dispersion Fourier processor;

– 9 – digital-analog converter of frequency-time samples;

– 10 – spectrum analyzer.

A priori frequency-temporal domain of the detected and reproduced signals with possible carrier frequency, spectrum width, and duration values was $\Pi \times T$=100 MHz×1,000 μs at the central frequency of f_0=750 MHz. The dispersion Fourier processors enabled the sampling of a frequency-time domain with a time resolution of ΔT=1 μs and the frequency of ΔF=1 MHz [28–30]. According to the proven generalization of the sampling theorem, this corresponds to sampling the frequency-time domain into elements with the specified parameters. The tests were carried out to detect and recover the continuous, simple pulse, and LFM radio signals. Generator of standard signals 7 was used as a source of the continuous and simple pulsed radio signals; laboratory simulator 2 – the LFM pulse radio signals. The parameters of the test signals from generator 7 were measured and controlled by spectrum analyzer 10 and frequency meter 1, those of the LFM radio signals – by the readings from the digital indicators of simulator 2.

Test signals with controlled parameters were sent to the input of the signal detection-recovery device. Direct dispersion Fourier-processor 3 [28, 29] formed a two-coordinate expansion of signals in the form of (26). After digitization and measurement of frequency-time samples in unit 4, they were recorded to unit 5's REM with simultaneous multiple representations on the frequency-time panorama of raster indicator 6.

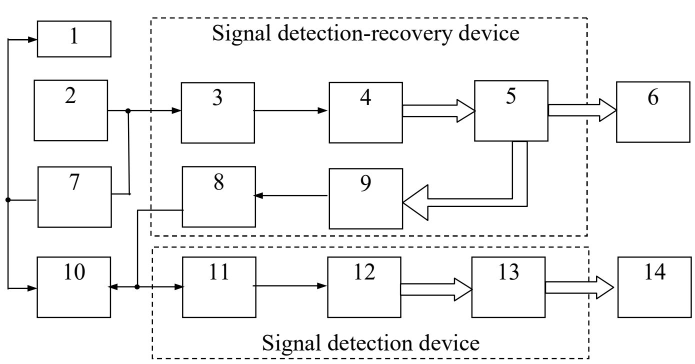
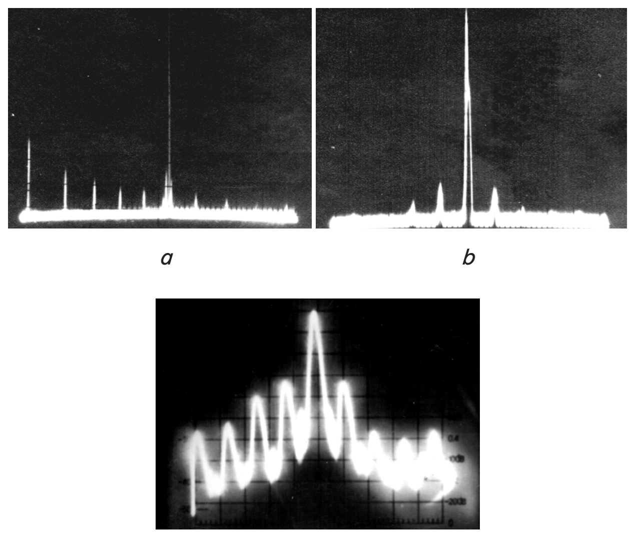


Fig. 3. Structural scheme of the laboratory set-up

Upon filling the REM, unit 5 was automatically switched to a multiple-read frequency-time sample mode to convert them in unit 9 to an analog form (27). The analog samples were used by reverse dispersion Fourier-processor 8 [28, 30] to form the reproduced signals in the form of (28). To monitor the processing results, the replicated signals were sent through a brancher to spectrum-analyzer 10 and for re-identifying similarly to the first in units 11–13. In addition, the signals reproduced were displayed on the frequency-time panorama of the second raster indicator 14.

The effectiveness of the devised frequency-time representation of signals for their identification and reproduction under the conditions of structural-parametric a priori uncertainty is illustrated by results from the experimental study shown in the photographs in Fig. 4–6.



a *b*

c

Fig. 4. The amplitude-frequency spectra of the continuous signal reproduced at the frequency of f_v=f_c=750 MHz, acquired from the screen of spectrum analyzer 10: *a* – frequency tags follow through 10 MHz; *b* – frequency tags follow through 1 MHz; *c* – the distance between the spectral lines corresponds to 1 MHz

The photographs in Fig. 4 show the amplitude-frequency spectra (AFS) of the reproduced continuous signal at the frequency f_v=f_c=750 MHz, acquired from the screen of spectrum analyzer 10. Large-scale frequency tags in the photograph in Fig. 4*a* follow in 10 MHz, and the frequency scale in the photographs in Fig. 4*b* is 1 MHz. Since the recovered signal was approximated by a sequence of radio pulses non stitched for phase according to (28), the reproduced AFS on the display of spectrum analyzer 10 is discrete. The distance between the spectral lines is determined by the time interval of sampling ΔT=1 μs and is equal to 1 MHz.

In addition, the photographs in Fig. 4, *a, b* show the spectral line of the continuous signal, directly sent from the output of generator 7 to the input of spectrum analyzer 10, coinciding with the central spectral line of the recovered continuous AFS.

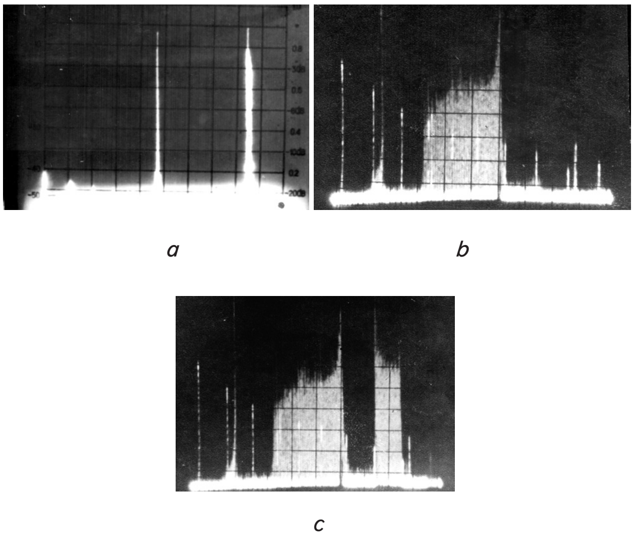


Fig. 5. The amplitude-frequency spectra of signal combinations on the screen of spectrum analyzer 10: *a* – the reproduction of a continuous and periodic pulse signal with the duration of τ_c=0.3 μs; *b* – reproducing a combination of sequences of simple pulse signals and LFM; *c* – reproducing a combination of sequences of two LFM signals

The photographs in Fig. 5 show the combinations of signals reproduced by AFS, acquired from the screen of spectrum analyzer 10 in a linear scale. The frequency scale tags in all photographs follow in 10 MHz. The photographs in Fig. 5*a* illustrate the AFS of the replay of a continuous signal at the frequency f_p=748 MHz and the sequence of pulse signals lasting τ_c=0.3 μs with a follow period of 1 ms at the frequency f_v=f_c=785 MHz. One can see that the AFS of the reproduced short pulse signals ($\Delta f_v \approx$3 MHz) are slightly wider than the AFS of the continuous signal reproduced. The AFS of the reproduced combination of simple pulse signals lasting τ_c=10 μs with a follow period of T=1 ms and the LFM signals with the base $\Delta f_c \times \Delta\tau_c$=32 MHz×32 μs, with a follow period of T=2 ms, are given in the photograph in Fig. 5, *b*. And, finally, the photograph in Fig. 5*c* shows the AFS of two detected and reproduced in the predefined frequency-time domain Π×T, LFM signals with the bases $\Delta f_c \times \Delta\tau_{c.}$=32 MHz×32 μs and 10 MHz×10 μs, respectively.

The effectiveness of the devised representation (25) to (28), implemented at the laboratory set-up, to identify and reproduce signals under the conditions of structural-parametric a priori uncertainty is illustrated by photographs of the results of the experimental study, shown in Fig. 6.

The photographs show the frequency-time signal panoramas [28] of the primary detection and re-detection after the reproduction of two LFM signals with parameters corresponding to Fig. 5, *c*, acquired from the screens of raster indicators, respectively, 6 and 14. The frequency scale on the panoramas in the form of horizontal light lines corresponds to 10 MHz/del. A shift in the frequency-time signal panoramas of the primary detection of signals and their re-identification after the reproduction in Fig. 6 is due to the difference in the central frequency of the signal detection device relative to the central frequency of the device for their detection – reproduction at 20 MHz. There are three digital displays at the bottom of the screens of raster indicators. The first group of numbers in the photographs shows the relative time of measurements with an accuracy to a television frame, the second group of numbers – the frequency with an accuracy to the resolution interval of 1 MHz, the third group of numbers – duration with an accuracy to the interval of sampling for time of 1 μs.

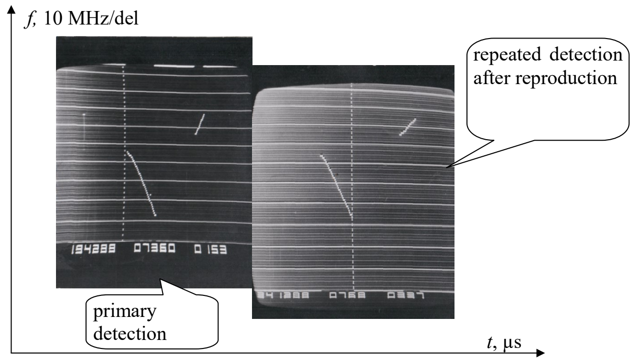


Fig. 6. Frequency-time signal panoramas of the results of the detection and reproduction of two LFM signals

The position of the frequency-time samples and, accordingly, the parameters of signals within the frequency-time panoramas, were measured in an interactive mode at a standard signal/noise ratio of 13.2 dB by combining the vertical dash-dot measurement mark with the brightness marks of the samples. The results of the measurements were displayed on digital displays. The measured frequency-time signal parameters at the input of the detection-recovery device, after initial detection and re-detection after reproduction, coincided with the accuracy of a single frequency-time sampling element.

6. Discussion of results of the signal detection and recovery based on the generalization of the sampling theorem

The practical value of signal processing based on the sampling theorem in a frequency-time domain is, during the detection (Fig. 6*a*), in the assessment of the frequency-time parameters, modulation functions, and the types of signals, and, when reproduced, in their recovery (Fig. 6, *b*). At the same time, the structural-parametric a priori uncertainty in terms of the frequency-time parameters and the type of processed signals in the specified frequency-time domain Π T is overcome. different signals and a visual representation of the results. This is confirmed by repeated tests and measurements involving different types of signals and by the visual presentation of the results.

Another aspect of the practical value of the resulting representation is the ability to overcome a priori uncertainty in terms of energy parameters when processing an ensemble of signals. In the case of REE assessment, it is always possible (and necessary) to determine the range of changes in the width of the spectrum of signals $\Delta f_{c\,\min} < \Delta f_c \leq \Delta f_{c\,\max} = \Pi$. This is the reason for the a priori knowledge on the frequency band of processing (of the device) but not on the spectra of signals. Ignorance of the ensemble of signals would imply the structural-parametric a priori uncertainty of carrying frequencies, spectrum width, the duration of pulse signals, and modulation functions. Even if these parameters are predetermined, the time of a signal arrival is unknown, or, under the mode of operation of the radiation source, may be changed to another SCS.

Sampling on the basis of the sampling theorem in the frequency-time domain makes it possible to identify and reproduce any signal entering the processing bandwidth, including a mixture of multiple signals. If one properly chooses the generalized parameters ΔT and $\Delta F=1/\Delta T$, the sampling of any signal would be close to optimal.

Consider the processing of an arbitrary signal with a spectrum width $\Delta f_{c\,\min}<\Delta f_c\leq\Delta f_{c\,\max}=\Pi$ and the duration $\tau_{c\,\min}\leq\tau_c\leq\tau_{c\,\max}=\mathrm{T}$ in the given a priori frequency-time domain $\Pi\times\mathrm{T}$. When sampling for time according to the WKS theorem (Fig. 1, *a,* 6, *a*), noises from the entire a priori frequency band Π within the duration of the signal τ_c are included in the processing. Similarly, when sampling a signal with the specified parameters for frequency according to the WKS theorem, noises would be processed (accumulated) within the maximum a prior duration T of the ensemble of signals (Fig. 1, *b*). This circumstance is caused by a priori uncertainty in terms of the frequency-time parameters of the ensemble of signals and leads to redundancy in the number of samples and deterioration of the signal/noise ratio for the sampling element. As a result, quality processing indicators would deteriorate. In accordance with the proven generalization of the theorem, obtaining the frequency-time samples in the processing of any signal entering a prior domain $\Pi\times\mathrm{T}$ would be carried out against the background of noises from the domain $\Delta f_c\times\tau_c$ occupied by the signal (Fig. 6). This fact makes it possible to obtain a win in relation to a signal/noise ratio compared to a single-coordinate sampling for time, equal to $\Pi/\Delta f_c$, and a one-coordinate sampling for frequency, T/τ_c.

Thus, the model of a discrete two-coordinate representation of signals by the system of orthogonal functions, obtained on the basis of generalization of the sampling theorem in the frequency-time domain, meets the requirements for:

– the adequate representation of real signals;

– the monopulse extraction, in the process of measuring, of information not only about parameters but also about the form (type) of signal;

– optimizing the processing of an ensemble of signals to solve the tasks of radio monitoring of radio emission and jamming radio-electronic devices.

This conclusion is illustrated by the results from processing real signals based on the derived representation and a dispersion Fourier transform when they are mapped onto frequency-time signal panoramas.

The main condition limiting the use of the sampling theorem in a frequency-time domain is the consistency of spectrum width or transmission Δf_c with the processing band, for example, in radar, radio navigation, and radio communication, which renders bulky two-coordinate sampling impractical. A certain restriction on the frequency-time sampling is imposed by a fundamental inability to obtain simultaneously high resolution capabilities for time ΔT and frequency ΔF.

Possible areas for further research are:

– optimizing the frequency-time sampling parameters, subject to $\Delta T\times\Delta F=1$, for various tasks of radio electronics and frequency ranges;

– studying errors in the truncation, overlay, tremor, and rounding [15] that can occur when a frequency-time sampling is applied in practice.

The results reported here are planned to be used in the future to devise methods, procedures, and algorithms for processing determinized and random signals under the conditions of varying degrees of a priori uncertainty.

7. Conclusions

1. The WKS sampling theorem has been generalized for a frequency-time domain. As a result of proving the theorem, a mathematical model has been built for the discrete frequency-temporal representation of signals in the form of a double series. The model, unlike known ones, allows for a two-coordinate expansion of signals with a finite spectrum on the systems of orthogonal functions sin x/x or the Woodward rectangular strobe functions with frequency filling by the harmonics of a Fourier series. At the same time, a priori frequency-time plane $\Pi\times\mathrm{T}$, within which the received signals can be found, is sampled into frequency-time elements with a frequency band $\Delta F=\Delta f_c/L$ and the duration $\Delta T=1/\Delta F$, depending on L. That makes it possible, under the conditions of structural-parametric a priori uncertainty, to form a sufficient number of samples to detect signals. In addition, the model explicitly contains the function of a phase-frequency-temporal modulation.

2. The properties of the derived representation have been investigated. The following has been proven:

– the orthogonality of sinx/x expansion functions and the rectangular Woodward strobe functions;

– the non-correlated samples at points $k\Delta T$ and $\ell\Delta F$;

– the equivalence of the accuracy of the approximation of signals by the basis functions sin x/x and the Woodward rectangular strobe functions;

– the compliance of the accuracy of the derived signal representation in the form of a double series for frequency and time to the accuracy of signal representation by a WKS series;

– the possibility to identify and reproduce the signal $N=2K\cdot L=2\tau_c\cdot\Delta f_c$, limited in its spectrum width and duration, by digital samples.

3. The experimental study on the detection-recovery of test signals has confirmed the theoretical results obtained and their practical significance for optimizing digital signal processing under the conditions of structural-parametric a priori uncertainty. The hardware errors in the detection-reproduction of signals did not exceed one frequency-time sampling element $\Delta F\Delta T=1$ MHz×1 μs. The study results show the practical value of generalizing the sampling theorem for a frequency-time domain and its mathematical description for devising effective methods, algorithms, and signal processing devices to overcome the structural-parametric a priori uncertainty.

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