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ESTABLISHING CONDITIONS FOR THE EXISTENCE OF BOUNDED SOLUTIONS TO THE WEAKLY NONLINEAR PULSE SYSTEMS

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Processes that involve jump-like changes are observed in mechanics (the movement of a spring under an impact; clockwork), in radio engineering (pulse generation), in biology (heart function, cell division). Therefore, high-quality research of pulse systems is a relevant task in the modern theory of mathematical modeling.

This paper considers the issue related to the existence of bounded solutions along the entire real axis (semi-axis) of the weakly nonlinear systems of differential equations with pulse perturbation at fixed time moments.

A concept of the regular and weakly regular system of equations for the class of the weakly nonlinear pulse systems of differential equations has been introduced.

Sufficient conditions for the existence of a bounded solution to the heterogeneous system of differential equations have been established for the case of poorly regularity of the corresponding homogeneous system of equations.

The conditions for the existence of singleness of the bounded solution along the entire axis have been defined for the weakly nonlinear pulse systems. The results were applied to study bounded solutions to the systems with pulse action of a more general form.

The established conditions make it possible to use the classical methods of differential equations to obtain statements about solvability and the continuous dependence of solutions on the parameters of a pulse system.

It has been shown that classical qualitative methods for studying differential equations are mainly naturally transferred to dynamic systems with discontinuous trajectories. However, the presence of a pulse action gives rise to a series of new specific problems.

The theory of systems with pulse influence has a wide range of applications. Such systems arise when studying pulsed automatic control systems, in the mathematical modeling of various mechanical, physical, biological, and other processes

Keywords: differential equations, pulse system, bounded solutions, Green-Samoilenko function, regular solutions

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1. Introduction

The modern development of natural science and technology contributes to the emergence of problems described by the systems of differential equations with discontinuous trajectories and, in particular, to the evolution of the mathematical theory of pulse systems.

When mathematically modeling this type of process, the duration of such perturbations can often be conveniently disregarded, believing that they have the nature of a pulse.

This idealization leads to the need to study the systems of differential equations whose solutions change in a jump-like fashion. However, it is not only the idealization of replacing short-term perturbations with «instantaneous» ones that leads to differential equations with discontinuous trajectories. Often, the disruptions of certain dependences in the system under study are a significant characteristic of it.

The theory of nonlinear systems of differential equations with a pulse influence, to which a series of problems related to natural science and technology are reduced, has been enriched with significant results in recent decades. Among the systems studied, there are systems with a pulse action of weak nonlinearity. The complexity of the mathematical statement of a problem for analytical research into this type of system is due to the non-smoothness of the corresponding dynamic processes. That necessitates devising research methods for weakly nonlinear systems of differential equations with pulse influences. Therefore, it is a relevant task to study the solutions to systems of a given type.

The results of the research could be successfully used in the study of oscillatory processes in various mechanical and electromechanical systems with discontinuous characteristics when investigating the multi-frequency fluctuation processes in discontinuous systems, as well as other models of natural science.

2. Literature review and problem statement

Paper [1] established the conditions that warrant the hyperbolicity of systems of differential equations with pulse action. The resulting hyperbolicity conditions make it possible to investigate the existence of bounded solutions to the heterogeneous multidimensional systems of differential equations with pulse perturbation. In [2], sufficient conditions were established for the existence of an asymptotically stable invariant toroidal manifold of the linear expansion of a dynamic system on the torus for the case of the matrix of the system switching with its integral. The proposed approach is applied to study the stability of invariant sets of a certain class of discontinuous dynamic systems.

Paper [3] reviews the most modern methods for studying the stability of solutions to pulse differential equations and their application to pulse control problems. The authors of [4] proved the exponential stability of the trivial torus for a single class of nonlinear extensions of dynamic systems on the torus. Their results were applied to investigating the stability of toroidal sets of pulsed dynamic systems. Work [5] considers the task of constructing an approximate adaptive control, including the case of pulse control, for a single infinite-dimensional problem with the Nemytsky-type objective functionality. The averaging method for obtaining approximate adaptive control was substantiated. The authors of [6] introduced the concept of a pulse non-autonomous dynamic system. The existence and properties of the pulse attracting set were investigated for it. The findings were applied to study the stability of a two-dimensional pulse-perturbed Navier-Stokes system. In [7], the recursive properties of almost periodic movements of pulse dynamic systems were explored. The results were applied to examine the qualitative behavior of discrete systems. The authors of [8] considered the properties of stability in relation to external (control) perturbations for the systems of differential equations with a pulse action at fixed time moments. They established the necessary and sufficient stability conditions for the classes of impulsive systems with the function of Lyapunov type. Paper [9] examines a non-autonomous evolutionary inclusion with pulse influences at fixed times. A corresponding non-autonomous multi-valued dynamic system is being built, for which the existence of a compact global attractor in phase space is proven. In [10], the existence of attracting sets of complex structure in the simplest evolutionary dissipative systems describing the dynamics of reaction-diffusion type was discovered. Work [11] proved the existence of global attractors in the discontinuous infinite-dimensional dynamical systems, which could have trajectories with an infinite number of pulse perturbations. The results were applied to study the asymptotic behavior of pulse systems generated by differential equations with a multi-digit right-hand part.

Studies [1–11] lay out the basics of the qualitative theory of differential equations with pulse action. In fact, the basics of the high-quality theory of pulse systems, based on the qualitative theory of differential equations, methods of asymptotic integration of such equations, the theory of difference equations and generalized functions were laid. However, the issues related to the existence of solutions to the weakly nonlinear pulse systems have not yet been fully investigated. In general, the presence of a pulse action significantly affects the behavior of the system. Even in the case of simple linear systems, the pulse effect could lead to significantly nonlinear behavior. Therefore, for each system with a pulse action, it is important to deeply investigate the behavior of its solutions, including the conditions for the existence of bounded solutions to such systems.

3. The aim and objectives of the study

The purpose of this study is to define conditions for the existence of bounded solutions along the entire real axis of the weakly nonlinear systems of differential equations with pulse perturbation at fixed time moments. The conditions to be established could make it possible to model and explore the dynamic systems of various evolutionary processes whose parameters may vary under the influence of external perturbations.

To accomplish the aim, the following tasks have been set:

– to find sufficient conditions for the existence of bounded solutions to a weakly nonlinear multidimensional system of differential equations with a pulse action;

– to establish the conditions for the existence of singleness of the bounded solution on the entire axis for the weakly nonlinear pulse systems;

– to test the possibility of investigating the solutions by using a SIR «pulse vaccination» model as an example.

4. The study materials and methods

The main object of this study is a weakly nonlinear system of differential equations with pulse action. Underlying the research into the systems of this type is the conditions for the existence and uniformity of bounded solutions. Knowing the conditions of existence and uniformity, we can investigate the behavior of solutions to the corresponding systems on manifolds. The use of a qualitative theory of differential equations makes it possible to establish a certain relationship between a class of linear and weakly nonlinear systems with pulse action. Applying the concept of regularity for the corresponding homogeneous system, we obtain an opportunity to find conditions for the existence and singleness of a solution through the coefficients of the original system. The obtained results are illustrated using a SIR model of pulse vaccination as an example. The solution to the model adequately matches the study results reported in this work.

5. The results of studying the weakly nonlinear systems of differential equations with pulse perturbations

5. 1. Sufficient conditions for the existence of bounded solutions to the weakly nonlinear system with a pulse action

Consider the system of differential equations with pulse perturbation:

$$\frac{dx}{dt} = A(t)x + f(t,x),\ t \neq \tau_i,$$

$$\Delta x\big|_{t=\tau i} = B_i x + I_i(x), \tag{1}$$

where the function $f(t,x)$, the matrix $A(t)$ is continuous and bounded for all $t\in R$, the matrices B_i are uniformly bounded for $i\in Z$ and such that:

$$imf\left|\det\left(E + B_i\right)\right| > 0,\ i\in Z. \tag{2}$$

The function $f(t,x)$ is piecewise continuous with respect to t with discontinuities of the first kind at points $t=\tau_i$ and satisfies the Lipschitz condition with respect to x uniformly with respect to $t\in R$. The same condition is satisfied by the function $I_i(x)$:

$$\|f(t,x)-f(t,y)\|\le L\|x-y\|,$$

$$\|I_i(x)-I_i(y)\|\le L\|x-y\|, \tag{3}$$

with all $t\in R$, $i\in Z$, and some $L>0$.

The sequence of moments of pulse perturbation $\{\tau_i\}$ is numbered by integers so that $\tau_i\to-\infty$ at $i\to-\infty$ and $\tau_i\to+\infty$ at $i\to-\infty$. We also assume that there is a finite boundary uniformly over $t\in R$:

$$\lim_{T\to\infty}\frac{i(t,t+T)}{T}=p<\infty. \tag{4}$$

We are interested in the existence of solutions to equations (1), bounded along the entire axis, on the assumption that the corresponding linear system:

$$\frac{dx}{dt}=A(t)x,\ t\ne\tau_i,\ \Delta x\big|_{t=\tau_i}=B_ix, \tag{5}$$

is weakly regular on R.

Consider the corresponding heterogeneous equation:

$$\frac{dx}{dt}=A(t)x+f(t),\ t\ne\tau_i,\ \Delta x\big|_{t=\tau_i}=B_ix+a_i. \tag{6}$$

Definition 1. A homogeneous system of equations (5) is termed a weakly regular one along the entire R axis if the heterogeneous equation (6) at each limited vector function $f(t)$ has at least one R-bounded solution.

A system of equations (5) is termed regular on R if this system has exactly one R-bounded solution at each fixed constrained function $f(t)$.

According to Theorem 1 [1], for the arbitrary R-bounded functions $f(t)$ and the sequence $\{a_i\}$, the system of equations has a single solution bounded along the entire axis.

Theorem 1. Let the system of equations (5) be weakly regular along the entire numerical line, and let the functions $f(t,x)$ and $I_i(x)$ satisfy, at all $t\in R$ and $i\in Z$, in some sphere $S_r=\{x\in R^n,\ \|x\|\le r\}$, the following conditions:

$$C\|f(t,x)\|\le r,\ C\|I_i(x)\|\le r, \tag{7}$$

where C is the constant of weak regularity (8). Then the system of equations (1) has at least one solution bounded along the entire R axis.

Proof. Let A be the manifold of the initial values of solutions to equations (5) bounded along the entire axis and let P: $R^n\to A$ be the projector that maps R^n onto A. According to theorem 1 [1], with any limited piecewise-continuous function $f(t)$ and a constrained sequence $\{a_i\}$, the system of equations (6) has a single solution bounded along the entire axis $x=\phi(t)$ that satisfies the condition $P\phi(0)=0$ and the following inequality:

$$\sup_{t\in R}\|\phi(t)\|\le C\max\left\{\sup_{t\in R}\|f(t)\|,\ \sup_{i\in Z}\|a_i\|\right\}. \tag{8}$$

In the space of all piecewise-continuous and limited functions with discontinuities of the first kind at points τ_i, we shall consider the set B_r of such functions $\psi(t)$, for each of which:

$$\sup_{t\in R}\|\phi(t)\|\le r,$$

where r is the number that appears in the condition of Theorem 2. Given the inequalities (7) for any function $\psi(t)\in B_r$, the function $f(t,\psi(t))$ is piecewise continuous and bounded throughout the numeric line. The $\{I_i(\psi(\tau_i))\}$ sequence would also be bounded.

On the set of functions B_r, we shall define the operator F, which assigns a bounded solution $\psi(t)$ to the following system of equations to each function $\psi(t)$ with B_r:

$$\frac{dx}{dt}=A(t)x+f(t,\psi(t)),\ t\ne\tau_i,$$

$$\Delta x\big|_{t=\tau_i}=B_ix+I_i(\psi(\tau_i)), \tag{9}$$

that satisfies the condition $P\phi(0)=0$, and, therefore, the following inequality:

$$\sup_{t\in R}\|\phi(t)\|\le C\max\left\{\sup_{t\in R}\|f(t,\psi(t))\|,\sup_{t\in R}\|I_i(\psi(\tau_i))\|\right\}. \tag{10}$$

Given the inequalities (7) and (10), we obtain that the set of functions B_r is invariant with respect to the operator F, that is, $FB_r\le B_r$. The set B_r is a convex, bounded, and closed set in the space of all piecewise-continuous functions having discontinuities of the first kind at points $t=\tau_i$. We shall make sure that the operator F: $B_r\to B_r$ is continuous, and, therefore, has at least one fixed point $\phi^*(t)=F\phi^*(t)$, that is, the system of equations (1) has at least one solution bounded along the entire axis.

Suppose the function sequence $\{\psi_m(t)\}\subset B_r$ $(m=1,2,\ldots)$ matches the function $\psi(t)$ at $m\to\infty$. It is directly checked that $\psi(t)\in B_r$. We shall show that the function sequence $\{\phi_m(t)\}=\{F\psi_m(t)\}$ $(m=1,2,\ldots)$ matches the function $\phi(t)=F\psi(t)$ at $m\to\infty$.

To this end, we denote the segment $I^{(q)}=\{t\in R:|t|\le q\}$ through $I^{(q)}$ for any natural number q.

Let us set the convergence of the $\{f_m(t)\}=\{f(t),\psi_m(t)\}$ $(m=1,2,..)$ function sequence to the function $f_0(t)=f(t,\psi(t))$ at $m\to\infty$ in the topology of space of all piecewise-continuous functions with discontinuities at points $t=\tau_i$. This is equivalent to the fact that the sequence of $\{f_m(t)\}$ functions coincides with the $f_0(t)$ function at $m\to\infty$ evenly on any segment $I^{(q)}$, $q=1,2,\ldots$.

Fix the $I^{(q)}$ segment. Points $\{\tau_i\}$ break this segment into the finite number of segments $I^{(q)}=I_1^{(q)}\cup I_2^{(q)}\cup\ldots\cup I_{rq}^{(q)}$. Consider an arbitrary segment $I_j^{(q)}\ (1\le j\le r_q)$. At any $x\in R^n$ and $m\in N$, the functions (t,x), $\psi_m(t)$, and $\psi(t)$ are continuous at an interval $\widehat{I}_j^{(q)}$, which coincides with the segment $I_j^{(q)}$ without its left end. Moreover, these functions are evenly continuous at interval $\widehat{I}_j^{(q)}$, since the narrowing of the specified functions to $\widehat{I}_j^{(q)}$ assume a continuous continuation on the entire segment $I_j^{(q)}$. It hence follows that the function $f(t,x)$ is evenly continuous on the set $\widehat{I}_j^{(q)}\times B_r$.

Note that the sequence $\{\psi_m(t)\}$ converges to the function $\psi(t)$ at $m\to\infty$ evenly at the interval $\widehat{I}_j^{(q)}$. Therefore, with any positive ε, there is such a natural number m_0 that for any $t\in\widehat{I}_j^{(q)}$ and $m\ge m_0$, $m\in N$, the following inequality holds:

$$\|f(t,\psi(t))-f(t,\psi_m(t))\|<\varepsilon.$$

Since the number of segments $I_j^{(q)}\left(1\le j\le r_q\right)$ is finite, the last inequality would hold at all fairly large $m\in N$ and at all $t\in I^{(q)}$. Therefore, the sequence of functions $\{f_m(t)\}$ converges to the function $f_0(t)=f(t,\psi(t))$.

Similarly, at any $i\in Z$, the sequence $\left\{a_i^{(m)}\right\}=\left\{I_i\left(\psi_m\left(\tau_i\right)\right)\right\}$, $m\in N$ converges to the vector $a_i=I_i(\psi(\tau_i))$ at $m\to\infty$. This follows from the fact that at any $i\in Z$ the function $I_i(x)$ is continuous and $\left\|\psi\left(\tau_i\right)-\psi_m\left(\tau_i\right)\right\|\to 0$ at $m\to\infty$.

Given the conditions (7), the functions $f_m(t)=f(t,\psi_m(t))$ and the sequences $\left\{a_i^{(m)}\right\}=\left\{I_i\left(\psi_m\left(\tau_i\right)\right)\right\}$, $i\in Z$, $m\in N$ are evenly bounded, causing a limit to the boundary function $f_0(t)=f(t,\psi(t))$ and the sequence $\{a_i\}=\{I_i(\psi(\tau_i))\}$.

In this case, equation (6) has a single bounded solution $\phi(t)=F\psi(t)$, which meets the condition $P\phi(0)=0$.

By the definition of operator F, the functions $\phi_m(t)=F\psi_m(t)$ are the solutions to the following equations:

$$\frac{dx}{dt}=A(t)x+f_m(t),\ t\ne\tau_i,\ \Delta x\big|_{t=\tau_i}=B_i x+a_i^{(m)}, \tag{11}$$

that meet the condition $P\phi(0)=0$ and the following inequality:

$$\sup_{t\in R}\left\|\phi_m(t)\right\|\le C\max\left\{\sup_{t\in R}\left\|f_m(t)\right\|,\ \sup_{i\in Z}\left\|a_i^{(m)}\right\|\right\}.$$

Thus, the sequence of functions $\{\phi_m(t)\}$, $m\in N$ is evenly bounded throughout the numerical axis, and the sequence of their derivatives $\left\{d\phi_m(t)/dt\right\}$, $m\in N$ is uniformly bounded at any interval (τ_i,τ_{i+1}), $i\in Z$.

Note that at any $i\in Z$, the narrowing, at the interval (τ_i,τ_{i+1}), of functions $\phi_m(t)$ and their derivatives assume a continuous continuation on the entire segment $[\tau_i,\tau_{i+1}]$. Thus, at any $i\in Z$, the sequence of the narrowing $\{\phi_m(t)/(\tau_i,\tau_{i+1})\}$ is compact.

So, at any $q\in N$, the sequence of $\left(\phi_m\setminus I^{(q)}\right),(m\in N)$ functions narrowing is compact; it follows, hence, the compactness of the $\{\phi_m(t)\}$ function sequence.

Since the sequence of functions $\{\phi_m(t)\}$, $m\in N$ consists of the solutions to corresponding equations (11) that satisfy the condition $P\phi_m(0)=0$, then any boundary function of this sequence satisfies the boundary equation (6) and the condition $P\phi_m(0)=0$. However, this property is characteristic of the only solution found earlier, $\phi(t)=F\psi(t)$. Thus, the sequence of functions $\{\phi_m(t)\}=\{F\psi_m(t)\}$, $m\in N$ converges to the function $\phi(t)=F\psi(t)$ at $m\to\infty$. We have obtained the continuity of operator F.

The compactness of the F operator is proven by similar considerations. To this end, it would suffice to show that the FB_r set is relatively compact. As shown above, the set of FB_r functions is evenly bounded throughout the axis, and, based on the definition of the operator F and condition (7), it follows from equation (9) that the set of derivatives of these functions is evenly bounded at any interval (τ_i,τ_{i+1}), $i\in Z$. Similarly, at any $q\in N$ sets of the narrowing of function $\left\{\left(\phi(t)\,|\,I^{(q)}\right):\phi(t)\in FB_r\right\}$ is relatively compact. So, we have obtained the relative compactness of the FB_r.

Then, according to Breuer's theorem about a fixed point, the operator F has at least one fixed point $F\phi(t)=\phi(t)$, and, therefore, equation (1) has at least one bounded solution along the entire axis. The theorem is proved.

5. 2. The existence of singleness of the bounded solution to the weakly nonlinear pulse systems

Theorem 2. Suppose that the system of equations (5) is weakly regular on the entire numerical axis R; the functions $f(t,x)$ and $I_i(x)$ satisfy inequality (3) with the Lipschitz constant $L<1/C$ (C is the weak regularity constant of equations (5)) in the sphere B_r, where the number $r>0$ satisfies the following inequality:

$$C\cdot\max\left\{\sup_{t\in R}\left\|f(t,0)\right\|,\sup_{i\in Z}\left\|I_i(0)\right\|\right\}+CLr\le r. \tag{12}$$

Then equation (1) has a unique solution $x=\phi(t)$, bounded along the entire axis, which satisfies the condition $P\phi(0)=0$ and $\sup_{t\in R}\left\|\phi(t)\right\|\le r$.

Proof. Note that if the Lipschitz constant $L<1/C$, then there is the number $r>0$, which satisfies inequality (12).

At any function $\psi(t)\in B_r$, the function $f(t,\psi(t))$ is piecewise-continuous and bounded. Bounded is the sequence $\{I_i(\psi(\tau_i))\}$. Indeed, it follows from inequalities (3) that:

$$\sup_{t\in R}\left\|f(t,\psi(t))\right\|\le\sup_{t\in R}\left\|f(t,0)\right\|+\left\|f(t,\psi(t))-f(t,0)\right\|\le$$
$$\le\sup_{t\in R}\left\|f(t,0)\right\|+\sup_{t\in R}L\left\|\psi(t)\right\|\le\sup_{t\in R}\left\|f(t,0)\right\|+L\cdot r, \tag{13}$$

accordingly:

$$\sup_{i\in Z}\left\|I_i\left(\psi\left(\tau_i\right)\right)\right\|\le\sup_{i\in Z}\left\|I_i(0)\right\|+\left\|I_i\left(\psi\left(\tau_i\right)\right)-I_i(0)\right\|\le$$
$$\le\sup_{i\in Z}\left\|I_i(0)\right\|+\sup_{i\in Z}L\left\|\psi\left(\tau_i\right)\right\|\le\sup_{i\in Z}\left\|I_i(0)\right\|+L\cdot r. \tag{14}$$

Therefore, the operator F that was built when proving the preceding theorem, can be considered on the set B_r. At the same time, if $\phi(t)=F\psi(t)$, where $\psi(t)\in B_r$, then, given the inequalities (10), (12) to (14), we have:

$$\sup_{t\in R}\left\|\phi(t)\right\|\le c\max\left(\sup_{t\in R}\left\|f(t,\psi(t))\right\|,\ \sup_{i\in Z}\left\|I_i\left(\psi\left(\tau_i\right)\right)\right\|\right)\le$$
$$\le c\cdot\max\left(\sup_{t\in R}\left\|f(t,0)\right\|,\ \sup_{i\in Z}\left\|I_i(0)\right\|\right)+c\cdot Lr\le r,$$

that is, $\sup_{t\in R}\left\|\phi(t)\right\|\le r$.

Therefore, $FB_r\subseteq B_r$.

The B_r set is bounded and closed. We shall show that the operator $F:B_r\to B_r$ is the compression operator. Indeed, let $\phi_1(t)$, $\phi_2(t)\in B_r$ and $\phi_1(t)=F\psi_1(t)$, $\phi_2(t)=F\psi_2(t)$, that is, the functions $\phi_1(t)$, $\phi_2(t)$ are the bounded solutions to the corresponding equations:

$$\frac{dx}{dt}=A(t)x+f\left(t,\psi_j(t)\right),\ t\ne\tau_i,$$
$$\Delta x\big|_{t=\tau_i}=B_i x+I_i\left(\psi_j\left(\tau_i\right)\right),\ (j=1,2). \tag{15}$$

The $\phi(t)=\phi_2(t)-\phi_1(t)$ function is a solution to the following equation, bounded along the entire axis,

$$\frac{dx}{dt}=A(t)x+f\left(t,\psi_2(t)\right)-f\left(t,\psi_1(t)\right),\ t\ne\tau_i,$$
$$\Delta x\big|_{t=\tau_i}=B_i x+I_i\left(\psi_2\left(\tau_i\right)\right)-I_i\left(\psi_1\left(\tau_i\right)\right),$$

that satisfies the condition $P\phi(0)=0$. Considering Theorem 1 [1] and conditions (3), we obtain an estimate:

$$\sup_{t\in R}\left\|F\psi_2(t)-F\psi_1(t)\right\|\le$$
$$\le c\max\left\{\sup_{t\in R}\left\|f\left(t,\psi(t)\right)-f\left(t,\psi_1(t)\right)\right\|,\right.$$
$$\left.\sup_{i\in Z}\left\|I_i\left(\psi_2(\tau_i)\right)-I_i\left(\psi_1(\tau_i)\right)\right\|\right\}\le$$
$$\le c\cdot L\cdot\sup_{t\in R}\left\|\psi_2(t)-\psi_1(t)\right\|.$$

If the Lipschitz constant $L<1/c$, then the operator $F: B_r\to B_r$ is the compression operator. According to Banach's theorem about the fixed representation point, F has a single fixed point. Therefore, there is a single solution $\phi(t)$, bounded along the entire axis, to equation (1), which satisfies the conditions $P\phi(0)=0$ and $\sup_{t\in R}\left\|\phi(t)\right\|\le r.$ The theorem is proved.

The proven theorem could be used when studying the existence of solutions, bounded along the entire axis, to the differential equations with a pulse effect of the following form:

$$\frac{dx}{dt}=A(t)x+f(t)+g(t,x,\varepsilon),\ t\ne\tau_i,$$
$$\Delta x\big|_{t=\tau_i}=B_ix+a_i+I_i(x,\varepsilon). \qquad (16)$$

Here, the matrices $A(t)$, B_i and the time constants τ_i are the same as in equations (1); $f(t)$ is the continuous (piecewise-continuous with discontinuities of the first kind at $t=\tau_i$) function bounded along the entire axis; $\{a_i\}$ is the bounded sequence. The function $g(t,x,\varepsilon)$ is continuous (piecewise-continuous with discontinuities of the first kind at $t=\tau_i$), continuous along x and ε, and, along x, they satisfy the Lipschitz condition. The $I_i(x, \varepsilon)$ functions are also continuous in the aggregate of their variables and satisfy the Lipschitz condition along x, ε – a small positive parameter. Assume the following:

$$\sup_{t\in R}\left\|g(t,0,\varepsilon)\right\|\le L(\varepsilon),\ \sup_{i\in Z}\left\|I_i(0,\varepsilon)\right\|\le L(\varepsilon),$$
$$\left\|g(t,x_1,\varepsilon)-g(t,x_2,\varepsilon)\right\|\le l(\varepsilon)\left\|x_1-x_2\right\|, \qquad (17)$$
$$\left\|I_i(x_1,\varepsilon)-I_i(x_2,\varepsilon)\right\|\le l(\varepsilon)\left\|x_1-x_2\right\|, \qquad (18)$$

for all x_1, x_2, such that $\left\|x_1\right\|\le r$, $\left\|x_2\right\|\le r$, where $L(\varepsilon)$ and $l(\varepsilon)$ are the non-positive ascending functions of the parameter ε, and $L(\varepsilon)\to 0$, $l(\varepsilon)\to 0$ at $\varepsilon\to 0$.

Theorem 3. Let the system of equations (5) be weakly regular along the entire real line and the number $r>C\cdot\max\left\{\sup_{t\in R}\left\|f(t)\right\|,\ \sup_{i\in Z}\left\|a_i\right\|\right\}$, where C is the constant of weak regularity of equations (5). Then we can specify such a positive number ε_0 that, for any $\varepsilon\in[0,\varepsilon_0]$, the system of equations (16) has a unique solution $\phi(t,\varepsilon)$, bounded along the entire axis, which satisfies the conditions $P\phi(0,\varepsilon)=0$ and $\sup_{t\in R}\left\|\phi(t,\varepsilon)\right\|\le r.$

In addition, the function $\phi(t,\varepsilon)$ is continuous along ε and $\lim_{\varepsilon\to 0}\phi(t,\varepsilon)=\phi_0(t)$, where $\phi_0(t)$ is the solution to equation (6), bounded along the entire axis, which satisfies the condition $P\phi_0(0)=0$.

The stated theorem is a consequence of the preceding theorem. Since $L(\varepsilon)\to 0$, $l(\varepsilon)\to 0$ at $\varepsilon\to 0$, there is such a number $\varepsilon_0>0$ that, at any $\varepsilon\in[0,\varepsilon_0]$, equation (16) satisfies the condition of Theorem 2.

Thus, equation (16) at any $\varepsilon\in[0,\varepsilon_0]$ has a single solution $\phi(t,\varepsilon)$, bounded at all $t\in R$, which satisfies the conditions $P\phi(0,\varepsilon)=0$ and $\sup_{t\in R}\left\|\phi(t,\varepsilon)\right\|\le r.$ In this case, at any $\varepsilon\in[0,\varepsilon_0]$, the function $\psi(t,\varepsilon)=\phi(t,\varepsilon)-\phi(t,0)$ is the solution to the system of the following equations, bounded along the entire axis,

$$\frac{dx}{dt}=A(t)x+g\left(t,\phi(t,\varepsilon),\varepsilon\right),\ t\ne\tau_i,$$
$$\Delta x\big|_{t=\tau_i}=B_ix+I_i\left(\phi(\tau_i,\varepsilon),\varepsilon\right),$$

that satisfies the condition $P\psi(0,\varepsilon)=0$.

Considering Theorem 1 [1], the following estimate holds:

$$\sup_{t\in R}\left\|\phi(t,\varepsilon)-\phi_0(t)\right\|\le C\cdot\max\left\{\sup_{t\in R}\left\|g\left(t,\phi(t,\varepsilon),\varepsilon\right)\right\|,\right.$$
$$\left.\sup_{i\in Z}\left\|I_i\left(\phi(\tau_i,\varepsilon),\varepsilon\right)\right\|\le C\cdot\left(L(\varepsilon)+l(\varepsilon)r\right).\right.$$

Because $L(\varepsilon)\to 0$, $l(\varepsilon)\to 0$ at $\varepsilon\to 0$, the function $\phi(t,\varepsilon)$, evenly with respect to $t\in R$, approaches $\phi_0(t)$ at $\varepsilon\to 0$.

5. 3. SIR-model of «pulse vaccination»

There are many mathematical models to effectively diagnose infectious disease, forecast, and examine the dynamics of the pathological process, which are widely used by modern medicine. One of these epidemiological models is the SIR model proposed in [12].

It divides the population into three groups:

– healthy individuals who are at risk and can catch an infection (denoted S – susceptible);

– infected persons who are carriers of the virus (denoted I – infected);

– recovered persons who have acquired permanent immunity to a given disease (denoted R – recovered).

We shall consider a SIR model for the «pulse vaccination» strategy This approach is to vaccinate a certain proportion p of the susceptible population S at regular intervals T. The basic idea of this strategy is to vaccinate a sufficient number of susceptible people and do so often to maintain the percentage of susceptible patients below the threshold required to start the epidemic (Fig. 1).

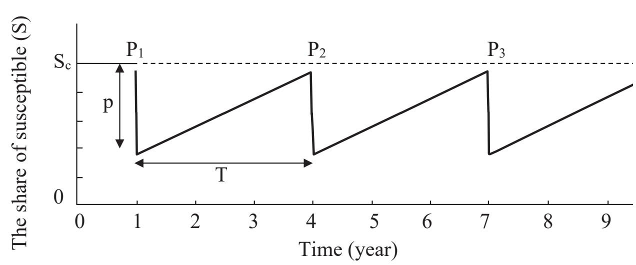

Fig. 1. Pulse vaccination scheme

Fig. 1 shows a scheme of pulse vaccination for the susceptible population over a certain time interval. Here, $P_1,\ldots,P_n$ are the vaccination pulses every T years, S_c is the threshold of the epidemic.

In terms of the SIR model, the mathematical component is described by a system of differential equations with a pulse influence:

$$\begin{cases} \dfrac{dS}{dt} = m - (\beta I + m)S, \\ \dfrac{dI}{dt} = \beta IS - (m+g)I, \\ \dfrac{dR}{dt} = gI - mR, \\ S(t_n) = (1-p)S(t_n - 0), \end{cases} \quad t_{n+1} = t_n + T.$$

where t_n is the time point in which we apply the n-th pulse of vaccination, t_n-0 is the time point immediately before the use of the n-th pulse, p is the proportion of the susceptible population vaccinated at time point $t=t_n$, T is the period between two consecutive vaccinations.

A typical solution to the SIR model employing the strategy of pulse vaccination is shown in Fig. 2. We can observe how the proportion of the population at risk $S(t)$ fluctuates in a stable cycle when using a pulse vaccine ($p=0.5$ and $T=2$). Those susceptible are involved in the periodic solution «without infection». Line $S_c \approx 0.0556$ denotes an «epidemic threshold».

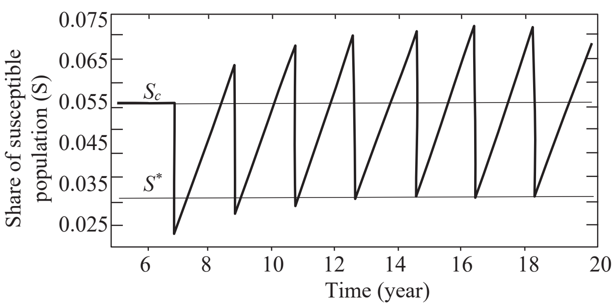

Fig. 2. Solution to the SIR-model employing a pulse vaccination strategy for risk group $S(t)$

In contrast, the proportion of infected population $I(t)$ is rapidly decreasing to zero, as shown in Fig. 3.

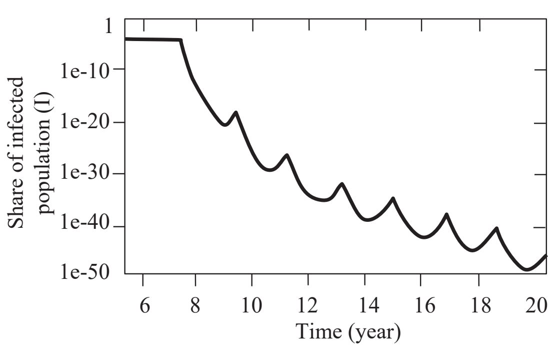

Fig. 3. Solution to the SIR model employing a pulse vaccination strategy for a group of infected individuals $I(t)$

The classic SIR model under real conditions cannot always accurately describe the results of an actual situation. The coefficients of the model, depending on the input data, may deviate from the classical condition. Therefore, there is a need to study the relevant systems characterized by weak nonlinearity. Our results could make it possible to determine and analyze solutions to the relevant systems.

Therefore, if we have enough vaccines to combat infectious disease, it would be reasonable that the susceptible population that is vaccinated every time is proportional to the number of susceptible people. However, such an approximation cannot reflect the real case. Usually, the number of susceptible persons requiring vaccination may exceed the possibility of local medical facilities due to a shortage of vaccines and doctors.

It is worth noting that work [13] reported an accurate analytical solution to the SIR model in a parametric form. Using an accurate solution, some explicit models corresponding to fixed parameter values were studied; it was shown that the numerical solution reproduces the analytical solution. It has been shown that the SIR model generalization described by the nonlinear system of differential equations can be reduced to an equation of the Abel type, which makes it much easier to analyze its properties.

6. Discussion of results of finding bounded solutions to the weakly nonlinear systems with a pulse action

The task to find and study bounded solutions for the nonlinear differential equations with a pulse action is important enough, though little studied in a general case. Classical works in this area focused mainly on the analysis of one-dimensional and two-dimensional systems of equations. In the study of multidimensional systems, it was necessary to devise another method based on the analysis of qualitative properties (regularity) of the corresponding linear homogeneous systems. These properties, on the one hand, could be effectively tested for wide classes of pulse systems, and, on the other hand, they make it possible to prove a series of quality properties for heterogeneous pulse-perturbed systems. The advantage of our method is that the conditions for the existence of bounded solutions to linear differential equations could be extended to classes of the weakly nonlinear pulse systems (Theorem 1). In addition, we have established conditions (Theorem 2) under which the existence and uniformity of bounded solutions for the weakly nonlinear systems of differential equations with a pulse action at fixed time points are warranted. The results from Theorem 2 have been extended to cover a more general case of piecewise-continuous functions with discontinuities of the first kind (Theorem 3).

Our results could be useful in studying the dynamics of pathological processes and to form pulse vaccination strategies in the compartmental SIR model. The established conditions for the existence of bounded solutions characterize the threshold values of the dynamics of pulse vaccination reaction against the spread of infections and could serve as markers for correcting the chosen vaccination strategy as a way to counteract the propagation of infection among the affected people.

In the future, based on the method devised, it is planned to study the stability of nonlinear pulse systems.

7. Conclusions

1. Using the concept of the regularity of a homogeneous system, we have established the conditions for the existence of a bounded solution for a weakly nonlinear system of differential equations with pulse action. The conditions obtained are sufficient. Note that the conditions are expressed directly through the coefficients of the original problem.

2. For a weakly nonlinear multidimensional system of differential equations with a pulse action, conditions for the

singleness of a bounded solution have been established. The obtained conditions make it possible to carry out qualitative analysis of the solutions to the relevant systems.

3. For the SIR model considered, a solution was derived for the case of pulse vaccination. Thus, for the case $p=0.5$ (part of the susceptible population that is vaccinated) and $T=2$ (a period between two consecutive vaccinations), it has been shown that the threshold of an epidemic is $S_c \approx 0.0556$. Part of the population $S(t)$, which is at risk, fluctuates in a stable cycle at pulse vaccination.

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