UDC 519:62-50
DOI: 10.15587/1729-4061.2021.241232

ANALYSIS OF ONE CLASS OF OPTIMAL CONTROL PROBLEMS FOR DISTRIBUTED-PARAMETER SYSTEMS

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In the paper, the method of straight lines approximately solves one class of optimal control problems for systems, the behavior of which is described by a nonlinear equation of parabolic type and a set of ordinary differential equations. Control is carried out using distributed and lumped parameters. Distributed control is included in the partial differential equation, and lumped controls are contained both in the boundary conditions and in the right-hand side of the ordinary differential equation. The convergence of the solutions of the approximating boundary value problem to the solution of the original one is proved when the step of the grid of straight lines tends to zero, and on the basis of this fact, the convergence of the approximate solution of the approximating optimal problem with respect to the functional is established.

A constructive scheme for constructing an optimal control by a minimizing sequence of controls is proposed. The control of the process in the approximate solution of a class of optimization problems is carried out on the basis of the Pontryagin maximum principle using the method of straight lines. For the numerical solution of the problem, a gradient projection scheme with a special choice of step is used, this gives a converging sequence in the control space. The numerical solution of one variational problem of the mentioned type related to a one-dimensional heat conduction equation with boundary conditions of the second kind is presented. An inequality-type constraint is imposed on the control function entering the right-hand side of the ordinary differential equation. The numerical results obtained on the basis of the compiled computer program are presented in the form of tables and figures.

The described numerical method gives a sufficiently accurate solution in a short time and does not show a tendency to «dispersion». With an increase in the number of iterations, the value of the functional monotonically tends to zero

Keywords: nonlinear boundary value problems, functional convergence, Pontryagin's maximum principle, minimizing sequence

Received date 25.08.2021
Accepted date 27.09.2021
Published date 29.10.2021

How to Cite: *Mamtiyev, K., Aliyeva, T., Rzayeva, U. (2021). Analysis of one class of optimal control problems for distributed-parameter systems. Eastern-European Journal of Enterprise Technologies, 5 (4 (113)), 26–33. doi: https://doi.org/10.15587/1729-4061.2021.241232*

1. Introduction

The foundations of theoretical researches and practical development in the field of distributed-parameter systems were first laid down in [1] more than half a century ago. Since then, over the years, control theory for distributed-parameter systems has been enriched with new ideas and results. Year by year, more and more works are published in its various sections. However, many important questions of the theory are not fully developed for optimal control problems for systems containing links with distributed parameters, the processes in which are described by boundary value problems for partial differential equations.

Optimal control theory is one of the main areas of practical use of mathematics. The rapid development of control theory for lumped-parameter systems is largely associated with the use of Pontryagin's maximum principle, Bellman's optimality principle, and Krasovsky's method of moments. At the same time, many real control objects have to be considered as distributed-parameter systems. The variety of spheres of application of the theory of distributed systems control, its methods and results is evidenced by its close connection with technical problems, with game theory and problems of positional control, with inverse problems of the dynamics of controlled systems.

There is no doubt that the Pontryagin maximum principle is one of the main mathematical tools for solving optimal control problems with lumped parameters. However, when considering a number of practically important problems, we have to examine distributed-parameter systems, where the process is described by boundary value problems for various types of partial differential equations. Such problems arise in the study of controlled processes of heat conduction, diffusion, filtration, etc. Despite the large flow of work on the control of distributed-parameter systems, many important questions still remain open. A review of works in the field of theory and applications of distributed-parameter control systems is described most extensively in [2]. This review covers the literature on numerical and approximate optimal control methods dating back to the 1960s. Of course, due to the large number of publications in this area, the bibliography of the presented paper is not exhaustive.

The results of the study can be used, in particular, in determining the optimal technological mode of gas well operation, subject to the depletion of the formation by a given time. Particular interest in this problem is presented for the operation of offshore fields, the service life of which is limited by the service life of the bottoms, that is, preliminarily preset.

2. Literature review and problem statement

The paper presents the results of certain system's study that provides a sequence of control actions for a certain class of control objects, which ensure the optimum of a given set of system quality criteria. Note that in most works devoted to the study of optimal control problems for systems containing objects with distributed parameters, the issues of numerical solution have not been sufficiently studied. In addition to the nonlinearity of boundary value problems, accounting various constraints imposed on the control actions and on the phase variables of the system necessitates the use of approximate optimization methods using computers.

It should be noted that the expediency of considering optimal control problems for systems containing links with distributed parameters is associated with the fact that control of an object with distributed parameters in real systems is carried out by control devices with lumped parameters. A cycle of papers [3–5] is devoted to the theoretical study of this type of optimal control problems.

In [3, 4], the problem of damping oscillations of a system described by a combination of a wave equation and an ordinary differential equation of the second order is considered, under the assumption that the control function and the object with lumped parameters act, respectively, on the left and right ends of the object with distributed parameters. The functions of the system states are related through the boundary conditions for the wave equation. The problem of oscillation control and the results are formulated, which determines the general solution of the boundary value problem. In [4], which is a continuation of [3], a solution of the boundary value problem was constructed and the control problem was solved. Further, using the method of straight lines, a finite-dimensional approximation of the boundary value problem is constructed and a criterion for the controllability of the system is found.

In [5], the problem of damping oscillations of a network consisting of *m* objects with distributed parameters is solved, in which a controlled object with lumped parameters acts through the boundary conditions at the point of connection of objects. The solution of the boundary value problem is constructed and the control problem is solved. Numerical calculations were not carried out, the work is presented as a purely theoretical study.

In [6], the problem of optimal control of the thermal regime of heated buildings is considered. The necessary optimality conditions are obtained, formulated in the form of the maximum principle. Computational aspects are analyzed and a method of approximate realization of optimal control is indicated. However, according to the method of successive approximations proposed in this work, which makes it possible to find the control action in the class of piecewise continuous and bounded functions, no calculations were carried out for specific initial data.

In [7], the problem of optimal control of processes described by a parabolic equation and sets of ordinary differential equations with controls of moving sources was investigated. Note that one of the main features of systems with control of moving sources is their nonlinearity with respect to the control that determines the motion law of the source. For the considered problem, the theorem of existence and uniqueness of optimal control is proved, necessary conditions of optimality are obtained in the form of point and integral maximum principles. Sufficient conditions for the Fréchet differentiability of the performance criterion are found and an expression for its gradient is obtained, however, the results are not applied to a specific applied problem of the type considered.

In [8], an approximate method is proposed for solving optimal control problems for one class of distributed systems, based on the use of perturbation theory. For a number of examples, the results of calculations of optimal processes are given. In this case, in order to construct an infinitesimal variation of controls, in addition to integrating the direct and adjoint boundary value problem, it is also necessary to solve linear programming problems. All the calculation formulas used in the construction of computational algorithms have been proved, although the convergence found by the proposed method of approximate solution has not been proven.

In [9], by regulating the bottomhole pressure in a certain interval, the technological mode of gas well operation was determined. The problem is reduced to the problem of optimal control of systems, the behavior of which is described by a one-dimensional equation of gas filtration in ~~a~~ porous media and an ordinary differential equation. Due to the nonlinearity of the gas filtration equations, it is not possible to prove the maximum principle and the use of the method of straight lines, the problem is reduced to a variational problem related to a system of ordinary differential equations. The calculation results are presented.

It should be noted that the above types of optimal control problems, especially in the case when the process is described by a set of nonlinear partial and ordinary differential equations, have been little studied. The main results obtained are related to the deduction of the necessary optimality conditions, the derivation of formulas for the gradient, and the proof of existence theorems. The practical use of these conditions, even in the simplest cases, leads to boundary value problems for partial differential equations, the exact solution of which cannot be obtained. Therefore, some authors, for example [10], usually restrict themselves to indicating some procedure for finding approximate solutions and it is not always proved that the approximate solution converges to the exact solution.

In this regard, when solving practically important above-mentioned types of optimal control problems using computational tools, it is very effective to use various approximate methods, in particular, the method of straight lines. This approach was used in [11], where, when approximating the heat conduction equations in phase variables, the problem associated with the choice of lumped, starting and distributed controls was reduced to solving a variational problem for systems of ordinary differential equations. In this paper and in the papers [6, 8, 10], the authors restricted themselves to indicating a method of approximate realization of the optimal control, while the convergence of the approximate solution was not proved. This suggests that it is advisable to analyze the above types of optimal control problems using computers.

It is also important to note that when studying optimal control problems for mixed nonlinear systems described by a set of ordinary and partial differential equations, it is not always possible to obtain optimality conditions in the form of the maximum principle. Moreover, when solving the problem approximately, the question of convergence of solutions with respect to control often remains open. The main mathematical difficulties in this case are directly related to the nonlinearity of boundary value problems for partial differential equations.

3. The aim and objectives of the study

The main goal of this study is the approximate solution of the optimal control problem for systems, the processes in which are described by rather general nonlinear boundary value problems of parabolic type in combination with the Cauchy problem for ordinary differential equations. Achievement of this goal makes it possible to choose such a solution (i.e., such a set of control functions) from all the solutions of the control problem that would be, in a sense, the most advantageous. Such systems have the best (in some way) properties compared to any other systems from a certain class.

This aim's achievement brings new challenges of solving the following objectives:

– to prove the convergence of the approximate solution of the approximating boundary value problem to the solution of the original one;

– to prove the convergence of the approximate solution of the approximating optimal problem in terms of the functional and to propose a constructive scheme for constructing a minimizing sequence of controls;

– to carry out the analysis of the numerical solution of the problem.

4. Materials and methods

In problems solved on high-speed computers, systems of nonlinear differential equations, either ordinary or partial, are most often encountered. The methods for the numerical solution of systems of ordinary differential equations are well developed, whereas the intensive creation of methods for solving partial differential equations began only after the advent of computers. In this paper, the method of straight lines – a universal method for solving systems of nonlinear partial differential equations is applied. The main idea of the method of straight lines is to reduce partial differential equations to solving a system of ordinary differential equations. This is the difference from the grid method, which directly reduces the solution of systems of partial differential equations to the solution of systems of algebraic equations. The method of straight lines can be used to solve partial differential equations of any type, but is mainly used to solve elliptic and parabolic equations. The adequacy of methods for the numerical solution of ordinary differential equations lies in the fact that they are well developed, while such methods for solving partial differential equations, which reduce them approximately to systems of ordinary differential equations (including the method of straight lines), acquire great practical importance. In the classical method of straight lines, the region of integration is divided into strips, usually by fixed straight lines, and the derivatives in one of the directions are replaced by finite-difference relations (usually linear). As a result, a system of ordinary differential equations is obtained, which is solved numerically. In the work, the numerical solution of the problem is accompanied by the development of appropriate software, which is correct, that is, ensures the solution of the problem. Using a computer, a numerical solution was obtained for one problem of this type related to thermal processes in a homogeneous rod. The method for solving the problem is based on the approximation of partial differential equations by ordinary differential equations.

A number of practically important problems lead to the need of studying optimal control problems for systems, the behavior of which can be described by various boundary value problems for nonlinear parabolic partial differential equations. Such problems, in particular, include the problems of optimal control of heating massive bodies [8], the problems of determining the technological mode of gas well operation, where the process is described by a nonlinear equation of unsteady gas filtration in porous media [9], and many others. In this case, the main difficulty, even in the presence of restrictions on the phase variables, is directly related to the nonlinearity of boundary value problems. Therefore, obtaining a numerical solution of such problems for both theoretical and practical purposes is of certain interest.

Now then, let in the area $Q=\{0\le x\le 1,\, 0\le t\le T\}$ some controlled processes be described by the boundary value problem of the following form:

$$u_t = a(x,t,u)u_{xx} + \\ + F(x,t,u,u_x,\alpha(x,t)),\, a(x,t,u)\ge a_0 = \mathrm{const}>0, \tag{1}$$

$$u_x(0,t)=\varphi^o(t,u(0,t),y^o(t),\beta^o(t)),\ t>0, \tag{2}$$

$$u_x(1,t)=\varphi^1(t,u(1,t),y^1(t),\beta^1(t)),\ t>0, \tag{3}$$

$$u(x,0)=u_0(x),\ 0\le x\le 1, \tag{4}$$

where $a(x,t,p)$, $F(x,t,p,q,\alpha)$, $k=0,1$ are given continuous and sufficiently smooth functions with a collection of their arguments, and besides a, a_p, F, F_p, F_{q_x}, F_α, φ_p^k, φ_y^k, φ_β^k are uniformly bounded and continuous by t. The function characterizing the initial state of a distributed object is continuous. Distributed and lumped controls take values from some closed areas, defined, for example, by the inequalities $|\alpha|\le 1, |\beta^k|\le 1$. The functions $y=y^k(t)$ satisfy the differential equations:

$$\dot{y}^k = f^k(t,u(k,t),y^k(t),\beta^k(t)), \tag{5}$$

with the initial conditions:

$$y^k(0)=y_o^k, \tag{6}$$

where the right-hand sides of equations (5) satisfy the same conditions as functions on the right-hand sides of conditions (2) and (3), and y_o^k are given constants.

The top three functions $P(x,t)=(a(x,t),\beta^o(t),\beta^1(t))$ will be called admissible control if:

– distributed and lumped controls take values from the respective areas;

– lumped controls have a finite number of discontinuity points of the first kind;

– distributed control has a finite number of non-intersecting smooth discontinuity lines in the area Q.

The problem of finding the function $u(x, t)$, $y^k(t)$ from conditions (1)–(6) with a fixed control is called a direct problem. We will assume that each admissible control corresponds to a unique solution of the direct problem (1)–(6), and a small change in control corresponds to a small change in its solution.

It is required to find such an admissible control $\bar{P}(x,t)=(\bar{\alpha}(x,t),\bar{\beta}^o(t),\bar{\beta}^1(t))$ and the corresponding solution of the problem (1)–(6), so that the functional:

$$S(P(x,t))=\int_o^1 G(x,u(x,T))\mathrm{d}x+\varphi(y^o(T),y^1(T)), \tag{7}$$

took the smallest possible value, where $G(x, p)$, $\varphi(y^0, y^1)$ are given continuously differentiable functions of their arguments.

Necessary optimality condition in the problem (1)–(7), in the case when the coefficient in front of the second-order derivative in (1) is constant. Boundary value problems similar to (1)–(6) describe many processes, including heating a massive body in an inertial furnace, processes of underground fluid dynamics and many others [6–9]. There is no doubt, that the problem of optimal control of systems, the behavior of which can be described by a set of ordinary and partial differential equations with additional conditions, represents a certain theoretical interest, especially for practical presentation.

The solution of the above problem implies the solution of the following tasks, which we have identified in separate subsections.

5. Results of research of optimal control problems for nonlinear distributed-parameter systems

The main difference between the considered problem and the previously presented ones is that in order to prove the convergence of the approximate solution, at least in terms of the functional, this paper shows, first of all, the convergence of the approximating boundary value problem to the solution of the original one.

5. 1. Convergence of the approximate solution of the approximating boundary value problem to the solution of the original one

In this section, the uniform convergence of the approximate solution of the approximating boundary value problem to the solution of the original one will be proven.

In order to approximately solve the problem (1)–(7), the method of straight lines is used. Let $\bar{\omega}_h$ be a uniform grid of straight lines in a given segment of the spatial variable with nodal points $x_h^i = ih,\ i = 0,1,\dots,n,\ nh = 1$. Let us denote by $\varphi_h^i(t)$ the value of an arbitrary function in the nodes x_h^i of the grid $\bar{\omega}_h$, and, at the nodes of this grid, we replace the direct problem (1)–(6) with the system of differential-difference equations:

$$\frac{du_h^0}{dt} = 2a\left(0,t,u_h^0\right)\left[\frac{u_h^1 - u_h^0}{h^2} - \frac{\varphi^0\left(t,u_h^0,y_h^0,\beta^0(t)\right)}{h}\right] +$$
$$+ F\left(0,t,u_h^0,\varphi^0\left(t,u_h^0,y_h^0,\beta^0(t)\right),\alpha_h^0(t)\right),$$
$$\frac{du_h^i}{dt} = a\left(x_h^i,t,u_h^i\right)\frac{\Delta^2 u_h^i}{h^2} +$$
$$+ F\left(x_h^i,t,u_h^i,\frac{\Delta_c u_h^i}{2h},\alpha_h^i(t)\right),\ i = 1,2,\dots,n-1, \quad (8)$$
$$\frac{du_h^n}{dt} = 2a\left(1,t,u_h^n\right)\left[\frac{u_h^{n-1} - u_h^n}{h^2} + \frac{\varphi^1\left(t,u_h^n,y_h^1,\beta^1(t)\right)}{h}\right] +$$
$$+ F\left(1,t,u_h^n,\varphi^1\left(t,u_h^n,y_h^1,\beta^1(t)\right),\alpha_h^n(t)\right),$$
$$\frac{dy_h^0}{dt} = f^0\left(t,u_h^k(t),y_h^0(t),\beta^0(t)\right),$$
$$\frac{dy_h^1}{dt} = f^1\left(t,u_h^n(t),y_h^1(t),\beta^1(t)\right),$$

with the initial conditions:

$$u_h^i(0) = u_0\left(x_h^i\right),\ i = 0,1,\dots,n,\ y_h^k(0) = y_0^k,\ k = 0,1. \quad (9)$$

Thus, the considered problem (1)–(7) is reduced to the choice of the function $\bar{P}_h^i(t) = \left(\bar{\alpha}_h^i(t),\bar{\beta}_h^0(t),\bar{\beta}_h^1(t)\right)$ from the conditions for the minimum of the functional:

$$S_h\left(P_h^i(t)\right) = h\sum_{i=0}^{n-1} G\left(x_h^i,u_h^i(T)\right) + \varphi\left(y_h^0(T),y_h^1(T)\right), \quad (10)$$

subject to constraints (8), (9).

Let us denote by:

$$\delta_h^i(t) = u\left(x_h^i,t\right) - u_h^i(t),\ i = 0,1,\dots,n,\ \eta_h^k(t) = y^k(t) - y_h^k(t),$$

where $u\left(x_h^i,t\right)$, $y^k(t)$ is the exact solution of the direct problem (1)–(6), and $u_h^i(t)$, $y_h^k(t)$ is the solution of the differential-difference problem (8), (9), and introduce the n+3-dimensional vector $\nu_h(t)$ with the components:

$$\nu_k^i(t) = \delta_k^i(t),\ i = 0,1,\dots,n,\ \nu_k^{n+1}(t) = \eta_k^0(t),\ \nu_h^{n+2}(t) = \eta_h^1(t).$$

Substituting the exact solution $u\left(x_h^i,t\right)$, $y^k(t)$ of the direct problem (1)–(6) into (8), (9) and subtracting (8), (9) from the obtained relations, compose the system of linear inhomogeneous equations for errors:

$$\frac{d\nu_h^0}{dt} = 2a\left(0,t,u_h^0\right)\left[\frac{\nu_h^1 - \nu_h^0}{h^2} - \frac{\tilde{\varphi}_u^0\nu_h^0}{h^2} - \frac{\tilde{\varphi}_y^0\nu_h^{n+1}}{h^2}\right] +$$
$$+\left[\tilde{a}_p\tilde{u}_{xx} + \tilde{F}_p + \tilde{F}_q\tilde{\varphi}_u^0\right]\nu_h^0 + \tilde{F}_q\tilde{\varphi}_y^0\nu_h^{n+1} + O(h),$$
$$\frac{d\nu_h^i}{dt} = a\left(x_h^i,t,u_h^i\right)\frac{\Delta^2\nu_h^i}{h^2} + \left[\tilde{a}_p\tilde{u}_{xx} + \tilde{F}_p\right]\nu_h^i +$$
$$+ \tilde{F}_q\cdot\frac{\Delta_c\nu_h^i}{2h} + O\left(h^2\right),\ i = 1,2,\dots,n-1, \quad (11)$$
$$\frac{d\nu_h^n}{dt} = a\left(1,t,u_h^n\right)\left[\frac{\nu_h^{n-1} - \nu_h^n}{h^2} - \frac{\tilde{\varphi}_u^1\nu_h^n}{h^2} - \frac{\tilde{\varphi}_y^1\nu_h^{n+2}}{h^2}\right] +$$
$$+\left[\tilde{a}_p\tilde{u}_{xx} + \tilde{F}_p + \tilde{F}_q\tilde{\varphi}_u^1\right]\nu_h^n + \tilde{F}_q\tilde{\varphi}_y^1\nu_h^{n+2} + O(h),$$
$$\frac{d\nu_h^{n+1}}{dt} = \tilde{f}_u^0\nu_h^0 + \tilde{f}_y^0\nu_h^{n+1} + O(h),$$
$$\frac{d\nu_h^{n+2}}{dt} = \tilde{f}_u^1\nu_h^n + \tilde{f}_y^1\nu_h^{n+2} + O(h).$$

with zero initial data:

$$\nu_h^i(0) = 0,\ i = 0,1,\dots,n+2, \quad (12)$$

where the sign «~» denotes the values of derivatives at intermediate points. Applying the known a priori estimate for solutions of a system of linear inhomogeneous ordinary differential equations for the solution of the system (11), (12), we have:

$$\max_{0\le i\le n+2}\left|\nu_h^i(t)\right| \le \frac{O(h)}{C}\left[e^{CT} - 1\right],\ C = \text{const} > 0. \quad (13)$$

From this estimate, we find that the solution of the differential-difference problem (8), (9) for $h\to 0$ converges at a rate $O(h)$ to the solution of the direct problem (1)–(6).

5. 2. Convergence of the approximate solution of the approximating problem by the functional and construction of a minimizing sequence

In this subsection, the convergence of the approximate solution of the approximating optimal problem with respect to the functional will be proven and the assumption that the sequence of controls constructed according to the proposed scheme is minimizing will be clearly established.

If we denote:

$$S_h^0 = \inf_{P_h^i} S_h\left(P_h^i\right) = S_h\left(\bar{P}_h^i(t)\right),$$

then, using the estimate (13) and the properties of the functions $G(x,u)$, $\varphi\left(y^0,y^1\right)$ in inequality:

$$\left|S\left(P(x,t)\right) - S_h\left(P_h^i(t)\right)\right| \le \sum_{i=0}^{n-1} \int_{x_h^i}^{x_h^{i+1}} \left| \begin{array}{l} G\left(x,u(x,T)\right) - \\ - G\left(x_h^i,u_h^i(T)\right) \end{array} \right| \mathrm{d}x +$$

$$+\left|\varphi\left(y^0(T),y^1\right)(T) - \varphi\left(y_h^0(T),y_h^1(T)\right)\right|,$$

it can be shown that:

$$\lim_{h\to 0} S_h^0 = \min_P S(P) = S\left(\bar{P}(x,t)\right),$$

that is, functional convergence takes place.

THEOREM. Let:

$$\bar{P}_h^i(t) = \left(\bar{\alpha}_h^i(t), \bar{\beta}_h^0(t), \bar{\beta}_h^1(t)\right),$$

be the optimal control in the approximating problem (8)–(10), and $\bar{\alpha}_h^i(x,t)$ is a continuation of the function $\bar{\alpha}_h^i(t)$ from the grid $\bar{\omega}_h$ for the whole area Q. Then the sequence $\bar{P}_h^i(x,t) = \left(\bar{\alpha}_h^i(x,t), \bar{\beta}_h^0(t), \bar{\beta}_h^1(t)\right)$ is minimizing for the functional (7) in the problem (1)–(7).

PROOF. Let h_m be some sequence of positive numbers that tends to zero as $m\to\infty$, and

$$\bar{P}_{h_m}^i(t) = \left(\bar{\alpha}_{h_m}^i(t), \bar{\beta}_{h_m}^0(t), \bar{\beta}_{h_m}^1(t)\right),$$

be the sequence of optimal solutions of the problem (8)–(10), corresponding to h_m. Let's continue the functions $\bar{\alpha}_{h_m}^i(t)$ from the grid of straight lines $\bar{\omega}_h$ for the whole area Q, assuming $\bar{\alpha}_{h_m}^i(x,t) = \bar{\alpha}_{h_m}^i(t)$, for $x_h^i \le x \le x_h^{i+1}$, $0 \le t \le T$, $i = 0,1,\ldots,n-1$.

Let us prove that the sequence:

$$\bar{P}_{h_m}^i(x,t) = \left(\bar{\alpha}_{h_m}^i(x,t), \bar{\beta}_{h_m}^0(t), \bar{\beta}_{h_m}^1(t)\right),$$

as $m\to\infty$ is minimizing for the functional (7).

For definiteness, we assume that the functional (7) has a finite infimum on the set of admissible controls. Let:

$$P_m(x,t) = \left(\alpha_m(x,t), \beta_m^0(t), \beta_m^1(t)\right),$$

be some minimizing sequence for the functional (7), that is:

$$\lim_{m\to\infty} S\left(P_m(x,t)\right) = \inf_P S(P) < +\infty. \quad (14)$$

In the approximating optimal problem (8)–(10) instead of:

$$P_h^i(t) = \left(\alpha_h^i(t), \beta_h^0(t), \beta_h^1(t)\right),$$

substitute $P_m\left(x_{h_m}^i,t\right) = \left(\alpha_m\left(x_{h_m}^i,t\right), \beta_m^0(t), \beta_m^1(t)\right)$.

Considering that:

$$\bar{P}_{h_m}^i(t) = \left(\bar{\alpha}_{h_m}^i(t), \bar{\beta}_{h_m}^0(t), \bar{\beta}_{h_m}^1(t)\right)$$

is the optimal control for the problem (8)–(10), we have:

$$S_{h_m}\left(\bar{P}_{h_m}^i(t)\right) \le S_{h_m}\left(P_m\left(x_{h_m}^i,t\right)\right). \quad (15)$$

Since the solution of the problem (8), (9) converges uniformly to the solution of the direct problem (1)–(6), the value of the approximating functional (10) converges to the value of (7). Therefore, for any small positive number, we can specify a natural number such that for any natural number greater than the indicated one, the following inequalities hold:

$$\left|S_{h_m}\left(\bar{P}_{h_m}^i(t)\right) - S\left(\bar{P}_{h_m}(x,t)\right)\right| < \varepsilon, \quad (16)$$

$$\left|S_{h_m}\left(P_m\left(x_{h_m}^i,t\right)\right) - S\left(P_m(x,t)\right)\right| < \varepsilon, \quad (17)$$

Considering in equality:

$$S\left(\bar{P}_{h_m}(x,t)\right) - S\left(P_m(x,t)\right) = S\left(\bar{P}_{h_m}(x,t)\right) -$$

$$-S_{h_m}\left(\bar{P}_{h_m}^i(t)\right) + S_{h_m}\left(\bar{P}_{h_m}^i(t)\right) - S\left(P_m(x,t)\right),$$

relation (15)–(17) for any $m>N$, we have:

$$S\left(\bar{P}_{h_m}(x,t)\right) \le S\left(P_m(x,t)\right) + 2\varepsilon. \quad (18)$$

Inequality (18) is valid for an arbitrary minimizing sequence; therefore, this implies that $\bar{P}_{h_m}(x,t)$ is also a minimizing sequence of controls for (7) in the problem (1)–(7), which is what we wanted to prove.

Thus, it has been proven that the sequence of controls constructed according to the proposed scheme is minimizing.

Comment. Note that similar results are also valid in the case of the first boundary value problem, if the boundary conditions for the functions states of a distributed object are taken as:

$$u(k,t) = \varphi^k\left(t, y^k(t)\right), \; k = 0,1, \; t > 0, \quad (19)$$

and $y^k(t)$ satisfies the equations:

$$\dot{y}^k = f^k\left(t, y^k(t), \beta^k(t)\right). \quad (20)$$

The outlined scheme remains applicable in the case when in the boundary conditions (2) and (3) the values of the states of the distributed object are set at one end of the segment, and the values of its change in the phase variable – at the other one.

Thus, the method of straight lines makes it possible to replace the original control problem with a variational problem for systems of ordinary differential equations.

5. 3. Analysis of the numerical solution of the heat transfer problem

This subsection provides a numerical solution to one variational problem related to thermal processes in a homogeneous rod.

Consider a problem that can be formulated in thermophysical terms as follows. Let there be a homogeneous rod with uniform length, the left end of which is heat insulated, and at the right end there is heat exchange with the external environment, and there are no heat sources in the rod. The temperature in the rod changes depending on the time and position of its coordinates. Let the ambient temperature be controlled by fuel consumption, which is related to the ambient temperature by virtue of a linear ordinary differential equation. It is assumed that the temperature distribution in the rod at the initial moment of time $t=0$ is equal to zero. It is required to the end of the process, by controlling the fuel consumption, without taking it beyond the maximum and minimum possible, to make the temperature distribution in the rod as close as possible to the given distribution.

The problem is mathematically reduced to the choice of the flow rate functions from the condition of the minimum of the quadratic functional:

$$S(\beta)=\int_0^1\left[u(x,T)-u^*(x)\right]^2 \mathrm{d}x, \tag{21}$$

under the following restrictions:

$$u_t=u_{xx},\ 0<x<1,\ 0<t\le T, \tag{22}$$

$$u(x,0)=0,\ 0\le x\le 1, \tag{23}$$

$$u_x(0,t)=0,\ 0<t\le T, \tag{24}$$

$$u_x(1,t)=\alpha\left[y(t)-u(1,t)\right],\ 0<t\le T,\ \alpha=\text{const}>0, \tag{25}$$

$$\nu y'+y=\beta(t),\ 0\le\beta(t)\le 1,\ 0<t\le T,\ \nu=\text{const}>0, \tag{26}$$

$$y(0)=1. \tag{27}$$

Note that this example is taken from [10], where a mathematical programming method is presented as applied to the problem (22)–(27). In [8], based on the use of the results of perturbation theory, a numerical solution of this problem was given without and with taking into account the constraints of the system phase coordinate.

When the boundary value problem (22)–(27) is approximated by the method of straight lines, the problem is reduced to solving a variational problem related by ordinary differential equations:

$$\begin{aligned}
&\frac{dz_h^0}{dt}=\frac{2}{h^2}\left[-z_h^0+z_h^1\right],\\
&\frac{dz_h^i}{dt}=\frac{1}{h^2}\left[z_h^{i-1}-2z_h^i+z_h^{i+1}\right],\ i=1,2,\dots,n-1,\\
&\frac{dz_h^n}{dt}=\frac{2}{h^2}\left[z_h^{n-1}-(1+\alpha h)z_h^n+\alpha h z_h^{n+1}\right],\\
&\frac{dz_h^{n+1}}{dt}=\frac{1}{\nu}\left[\beta(t)-z_h^{n+1}\right],
\end{aligned} \tag{28}$$

with zero initial conditions:

$$z_h^i(0)=0,\ i=0,1,\dots,n,\ z_h^{n+1}(0)=1. \tag{29}$$

Here $z_h=\left(z_h^0,z_h^1,\dots,z_h^n,z_h^{n+1}\right)$ is the n+2-dimensional vector with the components $z_h^i(t)=u_h^i(t),\ i=0,1,\dots,n,\ z_h^{n+1}(t)=y_h(t)$, and $u_h^i(t)$ are approximate values of $u\left(x_h^i,t\right)$ on the straight lines $x_h^i=ih,\ i=0,1,\dots,n.$

It is required to choose fuel consumption so that the sum:

$$S_h(\beta)=h\sum_{i=0}^{n-1}\left[z_h^i(T)-u^*\left(x_h^i\right)\right]^2 \tag{30}$$

takes a minimum value.

According to the usual presentation of the Pontryagin maximum principle, we compose the system of conjugate equations:

$$\begin{aligned}
&\frac{d\psi_h^0}{dt}=-\frac{\partial H}{\partial z_h^0}=-\frac{1}{h^2}\left[-2\psi_h^0+\psi_h^1\right],\\
&\frac{d\psi_h^1}{dt}=-\frac{\partial H}{\partial z_h^1}=-\frac{1}{h^2}\left[2\psi_h^0-2\psi_h^1+\psi_h^2\right],\\
&\frac{d\psi_h^i}{dt}=-\frac{\partial H}{\partial z_h^i}=-\frac{1}{h^2}\left[\psi_h^{i-1}-2\psi_h^{1i}+\psi_h^{i+1}\right],\ i=2,3,\dots,n-2,\\
&\frac{d\psi_h^{n-1}}{dt}=-\frac{\partial H}{\partial z_h^{n-1}}=-\frac{1}{h^2}\left[\psi_h^{n-2}-2\psi_h^{n-1}+2\psi_h^n\right],\\
&\frac{d\psi_h^n}{dt}=-\frac{\partial H}{\partial z_h^n}=-\frac{1}{h^2}\left[\psi_h^{n-1}-2(1+\alpha h)\psi_h^n\right],\\
&\frac{d\psi_h^{n+1}}{dt}=-\frac{\partial H}{\partial z_h^{n+1}}=-\frac{1}{h^2}\left[2\alpha h\psi_h^n-\frac{h^2}{\nu}\psi_h^{n+1}\right],
\end{aligned} \tag{31}$$

where H is the Hamilton-Pontryagin function of the problem (28)–(30).

It is important to note that the system of equations (28) can be obtained by approximating the boundary value problem conjugate to (22)–(27) by the method of lines:

$$\begin{aligned}
&\psi_t=-\psi_{xx},\ 0<x<1,\ 0\le t<T,\\
&\psi(x,T)=2\left[u(x,T)-u^*(x)\right],\ 0\le x\le 1,\\
&\psi_x(0,t)=0,\ t>0,\\
&\psi_x(1,t)+\alpha\psi(1,t)=0,\ \alpha=\text{const}>0,\ t>0,\\
&\nu\varphi'-\varphi=-\alpha\nu\psi(1,t),\ \nu=\text{const}>0,\ 0\le t<T,\\
&\varphi(T)=0,
\end{aligned} \tag{32}$$

that is, the system of equations (28) can be composed in two ways. Boundary conditions for the system of equations (31) and $\partial H/\partial\beta$ have the form:

$$\begin{aligned}
&\psi_h^i(T)=2h\left[z_h^i(T)-u^*\left(x_h^i\right)\right],\ i=0,1,\dots,n-1,\\
&\psi_h^n(T)=\psi_h^{n+1}(T)=0.
\end{aligned} \tag{33}$$

$$\partial H/\partial\beta=\frac{\psi_h^{n+1}}{\nu}. \tag{34}$$

For the numerical solution of the problem (28)–(30), the gradient projection scheme was used. In this case, the transition from $\beta^k(t)$ to the next iteration is carried out according to the rule:

$$\beta^{k+1}(t)=\begin{cases}\beta^k(t)-\delta\beta^k(t),\ \text{if}\ 0\le\beta^k(t)-\delta\beta^k(t)\le 1,\\ 1,\ \text{if}\ \beta^k(t)-\delta\beta^k(t)>1,\\ 0,\ \text{if}\ \beta^k(t)-\delta\beta^k(t)<0,\end{cases} \tag{35}$$

based on the construction of infinitesimal variations of controls, chosen by the formula:

$$\delta\beta^k(t)=\lambda\cdot\frac{\partial H^k/\partial\beta}{\left|\partial H^k{}_0/\partial\beta\right|},\ k=0,1,\ldots, \quad (36)$$

where the denominator of the fraction in formula (36) is the maximum element of the array consisting of the values of the derivative of the Hamilton functions with respect to control in a given time interval.

From the conditions $\psi_h^{n+1}(T)=0$ and expressions (35), (36), it follows that in the given example, variations of controls at the right end of the time interval are equal to zero. Therefore, when calculating by the formulas (35), (36), approximate optimal control of the values of the flow functions at the right end of the time interval during iterations will not change.

Some calculations were carried out according to schemes (35), (36) on the basis of programs written in the QBasic language. In our calculations, it was assumed that $\alpha=10$, $T=0.2$, $h=0.25$, $\nu=0.04$. The system of equations (28), (29) and the conjugate system (31), (33) were integrated by the Runge-Kutta method with a constant step. For the initial approximation, a linear function $\beta^0(t)$ was taken. To check the optimality of the optimal control found according to the scheme (35), (36), as $u^*\left(x_h^i\right)$ the solution of the problem (28), (29) was taken for a given control. In this case, the value of the minimum of the functional is equal to zero. In this case, the found approximately optimal control excluding the end of the time interval $0\le t\le T$ coincides with the given optimal control. This is due to the fact that for all iterations $\psi_h^{n+1}(T)=0$.

Fig. 1 shows the results of calculations of the problem (21)–(27) with the following functions $\beta^*(t)$, $\beta(t)$, $u(1, t)$, $y(t)$.

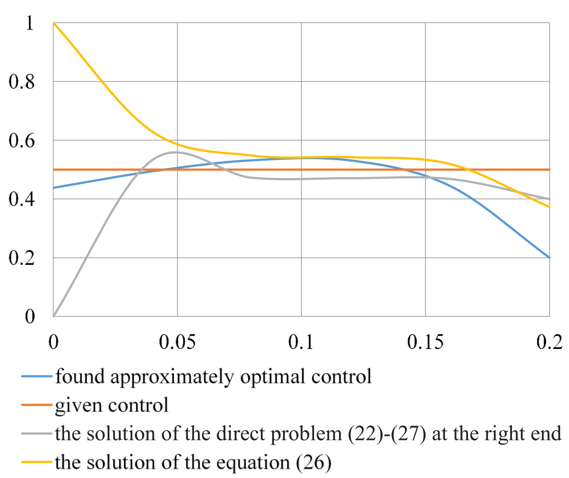

Fig. 1. Calculations of the problem (21)–(27)

The convergence of the process is as follows (Table 1).

Calculations were also carried out in the case when as $u^*\left(x_h^i\right)$ the solution of the problem (28), (29) was taken with relay optimal control with one switching point:

$$\beta^*(t)=\begin{cases}0, & \text{if } 0\le t\le 0.1,\\ 1, & \text{if } 0.1<t\le 0.2.\end{cases}$$

At the same time, the qualitative picture of the results did not change, the values of the functional decreased in approximately the same way as in the table. After 70 iterations, the value of the functional turned out to be practically zero, and the approximate optimal control obtained in this case, as can be seen from the given graphs (Fig. 2), with an increase in the number of iterations approaches the given relay control.

Table 1

Convergence of the process

No. of iteration	$S_h(\beta)$	$\min(\partial H/\partial\beta)$
0	$1.7870\cdot10^{-2}$	–0.61718
1	$1.7167\cdot10^{-2}$	–0.60452
5	$1.4496\cdot10^{-2}$	–0.55383
10	$1.1146\cdot10^{-2}$	–0.49029
20	$6.4572\cdot10^{-3}$	–0.36235
30	$2.8531\cdot10^{-3}$	–0.23250
40	$6.9710\cdot10^{-4}$	–0.10100
50	$7.3066\cdot10^{-5}$	–0.01401
60	$1.3448\cdot10^{-6}$	–0.00077

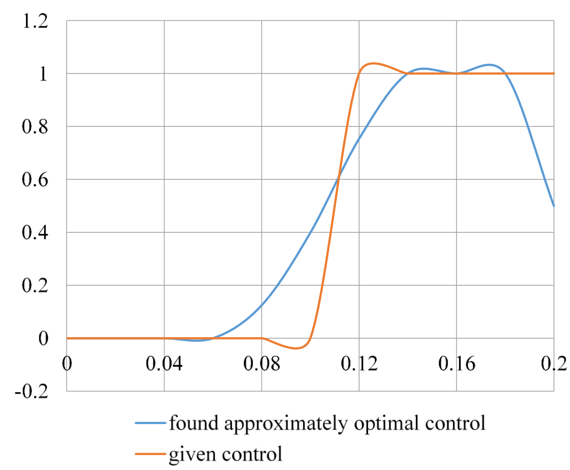

Fig. 2. Approximately optimal control

Note that in the process of solving the problem (28)–(30), it turned out that $\partial H/\partial\beta\le 0$, therefore, to avoid looping, as $\partial H_0/\partial\beta$ in formula (36), the minimum element of the array $\partial H/\partial\beta$ was taken by its absolute value for $0\le t\le T$ [11].

6. Discussion of the results of studying the solvability of one class of nonlinear optimization problems

In this paper, we considered a range of problems related to the direct problems of studying controlled distributed-parameter systems. Here, the direct problem was understood as a problem in which it was required to find controls providing a certain quality of changes in some dynamic system or its states. These tasks were based on numerous practical problems; they found ever wider application in solving various scientific, technical and national economic problems. This is due to the expanding possibilities of more adequate modelling of real processes, as well as the variety and depth of theoretical developments. The study of the solvability of nonlinear optimization problems and the development of constructive methods for their solution is one of the urgent problems, based on the theory of optimal control of distributed-parameter systems.

A discussion (or, in any case, a mention with an indication of the relevant literature) of methods for solving such problems is given, an analysis of their distinctive features and capabilities is carried out.

In addition to the result (formulated as a theorem) given for the case of ordinary differential equations, the following results are also reflected:

1. The approach based on the reduction of the original control system to the control problem described by ordinary differential equations seems to be promising as applied to the optimization of problems similar to (1)–(7). In this case, it is not very important to consider the conjugate boundary value problem for partial differential equations. It is important that the solution of the approximating boundary value problem converges to the solution of the original one, on the basis of which it is possible to prove the convergence of the approximate solution with respect to the functional.

2. In contrast to the linearity of the boundary value problem, which describes the control process when considering nonlinear boundary value problems, it is necessary to save its solutions in the RAM of the machine, since the integration of the adjoint system is impossible without this.

3. For problems in which the functional of the final state of the system is minimized, the gradient projection method gives a converging sequence even for ill-posed optimal problems.

In conclusion, we highlight some unsolved problems that may be of interest to researchers in this scientific field.

1. With the exception of some special cases, the boundary value problem (1)–(6), apparently, has not been studied in general form, and questions of its solvability for any admissible control are of interest.

2. The proof of the questions of convergence of the solution of the approximating optimal control problem is not always successful and is of particular interest in solving those applied problems where, in addition to the minimum of the objective functions, the control action itself is sought.

3. Obtaining the necessary optimality condition in the form of the Pontryagin maximum principle for the problem (1)–(7).

7. Conclusions

1. Uniform convergence of solutions of the approximating boundary value problem, expressed by differential-difference equations, to the solution of the direct problem was proven. This fact is of paramount importance in proving the convergence of an approximate solution, at least in terms of the functional.

2. The convergence of the approximate solution of the approximating optimal problem with respect to the functional was proven and the assumption that the sequence of controls constructed according to the proposed scheme was minimizing had been clearly established. In this case, the method of constructing distributed control over components $\overline{\alpha}_h^i(t)$ depends on the class of admissible controls. For example, if distributed control is sought in the class of functions that have the first derivative with respect to phase variables, then linear interpolation can be used to construct distributed control.

3. The analysis of the numerical solution of the control problem associated with the temperature distribution in a homogeneous thin rod was carried out. When the equations were approximated by the method of straight lines, the problem was reduced to solving a variational problem related to ordinary differential equations. For its numerical solution, a gradient projection scheme, based on the construction of infinitesimal variations of controls, was applied. With the variation of the initial data, the qualitative picture of the results did not change, and the gradient method for ill-posed optimal control problems gives a converging minimizing sequence. After 60 iterations of the gradient projection method, the value of the functional turned out to be ~$1.3 \cdot 10^{-6}$, and the found approximately optimal control excluding the end of the time interval coincides with the specified optimal control.

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