Solving the problem of continuum mechanics has revealed the defining generalizations using the function argument method. The aim of this study was to devise new approaches to solving problems of continuum mechanics using defining generalizations in the Cartesian coordinate system.

Additional functions, or the argument of the coordinates function of the deformation site, are introduced into consideration. The carriers of the proposed function arguments should be basic dependences that satisfy the boundary or edge conditions, as well as functions that simplify solving the problem in a general form.

However, there are unresolved issues related to how not the solutions themselves should be determined but the conditions for their existence. Such generalized approaches make it possible to predict the result for new applied problems, expand the possibilities of solving them in order to meet a variety of boundary and edge conditions.

The proposed approach makes it possible to define a series of function arguments, each of which can be a condition of uniqueness for a specific applied problem. Such generalizations concern determining not the specific functions but the conditions of their existence. From these positions, the flat problem was solved in the most detailed way, was tested, and compared with the studies reported by other authors.

Based on the result obtained, a mathematical model of the flat applied problem of the theory of elasticity with complex boundary conditions was built. Expressions that are presented in coordinateless form are convenient for analysis while providing a computationally convenient context. The influence of the beam shape factor on the distribution of stresses in transition zones with different intensity of their attenuation has been shown.

By bringing the solution to a particular result, the classical solutions have been obtained, which confirms its reliability. The mathematical substantiation of Saint-Venant's principle has been constructed in relation to the bending of a beam under variable asymmetric loading

Keywords: generalized approaches, function argument, Cartesian coordinates, Laplace equations, Cauchy-Riemann relations

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ADVANCING A GENERALIZED METHOD FOR SOLVING PROBLEMS OF CONTINUUM MECHANICS AS APPLIED TO THE CARTESIAN COORDINATE SYSTEM

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1. Introduction

Underlying the development of the economy of any state is the introduction and use of advanced technical solutions in various areas of production. Devising and implementing them pose new more complex theoretical and practical tasks for researchers that need to be solved. That necessitates the development of new, more effective approaches to obtaining the result and identifying simplifying generalizations in the process of solving the problem itself.

Such generalizations include the conditions for the existence of solutions to different types of differential equations of continuum mechanics, for example, in the form of differential invariant Cauchy-Riemann relations. This result is achieved by using a complex variable function argument method. A positive factor is that the presented method has the prospect of advancement in such areas of continuum mechanics as the theory of plasticity, elasticity, dynamic problems of the theory of elasticity. In this case, we are talking about finding more complex analytical solutions that could help abandon a series of simplifications in the problems, obtain a series of resolving functions that satisfy more complex and diverse boundary conditions.

2. Literature review and problem statement

One of the first works in this area was a publication on the theory of plasticity [1], which set out the basic approaches to solving the problem. In the future, there were developments using a combination of methods of argument functions and functions of a complex variable [2], invariant differential generalizations in the polar coordinate system [3]. It should be emphasized that the result obtained by simplification in those works is correlated with the results of the solutions by other authors.

It is of interest to consider studies in the literature that address defining generalizations in solving problems of continuum mechanics.

Determining the stressed state in the zones of elastic loading was reported in monograph [4]. Generalizations of the structure of numerical and analytical solutions to problems of the theory of elasticity [5], the method of integrated relations for estimating kinematic perturbations [6], based on variational inequalities, are considered.

The analysis of changes in the nature of loading by the thickness of the sample under the action of compact tension is shown in [7]. The maximum zone is located closer to the surface, which indicates the unevenness of the stressed state of the material. Taking into consideration the heterogeneity of the stressed-strained state of the alloy, in theory, is characterized by the introduction of coordinate functions into the consideration, or, in a given case, the argument of functions.

The local load problem at the discontinuity base is considered using a general approach determined by the state of the medium [8]. Repeated heterogeneity of the stressed state or a change in obvious conditions show the need to use coordinate functions in the solution in combination with periodic dependences. In the case of a method of argument functions, it is a combination of basic functions, including a trigonometric function, and a corresponding argument function.

Variable stresses and strains during loading are the main reasons for the decrease in the strength and durability of products [9]. That makes it relevant to solve applied problems characterizing the stressed state of articles using the approaches of classical equations from the theory of continuum mechanics.

The search for new methods for solving elastic and elastic-plastic problems is also relevant. The application of the mathematical apparatus of the theory of functions of a complex variable makes it possible to derive an analytical solution to the flat problem of the theory of elasticity.

It is shown in [10] that there are transition conditions for introducing into consideration additionally separated variables (an analog of the argument functions) when reformatting one type of differential equations into another. The very idea of transition is productive but the emergence of additional solutions does not mean determining the conditions for the existence of solutions.

The problem [11] presents the ability to predict one of the basic functions. The trigonometric function is implemented in the structural statement of a practical task. The solution does not consider the argument function as a closing component of the overall result.

The cyclic load in the case of a simple shift has been determined, which finds the corresponding response of internal stresses [12]. As before, the basic trigonometric function is introduced into consideration. Its use at different loads is shown. The possibilities of combining it with the argument functions are not given. An important aspect of the proposed solution is the choice of the basic trigonometric function, although a closing solution is not considered. The change in the external load causes a reaction from the medium according to the exponential law [13]. This is comparable to using a fundamental substitution in the method of argument functions. However, the functional purpose of the proposed dependence in the cited works is different, which does not make it possible to apply one of the argument functions in the solution.

In combination with basic functions, operating stresses are characterized during the loading of the part. In work [14], the method of R-functions is proposed, which, in terms of functionality, echoes the method of argument functions. However, the application of the R-function method does not lead to the establishment of certain ratios; they are involved in other schemes of finding solutions (for example, using variational principles).

As a result, it is shown that there are tendencies to use generalizing approaches in solving problems of continuum mechanics, and, in particular, the theory of elasticity. A significant field of problems united by some approaches in the formulation and solution of theoretical, practical tasks is covered: the use of the same basic functions; some additional dependences that can produce the result; selection of approaches in the implementation of predictive functions.

However, there are unresolved issues related to how not the solutions themselves should be determined but the conditions for their existence. Such generalized approaches make it possible to predict the result for new applied tasks, expand the possibilities of solving them in order to meet a variety of boundary and edge conditions in the problems of continuously changing production.

An option for overcoming such difficulties is the use of a complex variable function argument method, which has demonstrated its capabilities in solving diverse problems of continuum mechanics [15]. Those general regularities that have been identified make it possible to pose and solve new problems in the theory of elasticity: for example, the study of the stressed state in Cartesian coordinates using argument functions for more complex models of theoretical and applied problems.

3. The aim and objectives of the study

The aim of this study is to advance a generalizing method for solving problems of continuum mechanics, including the theory of elasticity, taking into consideration and using defining generalizations in the Cartesian coordinate system.

To accomplish the aim, the following tasks have been set:

– to propose an approach to the search for new more complex generalized solutions to the problems of the theory of elasticity in the Cartesian coordinate system using the complex variable function argument method;

– on the basis of the obtained result, to construct a mathematical model of the flat applied problem of the theory of elasticity with complex boundary conditions, taking into consideration the attenuation process of terminal loads with the transition to different zones of the stressed state;

– to show the influence of the beam shape factor on the distribution of stresses in the transition zones with different intensity of their attenuation;

– to obtain a mathematical substantiation of Saint-Venant's principle in relation to the bending of a beam with variable asymmetric loading.

4. The study materials and methods

When solving the problem of continuum mechanics, the defining generalizations using the method of argument functions have been revealed.

The statement of the flat problem of the theory of elasticity is well known in the literature [4, 16–18]. In this case, we obtain:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0,\quad \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0,$$

$$\nabla^2\left(\sigma_x+\sigma_y\right)=\nabla^2\left(2\cdot\sigma_0\right)=0. \tag{1}$$

Boundary conditions under stresses:

$$\tau_n=-\frac{\sigma_x-\sigma_y}{2}\sin 2\phi+\tau_{xy}\cos 2\phi, \tag{2}$$

where σ_0 is the average normal stress or hydrostatic pressure; σ, τ are the normal, tangential stresses; φ is the angle of inclination of the contact pad.

In works [1–3], boundary conditions are represented by a trigonometric form, such as:

$$\tau_n = -T_i \cdot \sin(\mathrm{A\Phi} - 2\phi), \tag{3}$$

which assumes for (3):

$$\tau_{xy} = T_i \cdot \sin(\mathrm{A\Phi}), \quad \sigma_x - \sigma_y = 2T_i \cdot \cos(\mathrm{A\Phi}), \tag{4}$$

where $T_i = T_i(x,y)$ is the coordinate function or the intensity of tangential stresses, $\mathrm{A\Phi}$ is the unknown coordinate function or the first argument function. For a flat problem, the following dependence holds:

$$T_i = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}. \tag{5}$$

Substituting (4) in (5), we obtain an identity, which indicates the correspondence and simplification of the main provisions of the conclusion and the boundary conditions of the problem being solved.

In this case, linearization is not just one advantage of trigonometric substitution. The latter functions are expressed through an exponential dependence, the indicator of which is represented by a complex variable. Fundamental substitution is used when the original system of equations is linearized. In this case, one can represent:

$$T_i = C_\sigma \cdot \exp(\theta), \tag{6}$$

where $\theta = \theta(x,y)$ is the unknown coordinate function or the second argument function. Taking into consideration (4), (6), the tangential stress can be represented in the basic functions:

$$\tau_{xy} = T_i \cdot \sin(\mathrm{A\Phi}) = C_\sigma \exp(\theta)\sin(\mathrm{A\Phi}). \tag{7}$$

In case (7), the statement of the problem changes, which can be formulated as follows: at what values of argument functions θ and $\mathrm{A\Phi}$ would the system of equations (1), (2) be identically satisfied? In work [2], the solution to this system of equations is proposed in the Cartesian coordinates in the following form:

$$\sigma_x = \pm\exp(\pm\theta)\begin{pmatrix} C_1\cos\mathrm{A\Phi} - \\ -C_2\sin\mathrm{A\Phi}\end{pmatrix} + \sigma_0 + f(x) + C,$$

$$\sigma_y = \mp\exp(\pm\theta)\begin{pmatrix} C_1\cos\mathrm{A\Phi} - \\ -C_2\sin\mathrm{A\Phi}\end{pmatrix} + \sigma_0 + f(y) + C,$$

$$\tau_{xy} = \exp(\pm\theta)(C_1\sin\mathrm{A\Phi} + C_2\cos\mathrm{A\Phi}),$$

$$\theta_x = \mp\mathrm{A\Phi}_y,\ \theta_y = \pm\mathrm{A\Phi}_x,$$

$$\theta_{xx} + \theta_{yy} = 0, \mathrm{A\Phi}_{xx} + \mathrm{A\Phi}_{yy} = 0. \tag{8}$$

Paper [3] proposes a more complex solution for polar coordinates. The analysis reveals that such generalizations can also take place in Cartesian coordinates. At the same time, at first, such a problem must be solved theoretically. Let us represent the intensity of tangential stresses in the following form:

$$T_i = C_{\sigma1}\exp(\theta) + C_{\sigma2}\exp(-\theta). \tag{9}$$

Expression (9) is somewhat reminiscent of the hyperbolic cosine. As can be seen from the presented solution to (8), that does not negate the minus in the exponent indicator. Taking into consideration (4), (9), write:

$$\tau_{xy} = \left[C_{\sigma1}\exp(\theta) + C_{\sigma2}\exp(-\theta)\right]\times$$
$$\times(C_1\sin\mathrm{A\Phi} + C_2\cos\mathrm{A\Phi}). \tag{10}$$

The representation of the tangent stress in form (10) expands the possibilities for the stresses to satisfy the boundary conditions at a deformation site.

5. Results of studying new approaches to solving problems of continuum mechanics in the Cartesian coordinate system

5. 1. Using the complex variable function argument method based on new generalized solutions

To substantiate the method of argument functions, one must show its capabilities in the process of solving the problem. Consider the integration of differential equilibrium equations taking into consideration expression (10) where the basic functions with the presence of two argument functions are indicated. Let us write down expression (10) through the function of a complex variable:

$$\tau_{xy} = C'_{\sigma1}\frac{\exp(\theta + i\mathrm{A\Phi}) - \exp(\theta - i\mathrm{A\Phi})}{2i} +$$
$$+C''_{\sigma1}\frac{\exp(\theta + i\mathrm{A\Phi}) + \exp(\theta - i\mathrm{A\Phi})}{2} +$$
$$+C'_{\sigma2}\frac{\exp(-\theta + i\mathrm{A\Phi}) - \exp(-\theta - i\mathrm{A\Phi})}{2i} +$$
$$+C''_{\sigma2}\frac{\exp(-\theta + i\mathrm{A\Phi}) + \exp(-\theta - i\mathrm{A\Phi})}{2}, \tag{11}$$

where $C'_{\sigma1} = C_{\sigma1}\cdot C_1$, $C''_{\sigma1} = C_{\sigma1}\cdot C_2$, $C'_{\sigma2} = C_{\sigma2}\cdot C_1$, $C''_{\sigma2} = C_{\sigma2}\cdot C_2$.

Equilibrium equations (1) are used to determine normal stresses in the following form:

$$\sigma_x = -\int\frac{\partial\tau_{xy}}{\partial y}\mathrm{d}x + \sigma'_0 + f(y) + C,$$

$$\sigma_y = -\int\frac{\partial\tau_{xy}}{\partial x}\mathrm{d}y + \sigma'_0 + f(x) + C. \tag{12}$$

Substituting in the expression for the partial derivative the value of the tangent stress (11), we obtain

$$\frac{\partial\tau_{xy}}{\partial x} = \frac{1}{2i}C'_{\sigma1}\begin{bmatrix}(\theta_x + i\mathrm{A\Phi}_x)\exp(\theta + i\mathrm{A\Phi}) - \\ -(\theta_x - i\mathrm{A\Phi}_x)\exp(\theta - i\mathrm{A\Phi})\end{bmatrix} +$$
$$+\frac{1}{2}C''_{\sigma1}\begin{bmatrix}(\theta_x + i\mathrm{A\Phi}_x)\exp(\theta + i\mathrm{A\Phi}) + \\ +(\theta_x - i\mathrm{A\Phi}_x)\exp(\theta - i\mathrm{A\Phi})\end{bmatrix} +$$
$$+\frac{1}{2i}C'_{\sigma2}\begin{bmatrix}(-\theta_x + i\mathrm{A\Phi}_x)\exp(-\theta + i\mathrm{A\Phi}) - \\ -(-\theta_x - i\mathrm{A\Phi}_x)\exp(-\theta - i\mathrm{A\Phi})\end{bmatrix} +$$
$$+\frac{1}{2}C''_{\sigma2}\begin{bmatrix}(-\theta_x + i\mathrm{A\Phi}_x)\exp(-\theta + i\mathrm{A\Phi}) + \\ +(-\theta_x - i\mathrm{A\Phi}_x)\exp(-\theta - i\mathrm{A\Phi})\end{bmatrix}.$$

Similarly, one finds a derivative along the y coordinate. Only the lower indices that denote partial derivatives by the corresponding coordinates differ.

Each derivative has four terms. Substituting partial derivatives into expressions for normal stresses (12), we obtain dependences written in a general form. The sub-integral functions in (12) are written through partial derivatives at coordinates opposite to that by which the integration is carried out. This excludes integration in a general form. If we follow a mathematical transition for the argument function from one variable to another using the Cauchy-Riemann relations:

$$\theta_x = -\mathrm{A}\Phi_y, \quad \theta_y = \mathrm{A}\Phi_x,$$

we shall get, after integration, analytical expressions in a general form.

Herewith:

$$\sigma_x = \pm\left[C_{\sigma1}\exp(\pm\theta) - C_{\sigma2}\exp(\mp\theta)\right]\times$$
$$\times\left(C_1\cos\mathrm{A}\Phi - C_2\sin\mathrm{A}\Phi\right) + \sigma_0 + f(y) + C,$$

$$\sigma_y = \mp\left[C_{\sigma1}\exp(\pm\theta) - C_{\sigma2}\exp(\mp\theta)\right]\times$$
$$\times\left(C_1\cos\mathrm{A}\Phi - C_2\sin\mathrm{A}\Phi\right) + \sigma_0 + f(x) + C,$$

$$\tau_{xy} = \left[C_{\sigma1}\exp(\pm\theta) + C_{\sigma2}\exp(\mp\theta)\right]\times$$
$$\times\left(C_1\sin\mathrm{A}\Phi + C_2\cos\mathrm{A}\Phi\right), \qquad (13)$$

$$\theta_x = \mp\mathrm{A}\Phi_y, \quad \theta_y = \pm\mathrm{A}\Phi_x,$$

$$\theta_{xx} + \theta_{yy} = 0, \quad \mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy} = 0.$$

It can be shown that expressions (13) identically satisfy the boundary conditions in (3), (4). At $C_{\sigma2}=0$, $C_{\sigma1}=1$, expressions (13) completely coincide with (8). Thus, the presented result (13) shows that the use of the method of argument functions makes it possible to obtain not only a solution option but also the possibility of its generalization using Cauchy-Riemann relations.

(13) shows not only the defining basic functions of the solution but also the conditions for the existence of a general solution – these are invariant differential generalizations between argument functions, including Laplace's equations. It follows that the argument function can be defined. The type of differential equations through which closing solutions are found becomes known. However, the Cauchy-Riemann transformations, accepted for the solution, are an assumption rather than proof of the existence of the solution. An unknown function is the hydrostatic pressure σ_0, which does not make it possible for the problem to be closed. For the ultimate result, it is necessary to have strict proof of the above provisions and determine the value of the hydrostatic pressure in (13). Let us use the condition of continuity of deformations (1), taking into consideration:

$$\sigma'_x + \sigma'_y = 2\sigma_0$$

or

$$\frac{\sigma'_x + \sigma'_y}{2} = \sigma_0,$$

where $\sigma'_x = \sigma_x - f(y) - C$, $\sigma'_y = \sigma_y - f(x) - C$.

Then:

$$\Delta^2\left(\sigma'_x + \sigma'_y\right) = \Delta^2\left(n\sigma_0\right) = 0. \qquad (14)$$

If the bracket in (14) is zero or constant, then the equation of continuity of deformations is identically satisfied. However, these are not the only solutions to the continuity equation. For certainty, let us use the integrated expressions present in each formula for normal stresses:

$$n\left[C_{\sigma1}\exp(\pm\theta) - C_{\sigma2}\exp(\mp\theta)\right]\pm$$
$$\pm\left(C_1\cos\mathrm{A}\Phi - C_2\sin\mathrm{A}\Phi\right) =$$
$$= n\left[C_{\sigma1}\exp(\pm\theta) - C_{\sigma2}\exp(\mp\theta)\right]\cdot\sin\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right). \qquad (15)$$

Expression (15) is of interest. Can it satisfy, as a solution, the equation of continuity of deformations (14), and under what conditions? We are talking about hydrostatic pressure σ_0. Translating (15) into a complex form of notation, taking into consideration the upper signs of the exponents, we obtain:

$$\sigma_0 = C_{\sigma1}\frac{1}{2i}\left\{\begin{matrix}\exp\left[\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] - \\ -\exp\left[\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right]\end{matrix}\right\} -$$
$$-C_{\sigma2}\frac{1}{2i}\left\{\begin{matrix}\exp\left[-\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] - \\ -\exp\left[-\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right]\end{matrix}\right\}. \qquad (16)$$

Determining the second derivatives by the coordinates of expression (16), we obtain:

$$\frac{\partial^2\sigma_0}{\partial x^2} = \frac{1}{2i}C_{\sigma1}\left\{\begin{matrix}\left[\theta_{xx} + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{xx}\right] + \\ +\left[\theta_x + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_x\right]^2\end{matrix}\right\}\exp\left[\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] -$$
$$-\frac{1}{2i}C_{\sigma1}\left\{\begin{matrix}\left[\theta_{xx} - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{xx}\right] + \\ +\left[\theta_x - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_x\right]^2\end{matrix}\right\}\exp\left[\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] -$$
$$-\frac{1}{2i}C_{\sigma2}\left\{\begin{matrix}\left[-\theta_{xx} + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{xx}\right] + \\ +\left[-\theta_x + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_x\right]^2\end{matrix}\right\}\exp\left[-\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] +$$
$$+\frac{1}{2i}C_{\sigma2}\left\{\begin{matrix}\left[-\theta_{xx} - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{xx}\right] + \\ +\left[-\theta_x - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_x\right]^2\end{matrix}\right\}\exp\left[-\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right];$$

$$\frac{\partial^2\sigma_0}{\partial y^2} = \frac{1}{2i}C_{\sigma1}\left\{\begin{matrix}\left[\theta_{yy} + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{yy}\right] + \\ +\left[\theta_y + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_y\right]^2\end{matrix}\right\}\exp\left[\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] -$$
$$-\frac{1}{2i}C_{\sigma1}\left\{\begin{matrix}\left[\theta_{yy} - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{yy}\right] + \\ +\left[\theta_y - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_y\right]^2\end{matrix}\right\}\exp\left[\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] -$$
$$-\frac{1}{2i}C_{\sigma2}\left\{\begin{matrix}\left[-\theta_{yy} + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{yy}\right] + \\ +\left[-\theta_y + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_y\right]^2\end{matrix}\right\}\exp\left[-\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] +$$
$$+\frac{1}{2i}C_{\sigma2}\left\{\begin{matrix}\left[-\theta_{yy} - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_{yy}\right] + \\ +\left[-\theta_y - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)_y\right]^2\end{matrix}\right\}\exp\left[-\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right].$$

Substitute the expressions into continuity equation (14) and group. We make sure that each operator includes the sums of squares that can be converted to the following form:

$$\left(\theta_x + \mathrm{A}\Phi_y\right)\left(\theta_x - \mathrm{A}\Phi_y\right) \pm 2i\left(\theta_x \mathrm{A}\Phi_x + \theta_y \mathrm{A}\Phi_y\right) + \\ +\left(\theta_y + \mathrm{A}\Phi_x\right)\left(\theta_y - \mathrm{A}\Phi_x\right).$$

Such a representation is an important step in solving the problem since the resulting nonlinearity can be eliminated by taking the products of parentheses equal to zero due to:

$$\theta_x = \mp \mathrm{A}\Phi_y, \quad \theta_y = \pm \mathrm{A}\Phi_x. \tag{17}$$

Substituting conditions (17) into the rest of the above expression, note that the parentheses at the complex unity are zero, that is:

$$\theta_x \mathrm{A}\Phi_x + \theta_y \mathrm{A}\Phi_y = \theta_x \mathrm{A}\Phi_x + \mathrm{A}\Phi_x\left(-\theta_x\right) = 0.$$

As a result of such transformations, the operators in the equations are simplified, and the equation of continuity of deformations takes the following form:

$$\Delta^2\left(n\sigma_0\right) = \frac{1}{2i}\cdot C_{\sigma 1}\left[\left(\theta_{xx} + \theta_{yy}\right) - i\left(\mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy}\right)\right]\times \\ \times \exp\left[\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] - \\ -\frac{1}{2i}\cdot C_{\sigma 1}\left[\left(\theta_{xx} + \theta_{yy}\right) + i\left(\mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy}\right)\right]\times \\ \times \exp\left[\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] - \\ -\frac{1}{2i}\cdot C_{\sigma 2}\left[-\left(\theta_{xx} + \theta_{yy}\right) - i\left(\mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy}\right)\right]\times \\ \times \exp\left[-\theta + i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] + \\ +\frac{1}{2i}\cdot C_{\sigma 2}\left[-\left(\theta_{xx} + \theta_{yy}\right) + i\left(\mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy}\right)\right]\times \\ \times \exp\left[-\theta - i\left(\mathrm{A}\Phi_0 - \mathrm{A}\Phi\right)\right] = 0. \tag{18}$$

From (18), one can see that all operators at different exponents contain the same differential relations, that is:

$$\theta_{xx} + \theta_{yy}, \quad \mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy}. \tag{19}$$

Taking into consideration the Cauchy-Riemann relations (17), it can be shown that the sums of the second derivatives (19) of the argument functions are zero. Indeed:

$$\theta_{xx} = \mp \mathrm{A}\Phi_{yx}, \quad \theta_{yy} = \pm \mathrm{A}\Phi_{xy},$$

$$\mathrm{A}\Phi_{xx} = \pm\theta_{yx}, \quad \mathrm{A}\Phi_{yy} = \mp\theta_{xy},$$

then

$$\theta_{xx} + \theta_{yy} = \mp \mathrm{A}\Phi_{yx} \pm \mathrm{A}\Phi_{xy} = 0,$$

$$\mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy} = \pm\theta_{yx} \mp \theta_{xy} = 0.$$

As a result, the equation of continuity of deformations is identically satisfied. Consequently, the hydrostatic pressure is determined from the following expression:

$$\sigma_0 = n\left[C_{\sigma 1}\exp\left(\pm\theta\right) - C_{\sigma 2}\exp\left(\mp\theta\right)\right]\times \\ \times\left(C_1 \cos \mathrm{A}\Phi - C_2 \sin \mathrm{A}\Phi\right), \tag{20}$$

and the conditions for the existence of a solution to equation (14), in the following form:

$$\theta_x = \mp \mathrm{A}\Phi_y, \quad \theta_y = \pm \mathrm{A}\Phi_x,$$

$$\theta_{xx} + \theta_{yy} = 0, \quad \mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy} = 0. \tag{21}$$

Comparing expressions (20), (21) with (13), we are convinced that these dependences close the problem and introduce sufficient certainty in the result. The derived solutions correlate with each other with the same unifying Cauchy-Riemann relations. It should be added that in determining the normal stresses, the Cauchy-Riemann relations were taken as an assumption but, in the case of (20), (21), they are obtained in the form of a correct derivation, which is perceived as the proof of solving the problem.

As a result, for the components of the stress tensor, one can write:

$$\sigma_x = \pm\left[C_{\sigma 1}\cdot\exp\left(\pm\theta\right) - C_{\sigma 2}\cdot\exp\left(\mp\theta\right)\right]\times \\ \times\left(C_1 \cos \mathrm{A}\Phi - C_2 \sin \mathrm{A}\Phi\right) + \sigma_0 + f\left(y\right) + C,$$

$$\sigma_y = \mp\left[C_{\sigma 1}\cdot\exp\left(\pm\theta\right) - C_{\sigma 2}\cdot\exp\left(\mp\theta\right)\right]\times \\ \times\left(C_1 \cos \mathrm{A}\Phi - C_2 \sin \mathrm{A}\Phi\right) + \sigma_0 + f\left(x\right) + C,$$

$$\tau_{xy} = \left[C_{\sigma 1}\cdot\exp\left(\pm\theta\right) + C_{\sigma 2}\cdot\exp\left(\mp\theta\right)\right]\times \\ \times\left(C_1 \sin \mathrm{A}\Phi + C_2 \cos \mathrm{A}\Phi\right),$$

$$\sigma_0 = n\cdot\left[C_{\sigma 1}\cdot\exp\left(\pm\theta\right) - C_{\sigma 2}\cdot\exp\left(\mp\theta\right)\right]\times \\ \times\left(C_1 \cos \mathrm{A}\Phi - C_2 \sin \mathrm{A}\Phi\right), \tag{22}$$

$$\theta_x = \mp \mathrm{A}\Phi_y, \; \theta_y = \pm \mathrm{A}\Phi_x,$$

$$\theta_{xx} + \theta_{yy} = 0, \; \mathrm{A}\Phi_{xx} + \mathrm{A}\Phi_{yy} = 0.$$

Consider that the differential relations in (22) determine the conditions for the existence of solutions to the problem. This can be shown by a specific example. We obtain several solutions to the Laplace equation for the argument of the trigonometric function, in the following form:

$$\mathrm{A}\Phi_1 = \mathrm{AA}_1 y, \; \mathrm{A}\Phi_2 = \mathrm{AA}_2 x,$$

$$\mathrm{A}\Phi_3 = \mathrm{AA}_3 xy, \; \mathrm{A}\Phi_4 = \mathrm{AA}_4\left(x^2 - y^2\right). \tag{23}$$

Substituting (23) in the Laplace equation, we are convinced that with all variants the arguments of trigonometric expressions are harmonic functions.

By substituting (23) in the Cauchy-Riemann relation, and then, integrating, we obtain the values for the second argument function:

$$\theta_1 = \mp \mathrm{AA}_1 x, \quad \theta_2 = \pm \mathrm{AA}_2 y, \; \theta_3 = \mp \mathrm{AA}_3 \frac{x^2 - y^2}{2}. \tag{24}$$

Expressions (24) satisfy Laplace's equation. Consequently, (23), (24) are the closing solutions to the statement equations of the theory of elasticity. From this example, one can see that the proposed approach makes it possible to define a series of argument functions, each of which can be a unique condition for a specific applied task. At the same time, the list of solutions is far from complete. Such generalizations concern determining not the specific functions but the conditions of their existence.

Expressions (22) can be represented through hyperbolic cosines and sinuses. In this case, one can write:

$$\sigma_x = \pm\left[(C_{\sigma 1}-C_{\sigma 2})\cdot\cosh(\pm\theta)+(C_{\sigma 1}+C_{\sigma 2})\sinh(\pm\theta)\right]\times$$
$$\times(C_1\cos \mathrm{A\Phi}-C_2\sin \mathrm{A\Phi})+\sigma_0+f(y)+C,$$

$$\sigma_y = \mp\left[(C_{\sigma 1}-C_{\sigma 2})\cdot\cosh(\pm\theta)+(C_{\sigma 1}+C_{\sigma 2})\sinh(\pm\theta)\right]\times$$
$$\times(C_1\cos \mathrm{A\Phi}-C_2\sin \mathrm{A\Phi})+\sigma_0+f(x)+C,$$

$$\tau_{xy} = \left[(C_{\sigma 1}-C_{\sigma 2})\cdot\sinh(\pm\theta)+(C_{\sigma 1}+C_{\sigma 2})\cosh(\pm\theta)\right]\times$$
$$\times(C_1\sin \mathrm{A\Phi}+C_2\cos \mathrm{A\Phi}),$$

$$\sigma_0 = \pm n\cdot\begin{bmatrix}(C_{\sigma 1}-C_{\sigma 2})\cdot\cosh(\pm\theta)+\\ +(C_{\sigma 1}+C_{\sigma 2})\sinh(\pm\theta)\end{bmatrix}\times$$
$$\times(C_1\cos \mathrm{A\Phi}-C_2\sin \mathrm{A\Phi}), \tag{25}$$

$$\theta_x = \mp \mathrm{A\Phi}_y,\ \ \theta_y = \pm \mathrm{A\Phi}_x,$$

$$\theta_{xx}+\theta_{yy}=0, \mathrm{A\Phi}_{xx}+\mathrm{A\Phi}_{yy}=0.$$

The result obtained in form (25) can be compared with the solutions obtained with the help of the Fourier series [4]. Let us show it. To this end, expressions must be simplified in order to obtain private solutions. Assuming C_3=C_4 in the solutions [4], we obtain:

$$\sigma_x = \sin\alpha x\left[C_1\alpha^2\cosh\alpha y+C_2\alpha^2\sinh\alpha y\right],$$

$$\sigma_y = -\alpha^2\sin\alpha x\left[C_1\cosh\alpha y+C_2\sinh\alpha y\right],$$

$$\tau_{xy} = -\alpha\cos\alpha x\left[C_1\alpha\sinh\alpha y+C_2\alpha\cosh\alpha y\right]. \tag{26}$$

Let us bring expressions (25) into correspondence with expressions (26). Herewith:

$$C_1=0,\quad C_2=-\alpha^2,\quad n=0,\quad f(x)=f(y)=C=0.$$

In accordance with the method of argument functions, one of the solutions to the problem is taken in the form of (23), (24):

$$\mathrm{A\Phi}_2=\mathrm{AA}_2x,\ \ \theta_2=\mathrm{AA}_2y,\ \ \mathrm{AA}_2=\alpha.$$

Taking into consideration the simplifications, expressions (25) are to be rewritten in the following form:

$$\sigma_x = -\sin\alpha x\left[C'_{\sigma 1}\cosh\alpha y+C'_{\sigma 2}\sinh\alpha y\right]\cdot(-\alpha^2),$$

$$\sigma_y = \sin\alpha x\left[C'_{\sigma 1}\cosh\alpha y+C'_{\sigma 2}\sinh\alpha y\right]\cdot(-\alpha^2),$$

$$\tau_{xy} = -\alpha\cos\alpha x\left[\cdot C_1\alpha\sinh\alpha y+C_2\alpha\cosh\alpha y\right]. \tag{27}$$

Comparing the formulas (26) and (27), we are convinced of their identity. This indicates that the special cases of both solutions obtained by different methods (the method of stress functions and the method of argument functions) coincide. The presented comparison determines the reliability of solution (22) and the possibility of its use in mechanical calculations of applied problems.

5. 2. The mathematical model of variable asymmetric loading of a pinched console as an applied problem of the theory of elasticity

Works [19, 20] report trends in the development of the calculation of the stressed state at bending, including attempts to implement generalized approaches to solving applied and general theoretical problems. Paper [19] deals with the problem of bending thick rectangular plates using reciprocity theorems based on Reisner's theory. Study [20] gives an accurate analytical solution to the problem of flat bending under the action of longitudinal normal loads. In this case, the load is set in the form of a trigonometric series. This work is to some extent the rationale for choosing the basic trigonometric function in the solution. The obtained results are confirmed by the results from finite element modeling.

Although works [19, 20] contain elements of certain generalizations but cannot be used in other areas of continuum mechanics, which limits the possibilities of their application.

The capabilities of the method of argument functions can be shown when solving an applied problem associated with the bending of the console loaded at its end [4]. This is one of the problems in which, in addition to solving the resistance of materials, the solutions from the mathematical theory of elasticity using the stress function are shown; Fig. 1.

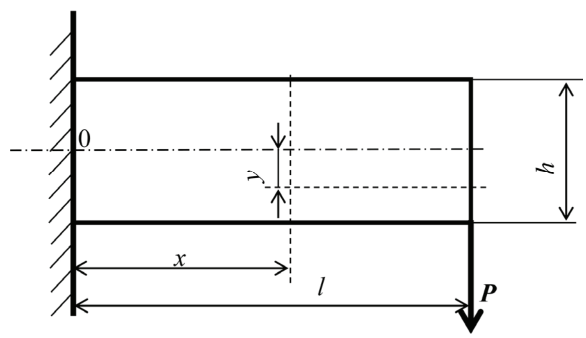

Fig. 1. Transverse bending of the console

The length of the console is denoted by l, the height – through h=2c. The console is sealed at the left end and loaded at the right end by the force P. We shall look for a solution using the method of argument functions taking into consideration working expressions (22) to (24), that is

$$\sigma_x = \left[C_{\sigma 1}\exp(\mathrm{AA}_1\cdot x)-C_{\sigma 2}\exp(-\mathrm{AA}_1x)\right]\times$$
$$\times(C_2\sin \mathrm{AA}_1y)+\sigma_0+f(y),$$

$$\sigma_y = -\left[C_{\sigma 1}\exp(\mathrm{AA}_1x)-C_{\sigma 2}\exp(-\mathrm{AA}_1x)\right]\times$$
$$\times(C_2\sin \mathrm{AA}_1y)+\sigma_0+f(x),$$

$$\tau_{xy} = \left[C_{\sigma 1}\exp(\mathrm{AA}_1x)+C_{\sigma 2}\cdot\exp(-\mathrm{AA}_1x)\right]\times$$
$$\times(C_2\cos \mathrm{AA}_1y),$$

$$\sigma_0 = n\cdot\left[C_{\sigma 1}\exp(\mathrm{AA}_1\cdot x)-C_{\sigma 2}\exp(-\mathrm{AA}_1x)\right]\times$$
$$\times(C_2\sin \mathrm{AA}_1y). \tag{28}$$

The conditions at the contour, at $y=\pm c$, $\sigma_y=0$, $\tau_{xy}=0$, at $x=l$, $\sigma_x=0$, at $x=l$, $y=0$, $\tau_{xy}=\tau_{max}=\tau_0$, next:

$$\int_{-c}^{+c}\tau_{xy}\,dy=-P. \tag{29}$$

Assuming $\sigma_y=0$, at $f(x)=0$, we obtain:

$$\sigma_y=-\left[C_{\sigma1}\exp(\mathrm{AA}_1x)-C_{\sigma2}\exp(-\mathrm{AA}_1x)\right]\times$$
$$\times(C_2\sin\mathrm{AA}_1y)+\sigma_0=0.$$

Hence

$$\sigma_0=\left[C_{\sigma1}\exp(\mathrm{AA}_1x)-C_{\sigma2}\exp(-\mathrm{AA}_1x)\right]\times$$
$$\times(C_2\sin\mathrm{AA}_1y). \tag{30}$$

Expression (30) corresponds to formula (28). Substituting (30) in (28) for the stress σ_x, we obtain:

$$\sigma_x=2\left[C_{\sigma1}\exp(\mathrm{AA}_1x)-C_{\sigma2}\exp(-\mathrm{AA}_1x)\right]\times$$
$$\times(C_2\sin\mathrm{AA}_1y)+f(y). \tag{31}$$

The next condition, at $x=l$, $\sigma_y=0$, is substituted in (31), we obtain:

$$0=2\left[C_{\sigma1}\exp(\mathrm{AA}_1l)-C_{\sigma2}\exp(-\mathrm{AA}_1l)\right]\times$$
$$\times(C_2\sin\mathrm{AA}_1y)+f(y)$$

or

$$f(y)=-2\left[C_{\sigma1}\exp(\mathrm{AA}_1l)-C_{\sigma2}\exp(-\mathrm{AA}_1l)\right]\times$$
$$\times(C_2\sin\mathrm{AA}_1y). \tag{32}$$

The boundary conditions, for the tangential stresses on the upper and lower faces of the console, take the form, at $y=\pm c$, $\tau_{xy}=0$. Hence:

$$\mathrm{AA}_1=\frac{\pi}{2c}. \tag{33}$$

Consider the next boundary condition, given (28), at $x=l$, $y=0$, $\tau_{xy}=\tau_{max}=\tau_0$,

$$\tau_0=\left[C_{\sigma1}\exp(\mathrm{AA}_1l)+C_{\sigma2}\exp(-\mathrm{AA}_1l)\right]C_2. \tag{34}$$

According to the literature [4], in the direction of sealing, normal stresses should increase and reach the maximum value at the zero point. This is possible when $C_{\sigma1}=C_{\sigma2}=C_\sigma$ because, in this case, the difference in the exponents of the square bracket for σ_x becomes zero and the normal stress reaches the maximum value. Then, taking into consideration (34):

$$C_\sigma\cdot C_2=\frac{\tau_0}{\exp(\mathrm{AA}_1l)+\exp(-\mathrm{AA}_1l)}. \tag{35}$$

Substitute in expressions (30), (31) the values of functions and constants (32), (33), (35). The formula to determine σ_x takes the following form:

$$\frac{\sigma_x}{\tau_0}=-2\times$$
$$\times\frac{\left[\exp\left(\frac{\pi}{2c}l\right)-\exp\left(-\frac{\pi}{2c}l\right)\right]-\left[\exp\left(\frac{\pi}{2c}x\right)-\exp\left(-\frac{\pi}{2c}x\right)\right]}{\exp\left(\frac{\pi}{2c}l\right)+\exp\left(-\frac{\pi}{2c}l\right)}\times$$
$$\times\sin\left(\frac{\pi}{2c}y\right). \tag{36}$$

The tangential stress:

$$\frac{\tau_{xy}}{\tau_0}=\frac{\left[\exp\left(\frac{\pi}{2c}x\right)+\exp\left(-\frac{\pi}{2c}x\right)\right]}{\exp\left(\frac{\pi}{2c}l\right)+\exp\left(-\frac{\pi}{2c}l\right)}\cos\left(\frac{\pi}{2c}y\right). \tag{37}$$

Expressions in the form of (36), (37) are convenient for analyzing the bending stress along the axis of the console and in the transverse direction. They are represented in a coordinateless form to provide a computationally convenient context. A similar approach is used in work [21].

5. 3. The influence of beam shape factor on stress distribution in transition zones

Expressions (36), (37) on the right-hand side are written as dimensionless quantities that vary within certain limits.

At $x=l$,

$$\sigma_x=0,\quad \frac{\tau_{xy}}{\tau_0}=\cos\left(\frac{\pi}{2c}y\right). \tag{38}$$

At $x=0$,

$$\frac{\sigma_x}{\tau_0}=-2\frac{\left[\exp\left(\frac{\pi}{2c}l\right)-\exp\left(-\frac{\pi}{2c}l\right)\right]}{\exp\left(\frac{\pi}{2c}l\right)+\exp\left(-\frac{\pi}{2c}l\right)}\sin\left(\frac{\pi}{2c}y\right).$$

$$\frac{\tau_{xy}}{\tau_0}=\frac{2}{\exp\left(\frac{\pi}{2c}l\right)+\exp\left(-\frac{\pi}{2c}l\right)}\cos\left(\frac{\pi}{2c}y\right). \tag{39}$$

If the length of the console is much higher than the height, then

$$\exp\left(-\frac{\pi}{2c}l\right)\to 0,$$

hence, expressions (39) can be rewritten in the following form:

$$\frac{\sigma_x}{\tau_0}=-2\frac{\left[\exp\left(\frac{\pi}{2c}l\right)\right]}{\exp\left(\frac{\pi}{2c}l\right)}\sin\left(\frac{\pi}{2c}y\right)=-2\sin\left(\frac{\pi}{2c}y\right),$$

$$\frac{\tau_{xy}}{\tau_0}=\frac{2}{\exp\left(\frac{\pi}{2c}l\right)}\cos\left(\frac{\pi}{2c}y\right). \tag{40}$$

As a result of gradual change, bending stresses can reach their extreme values for normal and tangential stresses. In the final form, for (40), the tendency of the tangential stress to zero is determined, at x=0, $l/2c\to\infty$; that is, $\tau_{xy}\to 0$.

Analyzing (38) to (40) reveals that the limiting maximum values of bending stresses when loaded with force P reach the following values:

$$\tau_{xy}=\tau_0\cos\left(\frac{\pi}{2c}y\right),\quad \sigma_x=-2\tau_0\sin\left(\frac{\pi}{2c}y\right). \tag{41}$$

Thus, at the free end of the console, the normal stresses, according to the boundary conditions, are zero, and the tangential ones reach a maximum value (41), thereby balancing the shear force P. In the hard seal zone, normal stresses (41) reach their maximum value [4], and tangential stresses tend to zero.

For clarity, based on expressions (36), (37), we plotted the bending stresses along the axis of the console and in the transverse direction at different values of $l/2c$ (Fig. 2–5).

In the figures, the ordinal axis shows the relative values of stresses σ_x/τ_0 (Fig. 2, 4) and τ_{xy}/τ_0 (Fig. 3, 5); the abscissa axis – the relative coordinates of length x/l (Fig. 2, 3) and the coordinates of the cross-section of the console y/h (Fig. 4, 5).

Figu. 2, 3 demonstrate the distribution of bending stresses along the x axis of the console, at different $l/2c$ values. Fig. 4, 5 illustrate the distribution of bending stresses in the transverse direction y, at $l/2c$=3 and y=c. Fig. 2, 3 display the transition zone in which changes in bending stresses occur, both for normal and tangential stresses. After reaching the extreme values of stresses for a long console, the stressed state of the curved beam stabilizes and resembles the stressed state of pure bending, in normal stresses of which only one variable y is indicated. There are no tangential stresses and the normal stresses no longer change along the length of the console (x coordinate), reaching the maximum value at the surface of the beam (41).

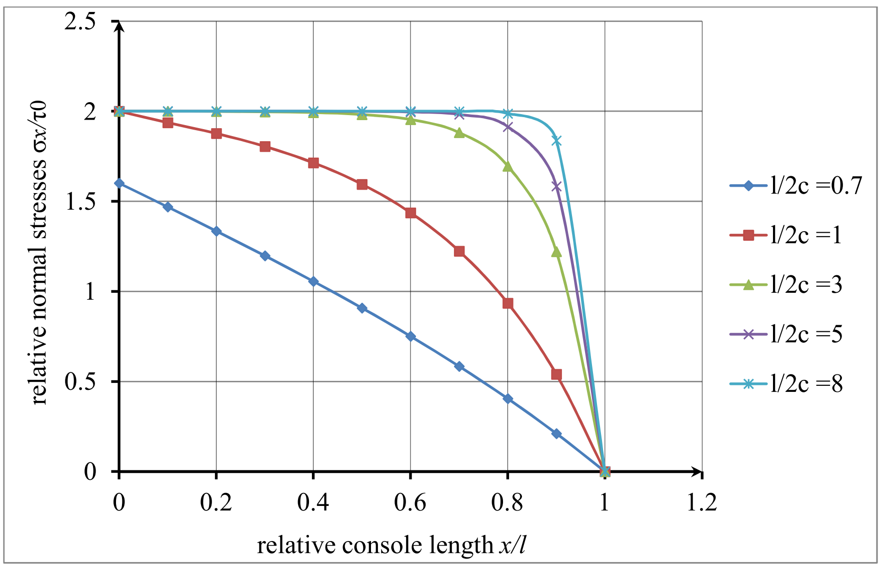

Fig. 2. Distribution of normal bending stresses along the length of the console at different values of the ratio $l/2c$

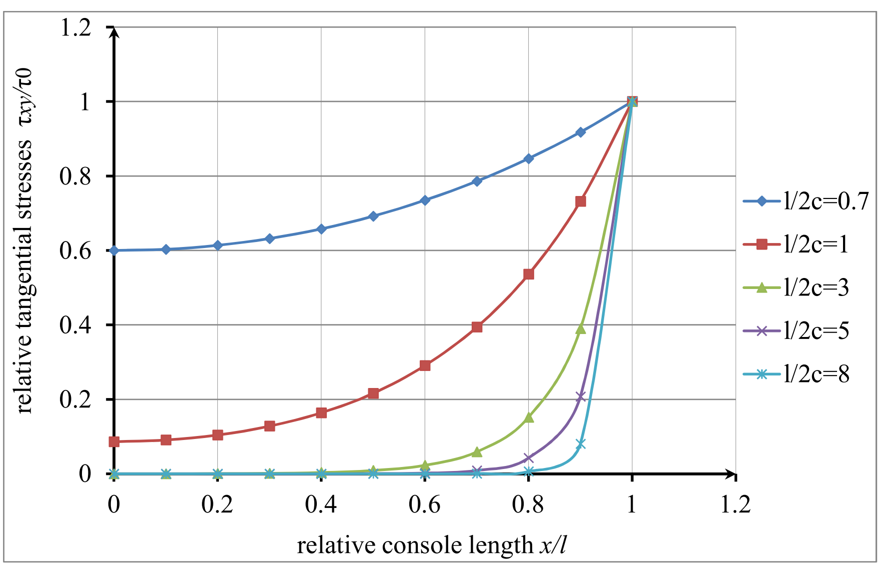

Fig. 3. Distribution of tangential bending stresses by the console cross-section at different values of the ratio $l/2c$

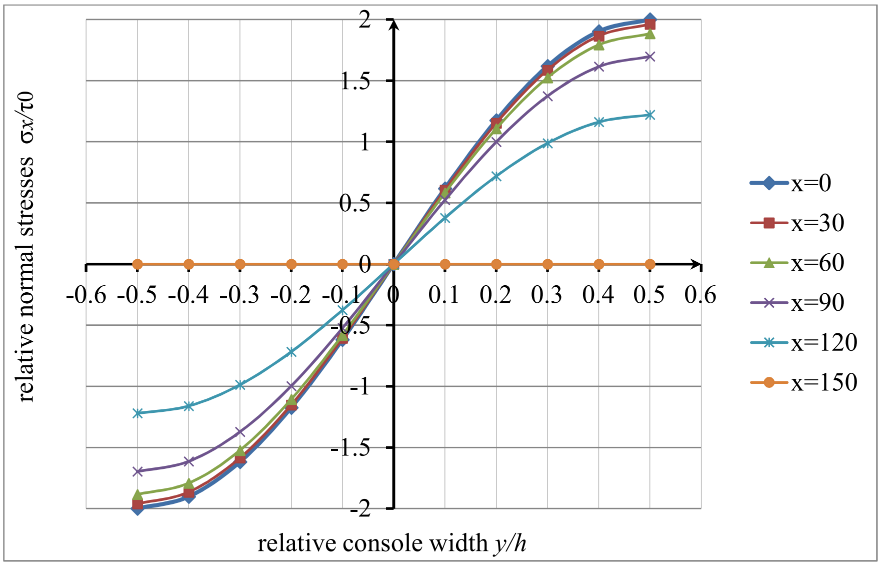

Fig. 4. Distribution of normal bending stresses along the console cross-section at different values of the relative coordinate x in the transition zone

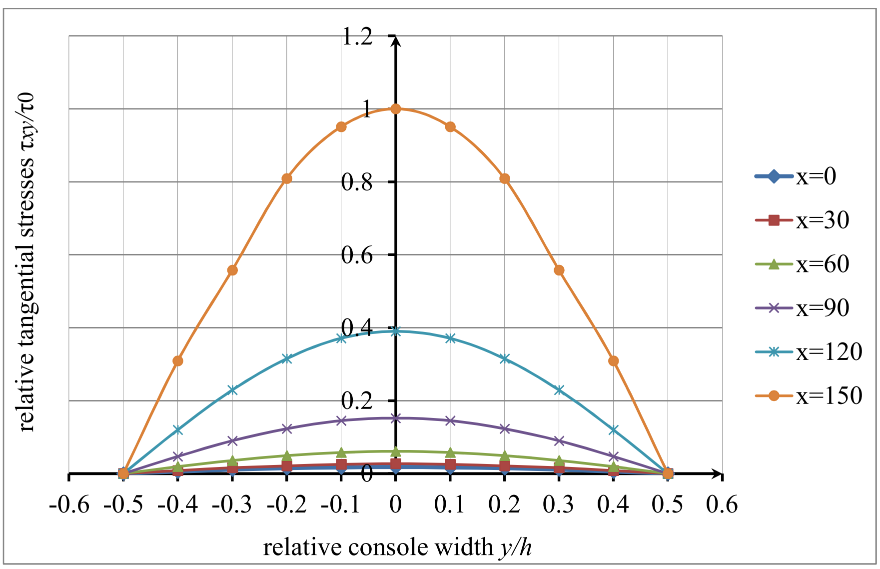

Fig. 5. Distribution of tangential bending stresses along the console cross-section at different values of the relative coordinate x in the transition zone

5. 4. Mathematical substantiation of the Saint-Venant's principle under variable asymmetric loading of a pinched console

Fig. 4, 5 show the change in bending stresses in the transition zone in the transverse direction y at different values of the relative coordinate x. There is a tendency to change the stresses from zero values to maximum. The presence of this section of transition from one stressed state of the beam to another is likely a feature of the proposed solution. It is generally accepted, by virtue of Saint-Venant's principle [22, 23], that a change in the load near the border leads to significant changes in stresses only near the end. In such cases, simple solutions can produce accurate enough results everywhere except for the vicinity of the border [4]. The resulting solution fully confirms this principle, with the only difference that the neighborhoods of the border can also be indicated by fading known stresses in one direction or another.

6. Discussion of results of studying the solution to the problem of continuum mechanics in the Cartesian coordinate system

The process of changing the stresses in the transition zone is determined by the $l/2c$ parameter, that is, the console shape factor. With a short console (up to $l/2c$=3), changes in the stress distribution cover the entire rod, Fig. 2–5. The influence of boundary conditions at the free end is decisive in the distribution of stresses along the entire length of the console.

For a long beam ($l/2c\geq5$), there is a rapid enough attenuation of the determining action of the boundary conditions of the end of the console. In this case, in the region of achieving extreme stress values, one can use the simplest solutions proposed in the resistance of materials or the theory of elasticity, Fig. 2–5.

In conclusion, the following should be emphasized: our different variants of one solution (22), (28), (35), (36) satisfy the system of equations of the theory of elasticity (1), boundary conditions (2), the boundary conditions of the applied problem (29). Based on solving the problem using the method of argument functions, Saint-Venant's principle has been specifically confirmed.

The conditions for solving the problem in the transition zone are specified. The comparison with the solutions by other authors [4] demonstrates that under certain conditions the function of stresses in the form of Fourier series after simplifications coincides with solution (27) given in the present work. This ensures the reliability of the result of the presented advancement.

A feature of the proposed solution is the identification of differential conditions of its existence using the argument functions, that is Cauchy-Riemann relations, Laplace equations, including Cartesian coordinates.

The study results can be explained by:

– using the complex variable function argument method;

– obtaining invariant differential generalizations in the form of Cauchy-Riemann relations, including a solution for Cartesian coordinates (13), (22), (25), (28), (36), (37);

– the results of the present work were compared with classical solutions to the problems of the theory of elasticity and with the solutions by modern authors. The analysis reveals that the proposed mathematical apparatus can be used in the theory of metal processing by pressure, geomechanics, the interaction of elastic bodies, non-stationary problems associated with the transmission of interaction in the form of a wave process.

Limitations include the limits of applicability of solutions. These approaches do not apply to solutions to the biharmonic equation using the argument functions in Cartesian coordinates.

The disadvantages of the study include the bulkiness and volume of the derivation. This is due, first of all, to the lack of accumulated material on this issue.

When solving problems of continuum mechanics, the defining generalizations in the method of argument functions have been revealed but this is not enough to use it in new problems. There is a need to expand the method, as well as the possibility of its use not only in the problems of continuum mechanics.

7. Conclusions

1. The possibility of using a complex variable function argument method as an approach to the search for new more complex generalized solutions to the problems of the theory of elasticity in the Cartesian coordinate system has been estimated. The proposed approach makes it possible to define a series of argument functions, each of which can be a condition of uniqueness for a specific applied task.

2. Based on our result, a mathematical model has been built of the flat applied problem of the theory of elasticity with complex boundary conditions, taking into consideration the attenuation process of terminal loads with the transition to different zones of the stressed state. The presence of a transition section from one stressed state of the beam to another is likely a feature of the proposed solution. A qualitative indicator of the study results is the application of the method to solving more complex problems of the theory of elasticity, predicting the result.

3. The influence of the beam shape factor on the distribution of stresses in transition zones with different intensity of their attenuation has been shown. Expressions that are represented in a coordinateless form are convenient for analyzing the bending stress along the axis of the console and in the transverse direction. They provide a computationally convenient context.

4. The mathematical substantiation of Saint-Venant's principle in relation to the bending of a beam with variable asymmetric loading has been obtained. The resulting solution fully confirms this principle, with the only difference that the neighborhoods of the border can also be indicated by fading known stresses in one direction or another. Bringing the solution to a particular result, the classical solutions have been derived, which confirms its reliability.

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