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ANALYZING THE CODE STRUCTURES OF MULTIDIMENSIONAL SIGNALS FOR A CONTINUOUS INFORMATION TRANSMISSION CHANNEL

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One of the directions to improve the efficiency of modern telecommunication systems is the transition to the use of multidimensional signals for continuous channels of information transmission. As a result of studies carried out in recent years, it has been established that it is possible to ensure high quality of information transmission in continuous channels by combining demodulation and decoding operations into a single procedure that involves the construction of a code construct for a multidimensional signal.

This paper considers issues related to estimating the possibility to improve the efficiency of continuous information transmission channel by changing the signal distance of the code structure.

It has been established that the code structures of such types as a hierarchical code construct of signals, a hierarchical code construct of signals with Euclidean metric, a reversible code construct of signals, a reversible code construct of signals with Euclidean metric have the potential, when used, to increase the speed of information transmission along a continuous channel. With a signal distance reduced by 10 percent or larger, it could increase by two times or faster.

The estimation of the effect of reducing a signal distance on the efficiency of certain types of code structures was carried out. It has been established that the hierarchical reversible code construct, compared to the hierarchical code construct, provides a win of up to two or more times in the speed of information transmission with a halved signal distance. Implementing the modulation procedure has no fundamental difficulties, on the condition that for each code of the code construct the encoding procedure is known when using binary codes. The results reported here make it possible to build an acceptably complex demodulation procedure according to the specified types of code structures

Keywords: continuous transmission channel, multidimensional signal, signal code construct, signal distance

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1. Introduction

One of the directions to improve the efficiency of modern telecommunication systems is the improvement of existing and the construction of new methods of modulation and interference-resistant coding for continuous signal transmission channels. The transition in telecommunication systems to the ensembles of multidimensional signals increases the speed of information transmission and ensures the transmission of large flows of information. At the same time, the issues of ensuring a high probability of transmitting information are resolved using powerful interference-resistant coding.

Signal transfer theory and interference-resistant coding theory have evolved independently for a long time. Increasing the requirements for the speed and volume of information transmission required the search for new promising theoretical principles for increasing the efficiency of modulation and decoding in continuous signal reception channels. One of these areas is research aimed at assessing the effectiveness of the use of code structures of multidimensional signals in combination with interference-resistant codes. In this case, the procedures for modulation/encoding and demodulation/decoding in such code constructs are carried out jointly and simultaneously. It is obvious that with rational construction, such code structures should combine the positive qualities of both ensembles of multidimensional signals and interference-resistant codes, provide simple implementation algorithms in continuous signal reception channels. That can provide significant movement towards improving the efficiency of telecommunication systems.

The use of multidimensional (with a large base, components, complex) signals can significantly improve the quality of message transmission along communication channels [1, 2]. The main characteristics of the signal system, that is, the set of signals and its mutually unambiguous mapping onto the dictionary of the source of messages, are the dimensionality of the signal, the frequency band used, the power of the set of signals, and the distance between the nearest signals. For many important types of channels, the limit characteristics, for example, power at the predefined dimensionality and minimum distance, are studied quite well.

However, the constructive theory of signals is developed mainly for a discrete, primarily binary channel, that is, within the coding theory. In the theory of coding, a discrete channel formed by a modulator of elementary signals, a continuous channel, and a demodulator of elementary signals, is considered the predefined one. At the same time, it is not possible to approach the potential characteristics of the continuous channel both due to the narrowing of the signal class and due to insufficient use in decoding information about signal distortion in a continuous channel. The latter drawback is overcome by the unifying of demodulation and decoding into a single procedure of reception in general, the so-called soft (or analog) decoding (or solution) and reception in the semi-safe channel [3, 4]. To overcome the first drawback, modulation, that is, the conversion of the word message into a signal at the input of a continuous channel should be considered as a single procedure that combines the encoding and modulation of elementary signals. The number of known signal constructs reflecting this approach is small.

The main approaches to the basic development of various code structures, the principles of their encoding and decoding are set out in [5, 6]. The basics of spatial modulation of code structures for Gaussian channels and their classification are considered in [7, 8]. Determining the lower limits for permutation codes when using them in multidimensional signals for transmission along continuous channels with damping is given in [9, 10]. Certain promising directions for the construction of new types of code structures, including Gaussian signals, are described in [11–13].

Discrete code is also associated with the type and principles of building code structures that are used in a particular multidimensional signal.

One of the directions of ensuring the high quality transmission of multidimensional signals in continuous channels is the combination of demodulation and decoding operations into a single procedure, which involves the creation of a code structure of a multidimensional signal. The question of assessing the possibility of the influence of parameters of code structures of multidimensional signals on the effectiveness of the use of a continuous channel of information transmission is a relevant scientific task. Currently, it is not sufficiently researched.

An important scientific task is to determine the type of code structures of multidimensional signals, the effectiveness of which is associated with a signal distance. This, in the future, will make it possible to analyze their capabilities for a more accurate accounting of the distribution of the signal distance to a set of elementary signals with the predefined power. Which, ultimately, will make it possible to assess the impact of changes in a signal distance on the effectiveness of a continuous information transmission channel.

2. Literature review and problem statement

The issues of determining the type of a multidimensional signal construct and assessing its capabilities for more accurate accounting of the distance distribution by a set of elementary signals with the predefined power are tackled in a series of works.

Paper [14] reports the results of research on the analysis and synthesis of code structures intended for use in modern and promising telecommunication systems. The types and features of the application of different types of code constructs are presented. However, the direct assessment of the effectiveness of their functioning in the direction of the impact of a signal distance on the effectiveness of the functioning of the continuous channel is not given.

Work [15] reports a study into the theory of signal-code structures and code coding. The cited work is good enough in considering the issues of applying different types of code constructs and defining the directions of their improvement. There is no direct question of estimation of the effectiveness of various code structures in the direction of reducing signal distance in it.

Paper [16] considers the issues of modeling of continuous communication channels if it is possible to use different code structures in them, but the issue of the relation between the channel efficiency and code construct is not considered in the paper. Accordingly, there are no estimates of the effectiveness of the application of code structures directly.

Work [17] explores the development of multidimensional signals for processing using a proposed hierarchical encoding algorithm based on an inseparable system. One of the types of a signal code construct is directly considered but without analyzing its effectiveness and impact on it exerted by the signal distance.

The authors of [18], against the background of research into the method of adaptive decoding of auto-orthogonal codes, consider certain issues of estimation of the effectiveness of their use in telecommunication networks. However, the issue of analyzing the entire complex of code constructs and assessing the impact on their effectiveness exerted by the signal distance is not considered.

The issue of increasing the effectiveness of multidimensional signals is considered in work [19]. When investigating the effectiveness of their use, the author analyzed the possibilities of using different types of code structures of such signals. However, there is no estimation of the impact exerted on the effectiveness of the multidimensional signal by the signal distance of its code structure.

Paper [20[investigated some non-standard sets of code structures for OFDM signals that solve the issue of taking into consideration the variety of different OFDM signals and the fluctuations in their amplitudes that affect their amplification. There is no analysis of the functioning and evaluation of the effectiveness of the code structures proposed in the work in terms of changes in the signal distance.

Work [21] reports a study into one of the types of promising code structures of a multidimensional signal. Namely, the work considers the code construct of the type "constellation design". Data from the studies into the efficiency of the code structure in question at the projected signal distance are given. However, there is no consideration of other types of code structures and the impact of their functioning on the effectiveness of the continuous transmission of information.

Paper [22] investigates the construction of a multiple access sparse code system (SCM). The basis of the system proposed in the paper is a promising signal code construct based on optimized unitary rotations on hypercubes. The issues of signal distance management in assessing the use of such a structure are not considered. There is also no comparative analysis of existing code structures and the proposed system of sparse code in terms of solving the scientific task set in the paper.

Thus, the scientific task, resolving which is tackled in this article, is to determine and analyze the types of code structures of multidimensional signals for continuous channels of information transmission. The structures to be defined, while ensuring properties regarding simplicity and versatility, should provide a greater speed of information transmission along a continuous channel due to more accurate accounting of the distribution of the signal distance.

3. The aim and objectives of the study

The purpose of this work is to assess the impact of signal distance of code structures of a multidimensional signal on the speed of information transmission along a continuous channel for different types of code constructs of a multidimensional signal.

To accomplish the aim, the following tasks have been set:

– to analyze existing types of code structures of multidimensional signals in terms of estimating the possibilities of changing their signal distance regarding their work efficiency;

– to employ mathematical modeling methods in order to assess the effect of changes in the signal distance on the rate of information transmission in a continuous channel according to certain types of code structures of multidimensional signals;

– to carry out the comparative analysis and evaluation of the effectiveness of reducing a signal distance regarding the speed of information transmission in a continuous channel of information transmission for a certain spectrum of code structures of multidimensional signals considered in this article.

4. The study materials and methods

A structural scheme of a single-channel information transmission system is considered, in which the code construct of a multidimensional signal for a continuous channel of information transmission is synthesized.

The structural diagram of the specified system is shown in Fig. 1 [14].

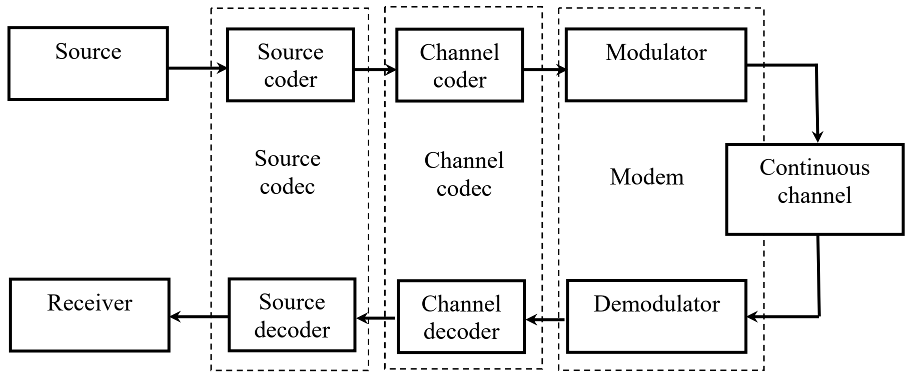

Fig. 1. Structural diagram of a single-channel information transmission system in a continuous channel

To study the code constructs of multidimensional signals for a continuous channel of information transmission, we shall apply methods from the theory of interference-resistant coding with correction of errors, as well as the theory of redundancy of radio signals. The operator methods of radio space transformation, statistical theory of communication, methods for determining a free distance of the invariant signal-code structures are also used.

5. Results of studying the impact of a signal distance of code structures on the functioning of a continuous information transmission channel

5. 1. Analyzing code structures to estimate the possibilities of influence exerted by a change in their signal distance on performance efficiency

Let us denote for a multidimensional signal the M- dimensionality code of length N with words and the minimum Hamming distance d through $(N, M, d)m$ or, at $M=M^K$, through $[N, K, d]M=(N, M^K, d)m$. The operator f of modulation of elementary signals is mapped to the symbol $q_n \epsilon \{0, ..., M-1\}$ of the word $q=(q_1, ..., q_N) \epsilon (N, M, d)_m$ elementary signal $x_n=f(q_n)$ from the set of elementary signals X of power $|X|=M$, contained in the full set of possible, at the input of a continuous channel, elementary signals. And the coding operator ϕ to the word u of the source dictionary U of the word q of the code. A pair of mappings f and ϕ specifies the mapping of the dictionary onto the set of signals, determining the design of the signal system, hereinafter referred to as code. Here is a constructive set of signals, presented in the form of a Cartesian power.

Suppose that for each pair of signals a measure of distinction D (x', x'') is defined, hereinafter referred to as a signal distance or simply a distance if misunderstandings are excluded. The signal distance is not necessarily a metric but in some cases of interest is a monotonic function of the metric. For many (but not all) types of channels, the signal distance is additive, that is, represented in the form [1]:

$$D(x', x'') = \sum_{n=1}^{N} D_0(x'_n, x''_n). \quad (1)$$

An example is the Euclidean distance square (not the metric) or the distances of Hamming and Lee (metrics). When using a code construct, the relationship between the minimum signal distance on the set of signals and the Hamming's distance d is given in conditions (1) by the obvious ratio [1]:

$$D=\min_{x',x''\in A, x'\neq x''} D(x',x'')\geq \delta d. \quad (2)$$

In most known code structures X is one-dimensional real (if the channel is low-frequency) or one-dimensional complex, that is, a two-dimensional real set (with amplitude and phase modulation of elementary signals in band channels). Fundamentally, the code construct is suitable at any dimensionality in the set of elementary X signals. However, it is successful only if all nonzero distances in X are the same, for example, when X is the correct simplex (in particular, consists of two signals) or a set of orthogonal signals with the same norms. Then the signal distance between the two signals from A is proportional to Hamming's distance between the words of the code – representations of these signals, and, with a good code, the dope is good. However, with a high power M of the set X, the increase of which is necessary to obtain high speed, the distances on X are significantly different. The code construct that replaces all nonzero distances $D_0(x'_n, x''_n)$ with the smallest of them, which can be interpreted as binary distance quantization, does not take into consideration these differences. At the same time, it has two important advantages – comparative simplicity and versatility. Under versatility, a fundamental possibility of obtaining signals systems of arbitrary dimensionality and with arbitrary signal distances is accepted. Simplicity is ensured by the regularity (for example, algebraic properties) of codes that combine the same-type elementary signals into multidimensional. Code structures retain these advantages in one way or another but make it possible to obtain more powerful signal systems due to more subtle accounting of the distribution of distances on X.

Code structures are based on the split of many elementary signals into continuous subsets, in each of which, with a successful split, the signal distance between the nearest signals is greater than all X. The most convenient hierarchical structure (HS) is the one in which the ideas of the generalized cascade code [23–25] are adapted for the signal system with arbitrary additive signal distance [26, 27]. The hierarchy means the set L of the breakdown of sets X into classes such that all classes of the same level (one partition) are equally powerful and can include classes of the previous level only entirely. That is, classes of the previous level are "nested" in classes of the next level, similar to the system of internal nested codes of generalized cascading code. The set of classes of the (l–1)-level, included in the class of the 1st level, is mapped mutually and unambiguously onto the set of characters of the M_l-dimension code N, M_l, $d_l)_{Ml}$ of the l-th level. This is the analog of the external code of the generalized cascading code, where $M_1M_2 \ldots M_L=M$. Since the signal distances between elementary signals of the l-level class increase with a decrease in l, the transition from a code structure with one M- dimensional code to a multicode HS makes it possible to increase the power of the set of signals without reducing the minimum signal distance. This is similar to when the transition from cascading to generalized cascading code makes it possible to increase code power without reducing the minimum Hamming's distance [24, 25].

The totality of L-convolutional codes makes it possible to get a convolutional analog of HS signals for continuous or discrete channels with additive signal distance based on the same hierarchy.

First of all, in terms of assessing the effect of signal distance on the efficiency of code structure operation, we are interested in those designs in which the character encoding at all levels of the hierarchy is carried out through the signal distance [1, 3, 14].

For further analysis and evaluation of efficiency, consider the following types of code structures of multidimensional signals [1–3, 14]:

– hierarchical signal code construct (HS);

– hierarchical signal code construct with Euclidean metric;

– reversible signal code construct;

– reversible signal code construct with Euclidean metric.

5. 1. 1. The hierarchical code construct of multidimensional signals

Suppose that on a set of elementary signals X of power $M=M_1M_2 \ldots M_L$ a hierarchy is defined – a set L of division into disparate classes. Each class of the l-th level of the hierarchy (l-th division) includes M_l classes of the (l–1) level, that is, it consists of $\mu l=M_1M_2 \ldots .M_l$ signals. The numbering of the classes of the (l–1)-th level, which are included in the class of the l-th level, sets a mutually unique mapping of the set of classes of the (l–1)-th level onto the set of digits {0, ..., M_l–1}. Therefore, the set $(q_{1n},\ldots,q_{ln})$, where it determines the only value of the n-th elementary signal, where f is the rule (operator) of modulation of elementary signals. We compare the l-th level and the minimum l-th signal distance in the class [1, 2]

$$\delta_l = \min_{q_{l+1,n,\ldots,q}\, {}_{Ln}(q'_{1n},\ldots,q'_{\ln})\neq(q''_{1n},\ldots,q''_{\ln})} D_0(x'_n, x''_n), \quad (3)$$

where

$$x'_n = f(q'_{1n}, q'_{\ln}, q_{l+1,n},\ldots,q_{Ln}),\quad x''_n = f(q''_{1n}, q''_{\ln}, q_{l+1,n},\ldots,q_{Ln}).$$

The L level class is the same as X, so $\delta_L=\delta$. In the hierarchy, one can also include a zero level with M classes of one signal in each and at $\delta_0=\infty$. Since the class of the next level can include the class of the previous level only entirely, then one can combine two levels by disregarding the (l–1)-th partitioning, so that one can take into consideration that $\delta_1>\delta_2>\ldots>\delta_L=\delta$.

Let $q_1=(q_{i1}, \ldots, q_{iN})$ be the word of code $(N, M_l, d_l)_{Ml}$ of the l-th level, the source dictionary is represented by the Cartesian product of the subsection of mutually unambiguous mapping onto the l-level code. Under the hierarchical structure, authentic authors understand the totality of the hierarchy on the set of elementary X signals, the mapping of the f sets of $(q_{in}, \ldots, q_{Ln})$ on the set of X, L level codes and L mapping ϕl. The scheme of the corresponding sequence of transformations (modulation) is as follows:

$$u\to(u_1,\ldots,u_L)\xrightarrow{\phi_l} q_l=(q_{l1},\ldots,q_{\ln})\xrightarrow{f} x_n.$$

Here, the left arrow means splitting a word into subsections (the word u can be a block in the sequence of characters of the source of information; if the power of the sets of blocks is less, then some dictionary words and signals are not used). Then each subword is encoded into the word code $(N_l, d_l)_{M|}$. The result is L code words of the same length N. The elementary signal modulator converts a set of n-characters of all words into the n-th elementary signal that enters the channel.

Assertion 1. The hierarchical structure sets the signal system with a power of the set of signals and with a minimum signal distance [3, 14]:

$$D > \min_{i \le l \le L}(\delta_l d_l). \tag{4}$$

The statement about the number of signals is obvious. It is also clear that HS sets the signal system, that is, the mutually unambiguous mapping of dictionary U on the set of signals A. To prove (4), consider a distance between the two signals $x' = (x'_1, ..., x'_N)$, $x'' = (x''_1, ..., x''_N)$, where $x'_n = f(q'_{1n}, ..., q'_{Ln})$, $x''_n = f(q''_{1n}, ..., q''_{Ln})$. There are at least one l and one n such as $q'_{1n} \ne q''_{1n}$ at $x' \ne x''$ and, then, there is $\lambda = \max\{l : q'_{1n} \ne q''_{1n}, 1 < l < L, 1 < n < L,\}$. Then δ_λ is the least of the nonzero distances between the elementary signals included in x' and x'' any of such distances $D_0(x'_n, x''_n) \ge \delta_\lambda \delta(q'_{1n}, q''_{1n})$, where $\delta(q'_{1n}, q''_{1n})$ is the Kronecker symbol (0 or 1). Hence:

$$D(x', x'') \ge \delta_\lambda \sum_{n=1}^{N} \delta(q'_{\lambda n}, q''_{\lambda n}) \ge \delta_\lambda d_\lambda, \tag{5}$$

since the signals correspond to different words of the λ-th level code. Since such λ is found for any pair of different signals from, (4) follows from (5).

With the predefined minimum signal distance D, the minimum Hamming's distance of codes should be chosen equal to [1, 2]:

$$d_l = \left] D / \delta_l \right[, \tag{6}$$

where $]D/\delta_l[$ is the smallest integer, not less than D/δ_l.

Since at l<L, the Hamming's distance d_l necessary for codes of all levels, except the last, can be, especially for the first levels, significantly less than $d=dL$, which makes it possible to increase the power of the set of signals in HS compared to the code construct.

Note 2. 1: The hierarchy is set by any L equivalence relationship on X if each breaks X into equal-power classes and the class of the next equivalence relation includes either all or no elements of the class of the previous relation. Thus, if X is mutually unambiguously mapped onto group G (for example, [8, 11]) and $G_1 \subset G_2 ... \subset G_L = G$ are its subgroups of orders μ_1, ..., μ_L, where $\mu_L = M_1 ... M_l$, then the adjacent class of group G on the subgroup G_l is mapped on the class of the l-level of the hierarchy. This type of HS with Hamming's distance as a signal includes generalized cascading codes. If X is given (as the region of values) by the function of integer arguments, then the equivalence ratio can be defined by the fixation of some arguments.

Let $(\lambda_1, ..., \lambda_\Lambda)$ be a permutation of indexes and $q_{ln} = p_\lambda$. Any such order of indices corresponds to the Λ-level hierarchy with the minimum distances δ_l, conditioned by (3). Instead of permutation, one can use another arbitrary mutually unambiguous match of sets $(p_{1n}, ..., p_{\Lambda n})$ and $(q_{1n}, ..., q_{Ln})$, given by L functions. We do not know the general method of such variable recoding, which leads to a successful hierarchy. Some of the recoding techniques are given in examples 8–10.

Note 2. 2: The signal system given by the hierarchical structure can be matched with other HSs, as any of the L! level permutations define any HS. Let q_{ln} becomes a symbol of the l-th level at the i-th permutation, at which, taking into consideration the corresponding changes in (3), the order of character fixation, the minimum distance equals $\delta_l^{(i)}$. From statement 1, the score from below follows $D \ge \min(\delta_l^{(i)} d_i)$. This score is true for all permutations, therefore $D \ge \max x_i \min_l (\delta_l^{(i)} d_i)$.

5. 1. 2. The hierarchical code construct of multidimensional signals with Euclidean metric and rotary modulation

Let the signal be a vector of N_V-dimensional real Euclid space composed of N_v-dimensional vectors (elementary signals).

The Euclid distance is not additive but its monotonous function is a square of the Euclid distance, which is defined as energy distance. The specified distance is not a distance in the generally accepted sense as it does not satisfy the "triangle axiom" but it is additive and can serve as a signal distance of HS. From monotony, it follows that the optimization of signals at minimum energy and Euclid distances is equivalent. Consider examples, given that the energy distance between two elementary signals [2, 3] is

$$D_0(x'_n, x''_n) = \sum_{i=}^{v} (x'_{in} - x''_{in})^2 = (\rho'_n)^2 + (\rho''_n)^2 -$$
$$-2\rho'_n \rho''_n \cos 2\varphi = (\rho'_n - \rho''_n)^2 + 4\rho'_n \rho''_n \sin^2 \phi, \tag{7}$$

where $\rho_n = |x_n|$ is the norm of the vector x_n, ϕ is half the angle between vectors x'_n and x''_n.

Let v=1 and the one-dimensional set X take the form $X = \{x = A + q_n a : 0 \le q_n \le M-1\}$, where A and a are the constants, and the symbol is a positional record as an L-bit number with a mixed base, that is [2, 3]:

$$q_n = q_{1n} M_2 ... M_L + ... + q_{L-1,n} M_L + q_{Ln}. \tag{8}$$

The l-th class of the hierarchy level is a subset with X corresponding to fixed, as a result of (3), (7), which, together with (6), determines the minimum Hamming's distance d_l required for the l-th level code. The corresponding HS differs from the structure in work [12] only by the rule of choosing the Hamming's distances d_l.

If all the values of each of the characters q_{ln} are considered equal, then the average energy of the elementary signal is minimal at $A = -a(M-1)/2$ and is equal to [2, 12]:

$$E = a^2 (M^2 - 1) / 12. \tag{9}$$

Calculations show that at N=32, r=1, the codes $(32, 26, 4)_2$ and $(32, 6, 16)_2$ match the signal system $M = 2^{32}$ with a minimum energy distance D=12.8 at a single average energy per coordinate. This is 3.2 times more than with no excess binary signals ±1. In order to obtain with the same signal system using a code construct, a quaternary code $(32, 16, 16)^4$ would be required, which does not exist (it exceeds Hamming's boundary [26]).

Note that the design of this code construct is suitable at 16 amplitude-phase modulation, which is matched by two-dimensional elementary signals in the form of a pair of one-dimensional quaternary ones [6].

The following are the signals with the same $N\upsilon$ energies (at a single average energy per coordinate), that is, signals in the sphere, the demodulation operator of which is invariant under known conditions to the scale of the signal. To use HS to build such signals, we adopt a stronger assumption that the energy of each υ-dimensional elementary signal is equal to

$$x_{1n}^2+...+x_{\nu n}^2=\nu. \tag{10}$$

Sometimes (10) also displays physical limitations on the signal, such as those associated with a peak factor.

Satisfying (10) the set of elementary signals X belongs to the radius of a sphere, that is, it is a polytope [28] (the polytope is understood both as the figure and the set of its vertices; its radius – the radius of the sphere). The vertices of such an elementary polytope can be obtained from one of the second turns in the υ-dimensional space, which is described by $\upsilon-1$ angles, so the corresponding modulation of elementary signals can be called rotary. Phase modulation (PM) is a separate case of the rotary one, at $\upsilon=2$. With the component of rotary modulation, a multidimensional signal is a set of N elementary signals with rotary modulation of each of them.

The polytope X allows mapping to the corresponding orthogonal real (or unitary complex) group of υ-dimensional space and the hierarchy can be described by a number of subgroups of this group. However, in these examples, one can do with more obvious geometric representations, without attracting the concepts of groups of movements. First, some elementary signal x_n is selected, which is taken as a (degenerate) elementary polytope of zero level. M_1 of its rotations (without vertex matches) give M_1-vertex polytope (class) of the first level. Combining M_2 turns of a polytope, we get a polytope – a second-level class, and so on. The energy distance δ_l is the square of the length of the smallest edge of the polytope. If the angle between the vertices on this edge, then, as a result (7), (10), we get [3, 28]:

$$\delta_1=4\nu\sin^2\psi_l. \tag{11}$$

At the constituent phase modulation (CPM) a $2N$-dimensional signal is formed by N signals of the M- dimensional PM. If $M=M_1...M_L$, then the suitable HS of signals of CPM (it is almost indifferent from the structure from work [1], as in Example 1), in which the class of the l-th level of the hierarchy is obtained by M_l turns of proper ($M1...M_{l-1}$)-angle shape. The minimum energy distance in the class is 0 [2, 12].

$$\delta_l=8\sin^2(\pi/M_1...M_l). \tag{12}$$

The CPM signal is determined by a set of phases (a_1, ..., a_N), where [2, 3]:

$$\begin{aligned}a_n&=\mathrm{arctg}(x_{in}/x_n)=\\&=a_{n0}+2\pi(q_{in}/M_1+...+q_{Ln}/M_1...M_L),\end{aligned} \tag{13}$$

$q_{ln}\epsilon\{0,..., M_l-1\}$, a_{no} is an arbitrary initial phase.

Here, the sum in parentheses or part of its terms, can also be represented using the Chinese residual theorem if the corresponding M_L are mutually simple [26]. At $M_l=2$ or 3, the Hamming's distance of the l-th level code is strictly proportional to the energy one. If, in addition, δ_l are treated as integers, then in (6) one can avoid losses for rounding. In this sense, two-level constructions with $M=4$ and 6 are interesting.

A signal at phase modulation of the type of PM-4 (QPSK – Quadrature Phase Shift Keying) quaternary PM can be considered as a pair of binary one-dimensional signals with values ±1. This makes it possible to compare the HS of PM-4 the type of CPM with the best PM-4 the type of CPM. Let $x_{n-1}=-(-1)p_{2n-1}$, $x_{2n}=-(-1)p_{2n}$, and the binary set belongs to the code $(2N, d)_2$, which defines the system from signals with a minimum energy distance $D=4d_1$. If this code is the best, then no four-point CPM can give a greater number of signals. On the other hand [3, 12],

$$\begin{aligned}a_n&=\mathrm{arctg}(x_{1n}/x_{2n})=\\&=-\pi/4+(\pi/2)(\rho_{2n-1}\otimes\rho_{2n})+\pi\rho_{2n},\end{aligned}$$

where Θ means addition modulo 2.

Comparing this expression with (13) at $L=2$, $M1=M2=2$, we see that $q_{1n}=p_{2n}$, $q_{2n}=p_{2n}\Theta p_{2n-1}$. From where it follows that, in this case, the use of HS is equivalent to narrowing the set of all binary codes of length $2N$ to its subset. This follows the Plotkin design, which is determined through the direct sum of codes [26].

It is known that Plotkin design leads to good or even optimal codes.

Example 1: $M=6$, $M_1=3$, $M_2=2$, $\delta_1=6$, $\delta_2=2$. At $D=6d_1$, codes are required here, and If $D=6$, then the code of the first level is no-redundant and, at $N=2^m-1$, the second code can be a Hamming code. Then, to get $D=12$, codes $[N, N-1, 2]_3$ of the first and second levels are required, for example, the BCH code $[32, 21, 6]_2$, in this case, that is, more than at a four-point PM at a distance greater than at binary PM.

If, at $M=6$, instead of (13), one uses a representation based on the Chinese residual theorem. That is, take $a_n=(2p_1+3p_2)\pi/3$, where p_1 is a three-way, p_2 binary character, identified with the characters of the codes of two levels, then the hierarchies of different example constructs can differ only by the permutation of levels.

The hierarchy is usually the better, the more vertices the polytope of the first levels has, unless, of course, the length of the smallest edge of each polytope $X^{(l)}$ is close to the maximum with the predefined number of vertices for all l.

At $\upsilon=3$, one of the many correct or semi-correct polyhedral shapes can be taken as an elementary polytope, each of which can usually be mapped to several hierarchies. For example, a cube (±1, ±1, ±1) can be represented as $M_2=4$ of the first level class, each containing $M_1=2$ opposite vertices. Then $\delta_1=4$, $\upsilon=12$, $\delta_2=4$. Another hierarchy is given by two turns of the tetrahedron ($M_1=4$, $M_2=2$, $\delta_1=8$, $\delta_2=4$). Of course, the cube will not make it possible to get a signal system, better than the best quaternary CPM. The most successful are the hierarchies, which are composed of the vertices of the icosahedron, the dodecahedron, and the union of these figures (oriented so that the centers of the faces of one and the vertex of the other lie on the same rays from the origin of the coordinates) and the Archimedes semi-proper polyhedron [5, 28].

Icosahedron, in particular, can be considered as six turns of diameter ($M_1=2$, $M_2=6$, $\delta_1=12$, $\delta_2=3\cos^2(\pi/10)$). And the dodecahedron – as five tetrahedron rotations ($M_1=4$, $M_2=5$, $\delta_1=8$, $\delta_2=16-\sin^2(\pi/10)$) [19, 20].

A series of proper and semi-proper four-dimensional shapes are described by the finite groups of quaternions [20] that have their own subgroups.

5. 1. 3. The reversible code structure of multidimensional signals

Let the set of X elementary signals be divided into R continuous subsets $X^{(0)},...,X^{(R-1)}$, P is the code of length N with the symbols $p_n \in \{0,...,R-1\}$. We shall match the word $p=(p_i,...,p_N)\in P$ with the constructive class of signals $X^{(p)}=X^{(p_1)}\times...\times X^{(p_N)}$ and assume that in the $X^{(p)}$ class the selected signal class $A^{(p)}\subseteq X^{(p)}$ of power $M^{(p)}$ with the minimum signal distance $D^{(p)}$, and the minimum distance between the constructive classes of signals is equal to $D^{(r,p)}=\min D\left(x^{(r)},x^{(p)}\right)$, $x^{(r)}\epsilon X^{(r)}$, $x^{(p)}\epsilon X^{(p)}$. Full set of signals [3, 4]:

$$A=\bigcup_{p\in P} A^{(p)}. \tag{14}$$

The design of the signal system, which is due to the set A and its mutually unambiguous mapping on the dictionary of the source shall be called a heterogeneous construct (since each class $X^{(p)}$ is kept in the Cartesian product $X^{(P)}$ of the dissimilar sets $X^{(Pn)}$). It is obvious.

Assertion 2. A heterogeneous construct determines the signal system of power [4]:

$$M=\sum_{p\in P} M^{(p)}. \tag{15}$$

with a minimum signal distance [2, 3]:

$$D\geq \min_{r,p\in P,r\neq p}\left\{D^{(p)},D^{(r,p)}\right\}. \tag{16}$$

Another estimate of the minimum signal distance using the minimum Hamming's distance d_p of the code P may be more convenient. Assume $\delta_x=\min\delta_x(i,j)$, $i\neq j$ is the least signal distance [2, 3]

$$\delta_x(i,j)=\min_{x_n^{(i)}\in X^{(i)},x_n^{(j)}\in X^{(j)}} D_0\left(x_n^{(i)},x_n^{(j)}\right),\ 0\leq i,j\leq R-1, \tag{17}$$

Between the subsets of the set of elementary signals. Then, from statement 2 and the additivity of the signal distance, we directly obtain.

Consequence. The minimum signal distance of the heterogeneous construct of a signal system satisfies the condition [3, 5]

$$D\geq \min_{p\in P}\left\{D^{(p)},\delta_x d_P\right\}. \tag{18}$$

In a general case, the heterogeneous construct does not differ from reversible. To make it regular, regular methods of constructing P codes and $A^{(p)}\subseteq X^{(p)}$. signal classes are needed. The hierarchical construct can be considered a separate case of a heterogeneous one, and if the character of the word $p\epsilon P$ is understood, for example, as a set $p_n=(q_{2n},...,q_{Ln})$, and $X^{(Pn)}$ as a class of the first level of the hierarchy. Class $A^{(P)}$, in this case, is obtained using the code structure, and the P code – using an $(L-1)$-level HS. More powerful are the structures that use as P a reversible (generalized equilibrium) code [7], and allow, unlike HS, the breakdown of X into unequal classes. The reversible code $P[W_0,...,W_{R-1}]$ is the R-dimensional code of length [3, 5]:

$$N=\sum_{i=0}^{R-1} W_i, \tag{19}$$

whose each word contains W_i characters with the value $i\epsilon\{0,..., R-1\}$. The binary reversible code is equilibrium (sometimes equilibrium codes are the codes with the same number of nonzero characters in each word; the reversible code is a separate case of equilibrium in this sense; hereafter, only binary code is understood as equilibrium).

Below is a mutually unambiguous representation of the reversible code $P=P[W_0,...,W_{R-1}]$ on the set of permutations of indexes 1,...,N. The word $p=(p_i,...,p_N)\in P$ can be mapped to $W_0!,...,W_{R-1}!$ index permutations because permutation $N_i^{(p)}=\{n:p_n=i\}$ does not change the word p. To select one of them, we shall map the natural permutation (1, ..., N) to any *initial word*, for example, $r=(r_i, ..., r_N)=(0,...0, 1, ..., 1, ..., R-1)$, and let $p=\pi_p(r)=\left(r_{i_p},...,r_{N_p}\right)$, where $(1_p, ..., N_p)$ is the permutation of indexes, and the action π_p is to be understood as a change in the order of passage of code characters, elementary signals, etc. The action π_p is defined unambiguously if we make it a condition that at $n',n''\in N_i^{(r)}$, and $n'<n''$ it follows that $n'_p<n''_p$.

We shall accept a constructive class of signals $X^{(r)}=X_0$ and a class of signals $A^{(r)}=A_0\subseteq X_0$ corresponding to the initial word r, as *initial classes.* Since the constructive class $X^{(p)}$ is a permutation of the Descartes product X_0 terms, that is, $X^{(p)}=\pi_p(X_0)$, it is natural to accept that the signal class is $A^{(p)}=\pi_p(A_0)=\{\pi_p(x):x\in A_0\}$. Then all signal classes would have the same power and minimum signal distances. Let the dictionary of the source be represented by the product $U=U_A\times U_p$ of the U_A sub-words of power $M_A=|A_0|$ and U_p, power $M_P=|P|$ and $F_A:U_A\to A_0$, $F_P:U_P\to P$, $F_\pi:P\to\{\pi_p:p\in P\}$ are the *mutually unambiguous representations***.** A heterogeneous construct with the reversible code P**,** predetermined by the initial class of signals $A_0\epsilon X_0$ and *mutually unambiguous representations* F_A**,** F_P**,** F_π is called the *reversible structure of a signal system* (RS). In RS, the signal at the input of the channel $x=\pi_p(F_A(u_A))$, where $\pi_p=F_\pi(F_P(u_P))$, $u_P\in U_P$, $u_A\in U_A$, $F_A(u_A)\in A_0$ (the product $F_\pi F_P$ can be replaced by one conversion).

Let $D^{(p',p'')}$ be the distance between the constructive classes (that is, the nearest class signals) $\pi_{p'}(X_0)$, $\pi_{p''}(X_0)$, $p',p''\in P$, and D_A be the minimum distance in the signal class A_0. Then, from statement 2 and its consequence, we have

Assertion 3. The reversible structure determines a signal system of power $M=M_AM_P$ with a minimum signal distance with a ratio satisfying [2, 3]:

$$D\geq \min_{p',p''\in P,p'\neq p''}\left\{D_A,D^{(p',p'')}\right\}, \tag{20}$$

$$D\geq \min\left\{D_A,\delta_x d_P\right\}. \tag{21}$$

Considering (21), Hamming's distance between the closest words of code P can be taken equal to

$$d_P=\left]D/\delta_x\right[. \tag{22}$$

Use this ratio if R=2 (that is, the P code is equilibrium) or if all nonzero signal distances $\delta_x(i,j)$, predetermined by (17), are the same or close to each other. With significantly different $\delta_x(i,j)$, it is more effective to use the reversible code in the form of composition or product of compositions of simpler codes. The reversible code $P=P[W_0,...,W_{R-1}]$ of length N is called the composition of the reversible (Q+1)-dimensional code $P_1=P[W_0,...,W_{Q-1},N_t]$, $Q<R-1$, $N_t=W_0+...+W_{R-1}$ of N length with characters from $\{\quad Q-\ t\}$ As the first characters Q of this set, choose any, not just the first Q elements from the {0,..., Q+1}) and (R–Q)-dimensional code $P_2=P[W_0,...,W_{R-1}]$ of length N_t with characters from $\{Q\ Q+\quad R-\ \}$ If each word of the code P_1 is mapped to the class of power $M_{P_1}=|P_2|$ of words of the code P_2, generated by replacing the characters t in the word $p=(p_i,...,p_N)\in P_1$. For example, in the order in which the numbers of their occupied places increase) the characters of each word $(t_1,...,t_{N_t})\in P_2$.

It is clear that the composition power of the two codes is equal to $M_P=M_{P_1}M_{P_2}$ and the minimum Hamming's distance [2, 3] is:

$$d_P=\min\{d_{P_1},d_{P_2}\}, \tag{23}$$

where d_{P_1}, d_{P_2} are the minimum Hamming's distances of the codes P_1 and P_2.

The composition or product of compositions can be used as a construct (perhaps not too close to the optimal) reversible code with the help of equilibrium ones. The fact is that although few good structures of equilibrium codes are known, there are almost no known constructs of non-binary reversible codes. No less significant is the fact that the compositional structure of the reversible code P for a RS signal system makes it possible, as mentioned above, to take into consideration the difference between the distances $\delta_x(i,j)$ between the subsets of elementary signals. Indeed, let $P_p\epsilon P$ be a class of words of the code P, to which the word $P\epsilon\mathbf{P_1}$ is matched. The minimum Hamming's distance in this class is likely to be d_{p2}. Consequently, two words $p',p''\in P_p$ correspond to constructive classes of signals that $X^{(p')}$, $X^{(p'')}$ differ no less than by d_{p2} subsets of $X^{(i)}$ elementary signals, from which it follows that

$$D^{(p',p'')}\geq d_{P_2},\delta_x,$$

where $\delta_x=\min\delta_x(i,j),\ \ i\neq j,\ \ i,j\in\{Q,Q+1,...,R-1\}$.

If the set $\{Q,Q+1,...,R-1\}$ (that is, any suitable subset from the domain of determining the characters of the code P) corresponds to a rather distant subset of elementary signals, then

$$d_{P_2}=\left]D/\delta_{x_2}\right[, \tag{24}$$

may be significantly smaller than the next ones from (22) or (22) and (23). Note that in the case in question, given (20),

$$D\geq\min\{D_A,d_{P_1}\delta_x,d_{P_2}\delta_x\}. \tag{25}$$

If the P code is selected, then it is also necessary to specify the design of the initial class of signals A_0 In one case, namely when all the weights of the code are multiples of N_0, that is,

$$W_i=w_iN_0,\ \ N=wN_0,\ \ w=w_0+...+w_{R-1}, \tag{26}$$

this class can be built using HS. To this end, as the initial constructive class of signals, a Cartesian power is taken [3]:

$$X_0=\tilde{X}^{N_0}, \tag{27}$$

where $\tilde{X}=\left(X^{(0)}\right)^{w_0}\times...\times\left(X^{(R-1)}\right)^{w_{R-1}}$ is the *constructed set of elementary signals*. To build a system (of the initial class A_0) of signals in X_0, the HS is suitable with N_0 elementary signals $x_n\in\tilde{X}$. The reversible construction of a signal system with a hierarchical structure of the initial signal class is to be called a hierarchical reversible structure (HRS). Some ways to construct a hierarchy on a set $\tilde{X}$ (suitable in other cases) are given in the following paragraph.

5. 1. 4. The reversible code construct of multidimensional signals with Euclidean metric

To construct, by using HS, signals on a sphere $S_{N_v}^{N_{v-1}}$, that is satisfying the condition [2, 3, 5]:

$$\sum_{n=1}^{N}p_n^2=N_v,\ \ p_n=\|x_n\|, \tag{28}$$

a stronger condition (10) was introduced in the previous chapter. The structures from the previous chapter make it possible to track the execution of (28) in a more general case.

Let $\{X^{(0)},...,X^{(R-1)}\}$ be a set of elementary v-dimensional polytopes and $p^{(i)}=\|X^{(i)}\|$. For certainty, we assume that $p^{(i)}<p^{(i+1)}$ (if $p^{(i)}=p^{(i+1)}$, it is possible and, as a rule, it is advisable to combine two polytopes into one with a larger number of vertices). As a result of (7), (17), a minimum energy distance between polytopes [3, 5] is:

$$\delta_x(i,j)=\left(p^{(i)}-p^{(j)}\right)^2+4p^{(i)}p^{(j)}\sin^2\psi_{ij}, \tag{29}$$

where ψ_{ij} is the half of the smallest angle between the vertices of polytopes. The mapping $X_n=X^{(p_n)}$, $p_n\in\{0,...,R-1\}$ defines a polytope to which the n-th elementary signal x_n belongs and thereby matches the word $p=(p_1,...,p_N)$ of the code of polytopes P to a constructive class $X^{(p)}$. The distance between the two classes $X^{(p')}$, $X^{(p'')}$ [3, 5]:

$$D^{(p',p'')}=\sum_{n=1}^{N}\delta_x(p'_n,p''_n). \tag{30}$$

Selecting the representation code P which on a set of sets of radii (p_1, ..., p_N) satisfies (28), and, by matching each word p of the code signal class $A^{(p)}$, we obtain a system of signals with equal energies, the parameters of which are determined by assertion 2 and the consequence of it. It is advisable (when possible) as adjacent polytopes $X^{(i)}$, $X^{(i-1)}$ to choose mutual ones [28, 29], orientating them so that the centers of the surface nests of one and the vertex of the other lie on common rays from the coordinate origin. Moreover, if polytopes are different, a larger radius should be in a polytope with a larger number of vertices.

5. 2. Estimating the influence of change in a signal distance on operational efficiency for certain types of code structures of multidimensional signals

Mathematical modeling methods to assess the effect of signal distance change on the efficiency of continuous transmission of information for certain types of code structures of multidimensional signals.

For a hierarchical code structure, the following calculations will be made. *Example 2*: Assume v=2, N=3, $X^{(0)}$ is the point at the coordinate origin (degenerate polytope), $X^{(1)}$ and $X^{(2)}$ are the mutually rotated hexagons with radii $p^{(1)}=\sqrt{2},\ \ p^{(2)}=\sqrt{6}$ and angles at the vertices [5, 28, 29]:

$$\alpha_n^{(i)}=(i-1)\pi/6+s_n\pi/3+t_n 2\pi/3,$$

where $i\in\{1,2\},\ \ s_n\in\{0,1\},\ \ t_n\in\{0,1,2\}$.

As a result of (29), here $\delta_x(0,1)=\delta_x(1,2)=2,\ \ \delta_x(0,2)=6.$ The four words of the polytope code P satisfy (28): (1,1,1) – it corresponds to a set of radii $(p_1,p_2,p_3)=(\sqrt{2},\sqrt{2},\sqrt{2})$ and cyclic shifts of the word (2,0,0), that is, a set of radii ($\sqrt{6}$,0,0). Let D=6. The energy distance between any pair of four constructive classes of signals $X^{(p)}$ is not less than D=6, which is checked by (29). Between the signals of each of the last three classes (with different dialing radii), the distance is also at least D=6, so all elements of these classes can be taken as signals. Such signals are 3·6=18. In the first constructive class, 63=216 signals. Of these, using an HS example 4, one can select 332=54 signal at D=6 The resulting set of M=72 signals corresponds to the most well-known 6-dimensional packing of equal spheres relating to their equal sphere [30]. Elements of this set can be encoded with four source characters $u=(u_1,u_2,u_3,u_4)$, from which u_1, u_2 are three, u_3 is binary, u_4 is quaternary. If $u_4\le 2,$ then $p_1=p_2=p_3=p^{(1)}=\sqrt{2},$ that is, the word p=(1,1,1) of the polytope code is passed, and the angles at the vertices of polytopes (hexagons) are equal to $\alpha_n=\alpha_n^{(1)}=s_n\pi/3+t_n 2\pi/3,$ where $t_1=u_1,\ \ t_2=u_2,$ $t_3=u_4,\ \ s_1=s_2=s_3=u_3.$ If $u_4=3,$ then $p_n=0$ at $n\ne 1+u_2,$ $p_{1+u_2}=\sqrt{6},\ \ \alpha_{1+u_2}=\pi/6+u_3\pi/3+u_1 2\pi/3.$ The resulting set of signals can be matched to the hierarchy and then use HS. The corresponding structure, however, wins a little compared to the example 1 construct.

Heterogeneous constructions similar to those described in recent examples are unacceptable at high dimensionality. Let us turn to the RS when, given (19), (28) [3, 5]:

$$\sum_{i=0}^{R-1}\left(p^{(i)}\right)^2 W_i=N_v,\ \ \sum_{i=0}^{R-1}\left(p^{(i)}\right)^2 w_i=wv, \tag{31}$$

where the latter formula refers to case (26), that is, to HRS.

For a hierarchical heterogeneous code construct, we perform the following calculations.

Example 3. Let v=1 and the ternary set $X=\{0,\pm 1\}$ be divided into R=2 subsets $X^{(0)}=\{0\}$ (the point at the beginning of the numerical axis, $p^{(0)}$=0) and $X^{(1)}=\{\pm p\}$ (zero-dimensional sphere of radius $p^{(1)}$=p), the only nonzero distance between which is equal, considering (31), to $\delta_x(0,1)=\delta_x=p^2=N/W_1,$ where W_1 is the weight of the equilibrium code $P=P[N-W_1,W_1]$ of length N. Since all W_1 of nonzero signal components belong to the same binary sets $X^{(1)}$, for the initial signal class, a code construct with a character code (defining the signs of nonzero components) (W_1,M_A,d_A) at $\delta=4p^2$. It follows from (2) and assertion 3 that such an RS would determine the signal system of power $M=M_A M_P$ with a minimum distance $D\ge\min\{4d_A p^2,d_P p^2\},$ where M_P and d_P are the power and Hamming's distance of code P, $p^2=n/W_1$. At N=2W_1, D=4, a code can be the entire set of words of length N of weight W_1 whose power is $M_P=C_N^{N/2}$, and the binary code of characters – non-redundant [5]. The power of the signal system at D=4 will be $M=M_A M_P=2^{N/2}C_N^{N/2}$.

At dimensionality 16 of HS, example 1 with codes $[4,4,1]_2$, $[4,3,2]_4$, $[4,1,4]_3$ produces 2^{18}·12 signals. The structure of example 3 with the same dimensionality and distance gives $(2^4-1)\cdot 2^8=2^8\cdot 15$ signals. The difference, in this case, is insignificant because the dimensionality of the signals is small and is caused by the fact that the RS allows any words of the code P of length $4N_0$ weight $2N_0$ with the required Hamming's distance, and HS – only those in which the weight of each four characters $(p_{4n-3},p_{4n-2},p_{4n-1},p_{4n})$ is equal to two.

The HRS discussed below allows a similar simplification when only such words of the reversible code are used, in which each group of w adjacent characters p_n, $n\in\{mw-w+1,...,mw\},\ \ 1\le m\le N_0=N/w,$ contains exactly w_i of characters i.

Example 4. We split the set of five one-dimensional elementary signals $X=\{0,\pm p,\pm 2p\}$ into R=3 subsets $X^{(0)}=\{0\},\ \ X^{(1)}=\{\pm p\},\ \ X^{(2)}=\{\pm 2p\}$. In designations (26), we accept w_0=w_2, w_2=2w_0, N=4N_0 and, taking into consideration (31), we obtain p^2=2/3, whence, given (29), $\delta_x=\delta_x(0,1)=\delta_x(1,2)=2/3,\ \ \ \delta_x(0,2)=8/3.$ The ternary permutable code $P=P[N_0,2N_0,N_0]$ here is obviously advantageous to construct by means of a composition of two equilibrium codes $P_2=P[N_0,N_0]$ with symbols from {0, 2} and $P_1=P[2N_0,2N_0]$ with symbols from {1, t} (then replacing t with symbols 0 or 2 of the code P_2 (chapter 5. 1. 1)) because the distance δ_x(0,2) is four times greater than δ_x. Let D=32/3. Then the Hamming's distance of the code P_2 should be equal to $d_{P_2}=4.$ At N_0=8, as P_2, one can choose $M_{P_2}=870$ words of weight 8 of the Hamming's code $[16, 11, 4]_2$ (on the upper estimate from [17], $M_{P_2}\le 1320$). The code $P_1=P[16,16]$ must have Hamming's distance $d_{P_1}=]D/\delta_x[=16$ and it can be an equilibrium (without zero and single words) code of power $M_{P_1}=2(2^5-1)=2\cdot 31.$ The required ternary reversible code of power $M_P=M_{P_1}M_{P_2}=2\cdot 31\cdot 870$ is built. The initial class of signals will be built using HS, taking as a composite elementary polytope a three-dimensional (since one of the coordinates is zero) parallelogram $\tilde{X}=X^{(0)}\times X^{(1)}\times X^{(1)}\times X^{(2)},$ that is, 8 sets $\left(0,p(-1)^{s_3},p(-1)^{s_2},p(-1)^{s_1}\right),$ where $s_i\in\{0,1\}.$ If

$$s_1=q_{1n},\ \ s_2=q_{1n}\oplus q_{2n},\ \ s_3=q_{1n}\oplus q_{2n}\oplus q_{3n},$$

then, from (3), we obtain

$$\delta_1=\left(4p^2+4p^2+16p^2\right)=24p^2,$$

$$\delta_2=\min\left\{4p^2+4p^2,16p^2\right\}=8p^2,$$

$$\delta_3=\min\left\{4p^2,4p^2,16p^2\right\}=4p^2.$$

Thus, we need three level codes with Hamming's distances $d_l=]D/\delta_l[,$ equal to d_1=1, d_2=2, d_3=4, that is, codes $[8,8,1]_2$, $[8,7,2]_2$, $[8,4,4]_2$, which corresponds to the power of M_A=2^{8+7+4}=2^{19}. In total, the system of signals of dimensionality of 32 built with the help of HRS has M=$M_A M_P$=2^{21}·3·5·29·31=$2^{34,72}$ signals at $D=16p^2=32/3.$

Codes $[8,8,1]_2$, $[8,7,2]_4$, $[8,4,4]_4$, $[8,1,8]_3$ produce a signal system of the same dimensionality and the same $D=16p^2=32/3,$ as the HRS from example 4, but with a number of signals $M=2^8 4^7 4^4 3=2^{31,585},$ that is, about three bits less.

Example 5. Assume N_0=10, D=9. Then the codes (in level ascending order) $[10,10,1]_3$, $[10,10,1]_3$, $[10,9,2]_2$, $[10,6,3]_2$,

$[10,6,4]_3$, where the fifth level code comes from the Galley ternary code and the sixth level is not used (the code consists of a single word), give the initial signal class of power $M_A = 3^{26}2^{15} = 2^{56{,}21}$. The above ternary code power score is, at d_P=12, M_P>$2^{16{,}38}$ whence M=M_AM_P>$2^{72{,}6}$.

The simulation showed that the fifth-level Reed-Solomon code $[10,7,4]_9$ and the code $[10,2,6]_3$ of the sixth level together with the codes of the first four levels give a simplified version of the HRS from example 10a, at M=$3^{34}\cdot2^{17}$=$2^{70{,}89}$, which is not much worse. Level 5 code $[40, 26.8]_9$, obtained by shortening the Reed-Solomon code over HS (81), and the code $[40,16,2]_2$ of the sixth level, together with five other codes, produce a simplified version of the HRS example 10b of power M=$3^{131}\cdot2^{79}$=$2^{286{,}64}$, that is, significantly less.

5. 3. The comparative analysis of reducing a signal distance while improving the operational efficiency of a continuous channel of information transmission

Generalized results of estimating the signal distance reduction effect with improving the operational efficiency of code structure of multidimensional signal are given in Table 1.

Table 1 lists the parameters of signal code constructs built using the results of solutions to the examples of the specified types of code structures.

The first column shows the dimensionality of a signal system. The second – the dimensionality of elementary or, in the form of the sum of dimensionalities, a composite elementary polytope. The third – the minimum energy (Euclid square) distance (with a single average energy per coordinate). The fourth column shows the speed R in bits per coordinate. Then the R_1 speed of the best quaternary CPM (or other PM specified in the footnote) with the same N_v and D upper Shannon speed limits R_{Sh} [1] and Kabatyansky-Levenstein speed R_{KL} [30]. The last column shows the example number and the structure (HeS stands for a heterogeneous structure).

Table 1

Results of computing the signal distances of code constructs of multidimensional signals

N_v	v	D	R	R_1	R_{Sh}	R_{KL}	Note
30	2	6	1.159	0.792[1]	2.213	1.592	1, HS
64	2	12	1.096	0.895	2.240	1.639	1, HS
40	4	16	0.963	0.825	2.190	1.082	1, HS
6	2	6	1.028	0.792[1]	1.225	1.065[5]	2, HeS
6	3	2.07	1.515	0.292[2]	1.887	–	2, HeS
16	1+1	4	1.353	1.000	2.163	1.428	3, RS
24	1+2	12	1.024	0.792	1.664	1.114[5]	3, RS
32	1+1+1+1	10.67	1.085	0.969[3]	1.854	1.217	4, HRS
32	4	10.67	0.987	0.969[3]	1.854	1.217	4, HRS
60	2+2+2	9	>1.21	0.983[3]	2.400	1.802	4, HRS
60	6	9	1.182	0.983[3]	2.400	1.802	5, HRS
240	2+2+2	18	>1.42	0.962[4]	2.878	2.307	5, HRS
240	6	18	4.194	0.962[4]	2.878	2.307	5, HRS

Note: [1] – non-redundant 3-dimensional PM; [2] – non-redundant 6-dimensional PM; [3] – at D=8; [4] – at D=16; [5] – from [31, 32].

6. Discussion of results of reducing the signal distance while increasing the operational efficiency of the code construct of a multidimensional signal

Our analysis of the results, summarized in Table 1, reveals the following. The win of HRS compared to HS is the more noticeable the larger the dimensionality. Note, for comparison, that a single (binary) PM corresponds to the speed R=0.5 and D=8, and the two-time (quaternary) – R=1.0, D=4.

This is explained by the fact that the HRS, due to the peculiarities of formation and the possibility of varying the changes in the components of the code construct, can provide all nonzero distances with the same value. First of all, this is an opportunity to form the correct simplex from two signals of a simple HRS. Or a set of orthogonal signals with the same norms in more complex HRS constructs. An additional advantage of the HRS, which is confirmed by the data in Table 1, is that it has the ability to replace all nonzero signal distances with the smallest of them. That is, the possibility of binary quantization of the signal distance. This gives significant advantages in the speed of information transmission. As shown by the data in Table 1.

We shall define the features of the proposed method of forming code structures of multidimensional signals. Implementing modulation procedures, as can be seen from their description in chapter 5. 1, 5. 2, does not encounter fundamental difficulties even in a general case, unless, of course, an acceptable encoding procedure is known for each code. Difficulties may rather arise due to the fact that HS and HRS usually require non-binary codes and even optionally codes over the prime number. Not much is known about specific codes of this type. Recently, significant progress has been made in the theory of non-binary codes. The new ternary and quaternary codes [30–34] are described. Attractive is the direction associated with the codes above the rings of deductions [35, 36]. Sometimes, the necessary codes can be built using HS. The possibilities of reversible structures are also limited primarily by a small number of known structures of reversible codes, especially non-binary ones. Obviously, the most promising HRS is with a reversible code in the form of a composition of equilibrium ones (chapter 5. 3).

Thus, this paper has established and substantiated the relation of a signal distance of the specified types of code structures and the speed of the continuous channel of transmission of multidimensional signal against the background of maintaining a predefined level of noise immunity.

The limitations inherent in the above study include the following. It is possible to build an acceptable procedure of maximum plausible demodulation only in exceptional cases. The simplest approximate procedure that implements the energy distance D (that is, leading to the correct solution at the energy of interference <D/4) can be constructed as a sequence of executables in order of reducing the levels of acceptance procedures in general for individual codes that determine the design of the signal system if each of them implements D (similar to the key algorithm for generalized cascading codes [23]). At the same time, the acceptance in general for the HS l-level code implies minimizing at the appropriate limits ([4]) of the sum $\sum_{n=1}^{N}\Delta_{ln}\left(q_{ln}\right)$, where $\Delta_{ln}\left(q_{ln}\right)=\min\limits_{q_{jn},j<l}\Delta_n\left(q_{1n},\dots,q_{ln},\hat{q}_{l+1,n},\dots,\hat{q}_{Ln}\right)$, $\Delta_n\left(q_{1n},\dots,q_{Ln}\right)$ is the energy distance (or other measures of difference) between the n-th elementary signal $x_n = f\left(q_{1n},\dots,q_{Ln}\right)$ and the corresponding observation z_n at the output of the channel, $\hat{q}_{in}$ is the symbol of the word-solution of the code of the i-th level, i>l. With HRS, the reception in general for the reversible code is initially performed, which can be interpreted as a code of the (L+1)-th level (with the compositional design of the code, demodulation begins with the P_1 code). With RS, after solving the word of the reversible code, the task is reduced to the demodulation of the initial class of signals. Proof of the implementation of distance D with such a simple algorithm is based on the fact that with the correct reception of words of previous levels, the value $\sum \Delta_{ln}\left(q_{ln}^{*}\right)$, where q_l^* is the transmitted word of the l-th level code, does not exceed the energy of the interference vector.

Thus, for the suitability of the described structures, in fact, it would suffice that, for each of the defining designs of codes, there are known coding procedures and at least suboptimal reception in general.

A certain disadvantage of the proposed coding methods devised in the present paper is that encoding procedures should be developed under binary codes. However, in turn, the development of methods for improving code structures such as HS and HRS implies the use of non-binary codes as well. In this work, the issues of applying such codes are not considered in detail.

Further research on the use of non-binary codes for the formation of code structures is proposed as a further promising area of research and development in this direction. Additionally, certain prospects of research are associated with the issue of assessing the conditions of suboptimality of signal reception under the conditions of influence of additive Gaussian noise and during the Rayleigh attenuation of the multidimensional signal on the transmission path.

Such research involves the development of theoretical provisions for the study of the effects of noise and perturbations in the formation of a code construct of a multidimensional signal. And, against the background of accounting for the Rayleigh signal attenuation, there may be difficulties in the construction of the necessary mathematical apparatus and its further implementation in software.

7. Conclusions

1. It was established that the code constructs that are based on the construction of a hierarchical structure set a system of signals with minimal signal and distances, which can be determined by the values of the minimum Hamming's distances in the structure of code codes.

These types of code structures, in this work, include a hierarchical code construct of signals; a hierarchical code construct of signals with Euclidean metric; a reversible code construct of signals; a reversible code construct of signals with Euclidean metric.

By varying the values of signal distances to the minimum, compared to Hamming's distances, for certain code structures of multidimensional signals, it is possible to significantly increase the volume and speed of transmitted information along a continuous channel.

2. As a result of modeling, it was established that, depending on the type of code structures, changes in a signal distance can significantly increase the speed of information transmission in bits per coordinate.

For a hierarchical code structure, this can be up to 20 percent with a halved signal distance.

For a hierarchical reversible structure, with a decrease in the signal distance by 10 percent or more, the increase in the speed of information transmission can reach up to 35 percent or more.

3. The hierarchical reversible code construct, in comparison with the hierarchical code structure, ensures a win up to two or more times in the speed of information transmission with a halved signal distance.

Implementation of the modulation procedure has no fundamental difficulties, on the condition that for each code of the code construct the encoding procedure is known when using binary codes.

The simplest approximate demodulation procedure that implements the signaling distance can be constructed as a sequence of procedures in descending order of reception procedures in general for individual codes that determine the design of the system as a whole.

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