

This paper reports a solution to the problem of determining the configuration parameters of PID controllers when arranging the poles of the transfer function of a linear single-circuit automated control system for a predefined set of control objects.

Unlike known methods in which the task to find the optimal settings of a PID controller is formed as a problem of nonlinear programming, in this work a similar problem is reduced to solving a system of linear algebraic equations.

The method devised is based on the generalized Viète theorem, which establishes the relationship between the parameters and roots of the characteristic equation of the automatic control system. It is shown that for control objects with transfer functions of the first and second orders, the problem of determining the configuration parameters of PID controllers has an unambiguous solution. For control objects with transfer functions of the third and higher orders, the generated problem is reduced to solving the redefined system of linear algebraic equations that has an unambiguous solution when the Rouché–Capelli theorem condition is met.

Such a condition can be met by arranging one of the roots of the characteristic equation of the system on a complex plane. At the same time, the requirements for the qualitative indicators of the system would not always be met. Therefore, alternative techniques have been proposed for determining the configuration parameters of PID controllers. The first of these defines configuration parameters as a pseudo solution to the redefined system of linear algebraic equations while the second produces a solution for which the value of the maximum residual for the system of equations is minimal.

For each case, which was used to determine the settings of PID controllers, such indicators of the control process as overshooting and control time have been determined

Keywords: automatic control, PID controller, system of equations, configuration parameters, process quality indicators

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DETERMINING CONFIGURATION PARAMETERS FOR PROPORTIONALLY INTEGRATED DIFFERENTIATING CONTROLLERS BY ARRANGING THE POLES OF THE TRANSFER FUNCTION ON THE COMPLEX PLANE

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1. Introduction

The task to synthesize an automated control system for a particular object is one of the most important in its design. The purpose of synthesis is to construct a system that would meet certain requirements, namely: the system must be robust; demonstrate the predefined control quality indicators such as overshoot and regulation time.

The system synthesis can be carried out using both frequency and time methods.

Frequency methods of synthesis are based on the use of frequency characteristics such as Bode, Nyquist diagrams, or coordinates built in polar systems. They make it possible to synthesize controlling devices in order to achieve certain quality indicators of the control process, such as the maximum value of the frequency performance amplitude, resonance frequency, bandwidth, and margin of stability for the amplitude and phase.

Methods of system synthesis in the time domain make it possible to evaluate the quality indicators of the system according to its transition characteristic, which depends on

the arrangement of zeros and poles in the transfer function of the system.

The synthesis of the automatic control system involves not only the choice of its structure but also determining its parameters, first of all, the parameters to configure PI and PID controllers.

Thus, it is a relevant task to determine the configuration parameters for PID controllers by arranging the poles on a complex plane since it makes it possible to reduce the problem to solving a system of linear algebraic equations.

2. Literature review and problem statement

PID controllers are the most common among all types of regulators. At present, up to 90–95 % of all controllers in operation employ PID control algorithms [1]. Such high popularity of PID controllers is due to the simplicity of their structure, low cost, and their suitability for solving most practical tasks.

The stability of an automatic control system and the quality of the control process largely depend on the configuration parameters of the PID controller.

One of the first studies that addressed solving the problem of determining the configuration parameters for PID controllers was reported in [2]. The essence of the Sigler-Nichols method is that the two parameters, L and R , were determined based on the transitional characteristic of the object. To determine them, the inflection point was determined on the acceleration curve and a straight line was drawn through it, which is tangent to the acceleration curve. Such a straight line cut from the axis of abscissa the value of L while the angle of inclination to the same axis defined the value for R . The Sigler-Nichols method can only be used for a limited class of objects; it is largely subjective in determining the L and R parameters. Further research on solving the task of finding optimal configuration parameters for PID controllers was carried out in two directions – the time [3] and frequency domains [4].

A characteristic feature of works [3] and [4] is that the task of determining the optimal configuration parameters of PID controllers is stated as a problem of nonlinear programming. In addition, the authors confined themselves to objects with transmission functions of the first order with a delay.

A method for determining the configuration parameters of PID controllers in the time domain is based on minimizing the function of the control rms error [5]. The configuration parameters for a PID controller, obtained in this way, typically ensure the stability of the automatic control system but the transition processes in such a system are oscillatory in nature, whose intensity of attenuation is unsatisfactory in most cases. Therefore, it is necessary to introduce an additional component to the criterion of optimality based on the speed of change in the control error with a certain weight coefficient, whose value is selected on the basis of the intuition of the researcher followed by the subsequent repeated machine experiments.

The task of determining the configuration parameters of PID controllers in the temporal domain is a nonlinear optimization problem; solving it requires genetic algorithms [6], neural network technologies [7], and the so-called bacteria reproduction algorithm [8] (Bacterial Foraging Optimization Algorithm, BFOA-algorithm), which is proposed by the authors of work [9].

The use of so many algorithms for finding a minimum of the control rms error indicates the difficulty of solving the task, which is even more complicated when considering the restrictions for the configuration parameters of PID controllers.

The second direction in determining the configuration parameters of PID controllers is based on the use of frequency characteristics of automatic control systems. Among the methods used, the most common is the method of extended amplitude-phase characteristics [10] and the method of determining the configuration parameters of PID controllers using conventional frequency characteristics [11]. The disadvantage of frequency methods is that they are graphic-analytical, which causes certain difficulties in their computer implementation.

A method for determining the parameters of PID controllers by arranging the poles of the transfer function of the closed automatic control system on a complex plane is proposed, which makes it possible to reduce the problem to finding a solution to the system of linear algebraic equations.

The task of calculating the configuration parameters for a PID controller by arranging the poles of the transfer

function of the automatic control system was partially considered in [12] for the oil separation process.

Thus, the issue of determining the configuration parameters for PID controllers remains relevant, as evidenced by numerous publications. The methods and algorithms offered by their authors are reduced to solving problems of nonlinear programming, which imposes certain restrictions on the class of mathematical models that describe the dynamics of control objects. At the same time, little attention is paid in scientific papers to methods based on determining the parameters of configuration of PID controllers by arranging roots on a complex p -plane.

3. The aim and objectives of the study

The purpose of this work is to build a method for determining the parameters of PID controllers by arranging poles on a complex plane (p -plane) of the transfer function of a single-circuit automatic control system. This will make it possible to develop effective algorithms for determining the configuration parameters of PID controllers for the pre-defined set of transfer functions of control objects.

To accomplish the aim, the following tasks have been set:

- to form the concept for determining the configuration parameters of PID controllers for control objects with transfer functions up to the third order inclusive;
- to devise a procedure for determining the configuration parameters of PID controllers, focused on the pre-defined class of transfer functions of control objects.

4. The study materials and methods

We consider a single-circuit automatic control system (Fig. 1), which consists of a control object with the transfer function $W_{ob}(p)$ and a controlling element whose transfer function is $W_{cl}(p)$.

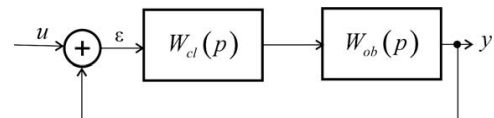


Fig. 1. Single-circuit automatic control system

Let the transfer function of the object be assigned as the ratio of two polynomials

$$W_{ob}(p) = \frac{\sum_{j=0}^m b_j p^{m-j}}{\sum_{i=0}^n a_i p^{n-i}}, \tag{1}$$

and the controller has the following transfer function:

$$W_{cl}(p) = C_1 + C_2 p + \frac{C_0}{p}. \tag{2}$$

In (1), (2), the following notations are adopted: a_i , $i = 0, n$; b_j , $j = 0, m$ are the parameters of the transfer function of the object; C_0 , C_1 , and C_2 are the controller's configuration options.

In addition, we assume $n < m$.

The transfer function of the open system (Fig. 1) takes the following form:

$$W_{os}(p) = \frac{(C_0 + C_1p + C_2p^2) \sum_{j=0}^m b_j p^{m-j}}{p \sum_{i=0}^n a_i p^{n-i}}.$$

5. Results of building a method for determining the configuration parameters of PID controllers

5.1. The concept of determining the configuration parameters of PID controllers

Knowing the transfer function $W_{os}(p)$ of the open system, we find the transfer function of the closed system relative to the input value u (Fig. 1)

$$W_{yu}(p) = \frac{(C_0 + C_1p + C_2p^2) \sum_{j=0}^m b_j p^{m-j}}{(C_0 + C_1p + C_2p^2) \sum_{j=0}^m b_j p^{m-j} + p \sum_{i=0}^n a_i p^{n-i}}. \tag{3}$$

Expression (3) is to be recorded in a slightly different form

$$W_{yu}(p) = \frac{(C_0 + C_1p + C_2p^2) \sum_{j=0}^m \beta_j p^j}{(C_0 + C_1p + C_2p^2) \sum_{j=0}^m \beta_j p^j + p \sum_{i=0}^n \alpha_i p^i}, \tag{4}$$

where $\beta_j = \overline{b_{m-j}}$, $j = \overline{0, m}$; $\alpha_i = \overline{a_{n-i}}$, $i = \overline{0, n}$.

The denominator of the transfer function $W_{yu}(p)$ contains the product of two polynomials. Multiplication of two polynomials is reduced to finding the product of their coefficients [13] according to the following formula:

$$s_k = \sum_{l=0}^k C_l \beta_{k-l}, \quad k = \overline{0, m+2}, \tag{5}$$

where $C_3 = C_4 = \dots = C_{m+2} = 0$; $\beta_{m+1} = \beta_{m+2} = 0$.

Thus,

$$\begin{aligned} s_0 &= C_0 \beta_0; \\ s_1 &= C_0 \beta_1 + C_1 \beta_0; \\ s_2 &= C_0 \beta_2 + C_1 \beta_1 + C_2 \beta_0; \\ s_3 &= C_0 \beta_3 + C_1 \beta_2 + C_2 \beta_1; \\ &\dots\dots\dots \\ s_m &= C_0 \beta_m + C_1 \beta_{m-1} + C_2 \beta_{m-2}; \\ s_{m+1} &= C_1 \beta_m + C_2 \beta_{m-1}; \\ s_{m+2} &= C_1 \beta_m + C_2 \beta_{m-1}. \end{aligned}$$

Thus, the denominator of transfer function (4) is the sum of two polynomials

$$Q(p) = Q_1(p) + Q_2(p),$$

where

$$Q_1(p) = s_0 + s_1p + s_2p^2 + \dots + s_m p^m + s_{m+1} p^{m+1} + s_{m+2} p^{m+2};$$

$$Q_2(p) = \alpha_0 p + \alpha_1 p^2 + \alpha_2 p^3 + \dots + \alpha_n p^{n+1}.$$

Considering the $Q_1(p)$ and $Q_2(p)$ values, we find that

$$Q(p) = s_0 + (s_1 + \alpha_0)p + (s_2 + \alpha_1)p^2 + \dots + (s_m + \alpha_{m-1})p^m + \alpha_m p^{m+1} + (s_{m+1} + \alpha_m)p^{m+1} + (s_{m+2} + \alpha_{m+1})p^{m+2} + \dots + \alpha_{n-1} p^n + \alpha_n p^{n+1}. \tag{6}$$

Characteristic polynomial (6) is recorded by the descending powers

$$Q(p) = \pi_0 p^{n+1} + \pi_1 p^n + \dots + \pi_{n-m} p^{m+1} + \pi_{n+1-m} p^m + \dots + \pi_n p + \pi_{n+1}, \tag{7}$$

where $\pi_0 = \alpha_n$; $\pi_1 = \alpha_{n-1}$; ...; $\pi_{n-m} = \alpha_m$; $\pi_{n-m+1} = s_m + \alpha_{m-1}$; ...; $\pi_n = s_1 + \alpha_0$; $\pi_{n+1} = s_0$.

Summarizing the result, we can write down:

$$\pi_i = \begin{cases} \alpha_{n-i} & \text{at } i = \overline{0, n-m-2}, \\ s_{n+1-i} + \alpha_{n-i} & \text{at } i = \overline{n-m-1, n+1}. \end{cases} \tag{8}$$

For ratio (8), the following condition must be met:

$$n - m - 2 \geq 0. \tag{9}$$

If $n - m - 2 < 0$, then the π_i value should be calculated using the following formula:

$$\pi_i = s_{n+1-i} + \alpha_{n-i} \quad \text{at } i = \overline{0, n+1}. \tag{10}$$

We assume that p_1, p_2, \dots, p_{n+1} roots of characteristic polynomial (7) contain no multiples.

Then, in accordance with Vieta's theorem, we obtain the following system of relations between the roots of characteristic equation (7) and its coefficients:

$$\begin{aligned} p_1 + p_2 + \dots + p_{n+1} &= -\frac{\pi_1}{\pi_0}, \\ p_1 p_2 + p_1 p_3 + \dots + p_n p_{n+1} &= \frac{\pi_2}{\pi_0}, \\ p_1 p_2 p_3 + p_1 p_2 p_4 + \dots + p_{n-1} p_n p_{n+1} &= -\frac{\pi_3}{\pi_0}, \\ &\dots\dots\dots \\ \sum_{i_1 < i_2 < \dots < i_k} p_{i_1} p_{i_2} \dots p_{i_k} &= (-1)^k \frac{\pi_k}{\pi_0}, \\ &\dots\dots\dots \\ p_1 p_2 \dots p_{n+1} &= (-1)^{n+1} \frac{\pi_{n+1}}{\pi_0}. \end{aligned} \tag{11}$$

In the system of equations (11), its right-hand sides are the functions of the controller configuration parameters, which makes it possible to select the $C_0, C_1,$ and C_2 values

so that the quality of the control process meets certain criteria.

In this case, we shall consider the following. For the automatic control system (Fig. 1) to be stable, it is necessary and sufficient that all the roots of characteristic equation (6) should belong to the left semi-plane of the p -plane (the complex plane of roots).

Therefore, first, the C_0 , C_1 , and C_2 values must be such to ensure that the system is robust; second, the system should have the desired properties [14] – the degree of stability η and fluctuations μ .

The degree of stability is determined by the nearest left real root to the imaginary axis $\eta = \min_k |\operatorname{Re} p_k|$, $k \in \{1, 2, \dots, n\}$. Designate the nearest left complex-coupled root as $p_c = -\alpha_c \pm j\zeta_c$. Then $\mu = \frac{\zeta_c}{\alpha_c}$.

In a general case, when the $n < m$ condition is met, the characteristic polynomial (6) of the automatic control system has $n+1$ simple roots. Therefore, in accordance with ratios (11), we obtain a system of linear algebraic equations whose dimensionality is $n+1$. Such a system contains three unknown C_0 , C_1 , and C_2 , that is, the system of equations (11) belongs to the class of redefined equations [15]. In the case where the characteristic polynomial of the controlled object has a second power, then the system of equations (11) is a second-order system with three unknowns. Such a system is indeterminate [15]. And, only for the characteristic second-order polynomial, we obtain a system of three equations with three unknowns.

If we form matrix A of the coefficients of unknowns in the system of equations (11), then the rank of such a matrix

$$\operatorname{rang}(A) = \min(n+1, 3), \tag{12}$$

where matrix A has a dimensionality of $(n+1) \times 3$.

Matrices for which condition (12) is met are the matrices of full rank [15]. The system of linear algebraic equations with matrices of full rank for $n=1$ is indeterminate and always compatible but does not have a single solution. If $n=2$, then, for the non-special matrix A , we obtain a single solution. The redefined system ($n>3$) is only compatible if the Rouché–Capelli theorem [16] is met, that is, the following equality should hold:

$$\operatorname{rang}(A) = \operatorname{rang}([A \mid B]), \tag{13}$$

where B is the vector of free terms of the right-hand sides of equation system (11).

5. 2. Procedure for determining the configuration parameters of PID controllers focused on the predefined class of transfer functions of control objects

5. 2. 1. An object’s first-order transfer function

Assume the transfer function of a controlled object is as follows:

$$W_{ob}(p) = \frac{k}{Tp+1},$$

where k is the transmission coefficient;

T – an object’s time constant.

The dynamic properties of the controller are characterized by transfer function (2).

In the considered case, the transfer function of the closed system (Fig. 1) takes the following form:

$$W_{yu}(p) = \frac{k(C_0 + C_1p + C_2p^2)}{k(C_0 + C_1p + C_2p^2) + p(Tp+1)}. \tag{14}$$

From formula (5), we find: $s_0 = C_0\beta_0$; $s_1 = C_1\beta_0$; $s_2 = C_2\beta_0$. Since $\beta_j = b_{m-j}$, $a_i = a_{n-1}$, where $m=0$, $n=1$, then $\beta_0 = b_0 = k$; $a_0 = a_1 = 1$, $a_1 = a_0 = T$.

Given that $n=1$, the power of a characteristic polynomial is equal to two. So

$$Q(p) = \pi_0p^2 + \pi_1p + \pi_2. \tag{15}$$

Since $n-m-2 < 0$, therefore, to determine the coefficients π_i , $i=0, 1, 2$ of the characteristic polynomial $Q(p)$, we use formula (10). Then: $\pi_0 = s_2 + \alpha_1$, $\pi_1 = s_1 + \alpha_0$, $\pi_2 = s_0$.

If we take into consideration the corresponding values s_0 , s_1 and s_2 , and α_0 i α_1 , we obtain

$$Q(p) = (kC_2 + T)p^2 + (kC_1 + 1)p + kC_0. \tag{16}$$

Of course, in this simple case, the characteristic polynomial (16) could be obtained directly from transfer function (12). The built algorithm for determining the characteristic polynomial shows its superiority over the technique of multiplication of polynomials (opening brackets) in the cases where we have high-power polynomials in the numerator and denominator of transfer function (3).

For characteristic polynomial (15), the following system of equations is constructed:

$$p_1 + p_2 = -\frac{\pi_1}{\pi_0},$$

$$p_1p_2 = \frac{\pi_2}{\pi_0}.$$

Hence, we find

$$\pi_0(p_1 + p_2) = -\pi_1,$$

$$\pi_0p_1p_2 = \pi_2.$$

Considering values for π_0 , π_1 , and π_2 , the following ratios have been derived:

$$(kC_2 + T)(p_1 + p_2) = -(kC_1 + 1),$$

$$(kC_2 + T)p_1p_2 = kC_0. \tag{17}$$

We have two equations and three unknowns. One of them can be chosen arbitrarily. Let $C_2=0$. Then, from the system of equations (17), we find:

$$C_1 = -\frac{1}{k}(T(p_1 + p_2) + 1), \tag{18}$$

$$C_0 = \frac{1}{k}Tp_1p_2. \tag{19}$$

Thus, formulas (18) and (19) make it possible to determine the setting parameters for a PI-controller by arranging the roots p_1 and p_2 of the characteristic equation on the complex p -plane. Obviously, to ensure that the system is robust, the roots p_1 and p_2 must be left.

Accept $p_1 = -\alpha + j\zeta$ and $p_2 = -\alpha - j\zeta$, where $\alpha > 0$ i $\zeta > 0$. Then, as it follows from formulas (18) and (19), we have:

$$C_1 = \frac{1}{k}(2T\alpha - 1), \tag{20}$$

$$C_0 = \frac{1}{k}T(\alpha^2 + \beta^2). \tag{21}$$

In most cases, the transition process can be considered complete if the component of the transition process, which is determined by the degree of stability η , is damped. That means that $\alpha = \eta$ and the degree of damping is $\mu = \frac{\zeta}{\alpha}$ or $\mu = \frac{\zeta}{\eta}$. Hence, $\zeta = \mu\eta$.

Taking into consideration the found values, formulas (20), (21) take the following form:

$$C_1 = \frac{1}{k}(2\eta T - 1), \tag{22}$$

$$C_0 = \frac{1}{k}T\eta^2(\mu^2 + 1). \tag{23}$$

In a general case, the time of control t_p can be approximately estimated [14] by the following formula:

$$t_p = \frac{1}{\eta} \ln \frac{1}{\chi},$$

where $\chi = 0,05 \div 0,1$.

After selecting the desired t_p value, one can find the degree of stability of the system:

$$\eta = \frac{1}{t_p} \ln \frac{1}{\chi}. \tag{24}$$

A numerical experiment was carried out. The programming environment MATLAB (The MathWorks, USA) was used to develop a program for building the transition characteristics of the system at a single jump-like input action. The parameters of the system's transfer function were as follows: $k=2.5$; $T=12$ s; $t_p=18$ s; and the values C_0 and C_1 were calculated from formulas (22), (23).

The results from the numerical experiment are given in Table 1.

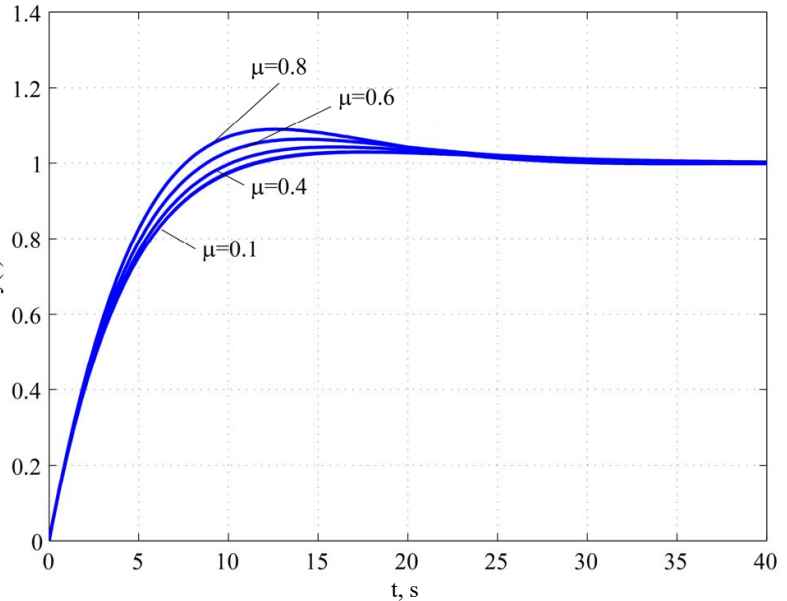


Fig. 2. Transitional characteristics of the system at different μ values

Analysis of Table 1 shows that the synthesized automatic control system allows for a satisfactory quality of control – the overshoot value does not exceed the permissible norms. The control time is slightly higher than the specified value. This is because (24) is approximate.

5. 2. 2. An object's second-order transfer function

Now a more complicated case is being considered. Let the transfer function of the control object be as follows:

$$W_{ob}(p) = \frac{b_0 p + b_1}{a_0 p^2 + a_1 p + a_2}. \tag{25}$$

The dynamic properties of the controller are characterized by transfer function (2).

The characteristic equation of a closed system is to be written in the form of ratio (7). Since $n=2$, ratio (7) is a third-power polynomial:

$$Q(p) = \pi_0 p^3 + \pi_1 p^2 + \pi_2 p + \pi_3. \tag{26}$$

Next, we find the polynomial's coefficients (26). The left-hand side of inequality (9) at $n=2$ and $m=1$ will be negative. Therefore, the values $\pi_i, i=0, 1, 2, 3$ are calculated from formula (10). We obtain

$$\pi_0 = s_3 + \alpha_2, \quad \pi_1 = s_2 + \alpha_1, \quad \pi_2 = s_1 + \alpha_0, \quad \pi_3 = s_0,$$

where $s_0 = C_0\beta_0, s_1 = C_0\beta_1 + C_1\beta_0, s_2 = C_1\beta_1 + C_2\beta_0, s_3 = C_2\beta_1$. Since $\beta_0=b_1, \beta_1=b_0, \alpha_0=a_2, \alpha_1=a_1, \alpha_2=a_0$, then

$$\pi_0 = C_2 b_0 + a_0,$$

$$\pi_1 = C_1 b_0 + C_2 b_1 + a_1,$$

$$\pi_2 = C_0 b_0 + C_1 b_1 + a_2,$$

$$\pi_3 = C_0 b_1.$$

Table 1
Values of the PI-controller configuration parameters and the control process quality indicators

The degree of fluctuation, μ	Configuration parameter		Control quality indicator	
	C_0, s^{-1}	C_1	Over-shoot, $\sigma, \%$	Control time, t_p, s
0.2	0.35	3.00	2.64	28.32
0.4	0.19	1.50	4.04	29.12
0.6	0.15	1.00	6.30	28.64
0.8	0.15	0.75	9.08	26.40

Ratios (11) for a third-power polynomial are:

$$\pi_0(p_1 + p_2 + p_3) = -\pi_1,$$

$$\pi_0(p_1p_2 + p_1p_3 + p_2p_3) = \pi_2,$$

$$\pi_0p_1p_2p_3 = -\pi_3.$$

Considering the values of $\pi_i, i=0, 1, 2, 3$, we obtain

$$\begin{aligned} (C_2b_0 + a_0)(p_1 + p_2 + p_3) &= -(C_1b_0 + C_2b_1 + a_1), \\ (C_2b_0 + a_0)(p_1p_2 + p_1p_3 + p_2p_3) &= C_0b_0 + C_1b_1 + a_2, \\ (C_2b_0 + a_0)p_1p_2p_3 &= -C_0b_1. \end{aligned} \quad (27)$$

Assume that one root of polynomial (26) is left and real, and the other two are the left and complex-coupled ones: $p_1 = -\alpha_{c,1}, p_2 = -\alpha_{c,2} + j\zeta_{c,2}, p_3 = -\alpha_{c,2} - j\zeta_{c,2}$. Then

$$\begin{aligned} p_1 + p_2 + p_3 &= -\alpha_{c,1} - 2\alpha_{c,2}, \\ p_1p_2 + p_1p_3 + p_2p_3 &= 2\alpha_{c,1}\alpha_{c,2} + \alpha_{c,2}^2 + \zeta_{c,2}^2, \\ p_1p_2p_3 &= -\alpha_{c,1}(\alpha_{c,2}^2 + \zeta_{c,2}^2). \end{aligned}$$

The results make it possible to write down a system of equations (27) in the following form:

$$\begin{aligned} (C_2b_0 + a_0)(\alpha_{c,1} + 2\alpha_{c,2}) &= C_1b_0 + C_2b_1 + a_1, \\ (C_2b_0 + a_0)(\alpha_{c,2}^2 + \zeta_{c,2}^2 + 2\alpha_{c,1}\alpha_{c,2}) &= C_0b_0 + C_1b_1 + a_2, \\ \alpha_{c,1}(C_2b_0 + a_0)(\alpha_{c,2}^2 + \zeta_{c,2}^2) &= C_0b_1. \end{aligned} \quad (28)$$

We obtained a system of three linear algebraic equations with three unknowns. From the last equation of system (28), we determined

$$C_0 = \frac{\alpha_{c,1}}{b_1}(C_2b_0 + a_0)(\alpha_{c,2}^2 + \zeta_{c,2}^2). \quad (29)$$

The C_0 value is put in the second equation of system (28). As a result, we have a system of two equations with two unknowns

$$\begin{aligned} (C_2b_0 + a_0)(\alpha_{c,1} + 2\alpha_{c,2}) &= C_1b_0 + C_2b_1 + a_1, \\ (C_2b_0 + a_0)(\alpha_{c,2}^2 + \zeta_{c,2}^2 - 2\alpha_{c,1}\alpha_{c,2}) &= \\ = \frac{b_0}{b_1}\alpha_{c,1}(C_2b_0 + a_0)(\alpha_{c,2}^2 + \zeta_{c,2}^2) + C_1b_1 + a_2. \end{aligned}$$

After obvious algebraic transformations, the transition to the following system of linear algebraic equations is carried out:

$$A_{11}C_1 + A_{12}C_2 = B_1, \quad (30)$$

$$A_{21}C_1 + A_{22}C_2 = B_2. \quad (31)$$

where $A_{11} = -b_0; A_{12} = (\alpha_{c,1} + 2\alpha_{c,2})b_0 - b_1; A_{21} = -b_1^2;$

$$A_{22} = b_0(b_1(\alpha_{c,2}^2 + \zeta_{c,2}^2 + 2\alpha_{c,1}\alpha_{c,2}) - \alpha_{c,1}b_0(\alpha_{c,2}^2 + \zeta_{c,2}^2));$$

$$B_1 = a_1 - a_0(\alpha_{c,1} + 2\alpha_{c,2});$$

$$\begin{aligned} B_2 &= a_0\alpha_{c,1}b_0(\alpha_{c,2}^2 + \zeta_{c,2}^2) + \\ &+ b_1(a_2 - a_0(\alpha_{c,2}^2 + \zeta_{c,2}^2 + 2\alpha_{c,1}\alpha_{c,2})). \end{aligned}$$

Solving the system of equations (30), (31) produces:

$$C_1 = \frac{A_{22}B_1 - A_{12}B_2}{A_{11}A_{22} - A_{12}A_{21}},$$

$$C_2 = \frac{A_{11}B_2 - A_{21}B_1}{A_{11}A_{22} - A_{12}A_{21}}.$$

The known C_2 value is used to find C_0 from formula (29).

The next step in calculating the parameters of the controller is to select the value for the roots of the characteristic equation in such a way that, first, the system is robust, and, second, the system should demonstrate the desired quality indicators of the control process.

The first condition is met automatically by arranging the roots of the characteristic equation in the left semi-plane of the p -plane.

To form the conditions for meeting the second requirement, consider the p -plane (Fig. 3), which hosts the roots of characteristic polynomial (24).

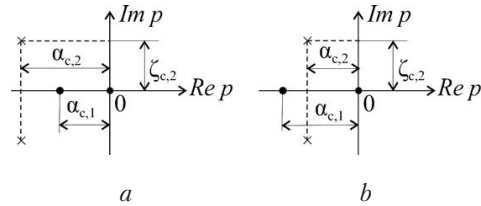


Fig. 3. Arranging the roots of a characteristic polynomial on the p - plane

Let the following ratio between the $\alpha_{c,1}$ and $\alpha_{c,2}$ values hold (Fig. 3).

$$\alpha_{c,2} = k_\alpha \alpha_{c,1}. \quad (32)$$

At $k_\alpha > 1$, the real root is closer to the imaginary $Im p$ axis (Fig. 3, a); in the case where $k_\alpha < 1$ (Fig. 3, b) – a complex-coupled root. As before, the degree of system stability η is determined from formula (24), and the degree of fluctuation is $\mu = \frac{\zeta_{c,2}}{\alpha_{c,2}}$.

Since $\alpha_{c,1} = \eta$, and $\alpha_{c,1} = k_\alpha \alpha_{c,1}$, then $\alpha_{c,2} = k_\alpha \eta$. If we consider (24), we obtain: $\alpha_{c,1} = \frac{1}{t_p} \ln \frac{1}{\chi}$ and, accordingly, $\alpha_{c,2} = \frac{k_\alpha}{t_p} \ln \frac{1}{\chi}$. Knowing $\alpha_{c,2}$, we derive the values $\zeta_{c,2} = \mu \frac{k_\alpha}{t_p} \ln \frac{1}{\chi}$.

Let the parameters of transfer function (25) be taken as follows: $b_0 = 4, b_1 = 7; a_0 = 20, a_1 = 6, a_2 = 1$. The other parameters were $k_\alpha = 1, 2$ and $k_\alpha = 0, 8, t_p = 20, \chi = 0, 05$.

For values of the degree of fluctuations $\mu \in \{0,2; 0,4; 0,6; 0,8\}$, by using the software developed in the MATLAB programming environment, we built the plots of the transition characteristics of the system at a single jump-like input value (Fig. 4). Since, for this case, $k_\alpha > 1$, the closest to the imaginary axis is the real root p_1 .

Fig. 4 shows that increasing the degree of fluctuation μ entails an increase in overshoot (Table 2).

In the case where $k_\alpha = 0,8$ ($k_\alpha < 1$), the transition characteristics of the system for different values of μ (Fig. 5) are constructed. This is the case when the closest to the imaginary axis is the left complex-coupled root.

Analysis of the results given in the Table 2 shows that the overshoot value increases at $k_\alpha < 1$. The maximum value of σ increased from 14.7032 % at $k_\alpha = 1,2$ to 27.8766, that is the increase in σ was 10.2 %. The control time, in this case, increased in comparison with the case where $k_\alpha > 1$.

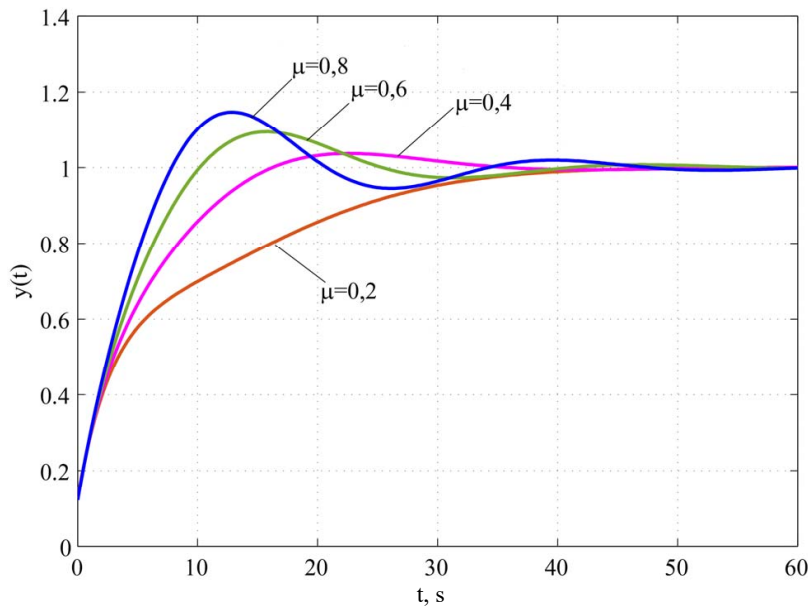


Fig. 4. Transition characteristics of the system at $k_\alpha = 1,2$

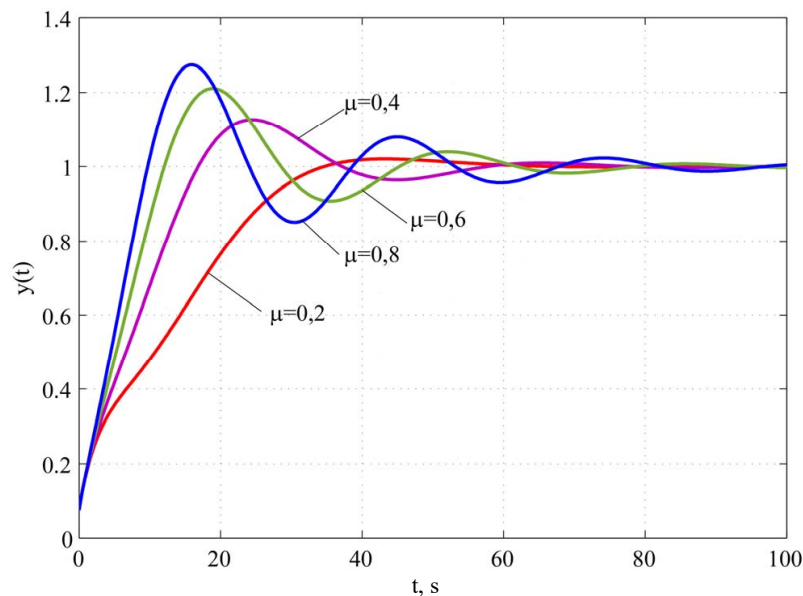


Fig. 5. Transition characteristics of the system at $k_\alpha = 0,9$

Table 2
Values of the PI-controller configuration parameters and the control process quality indicators at $k_\alpha = 0,9$

The degree of fluctuation, μ	Configuration parameter			Control process quality indicator	
	C_0, s^{-1}	C_1	C_2, s	Overshoot, $\sigma, \%$	Control time, t_p, s
0.2	0.0125	0.0358	0.4218	2.0598	54.2400
0.4	0.0250	0.0418	0.4172	12.6662	54.2400
0.6	0.0375	0.0518	0.4097	20.9134	74.6400
0.8	0.0498	0.0658	0.3992	27.8766	80.4000

5. 2. 3. An object’s third-order transfer function

We shall apply the method built for determining the parameters of setting the controller with transfer function (2) when the object (Fig. 1) is characterized by the following transfer function:

$$W_{ob}(p) = \frac{b_0 p^2 + b_1 p + b_2}{a_0 p^3 + a_1 p^2 + a_2 p + a_3}. \quad (33)$$

Using an earlier constructed algorithm, we find a characteristic polynomial of the system depicted in Fig. 1. The power of such a polynomial is equal to four. So

$$Q(p) = \pi_0 p^4 + \pi_1 p^3 + \pi_2 p^2 + \pi_3 p + \pi_4. \quad (34)$$

Since $n=3, m=2$, then $n-m-2 < 0$. That is why the value for $\pi_i, i=0, \dots, 4$ are found from formula (10). We obtain

$$\pi_0 = C_2 b_0 + a_0,$$

$$\pi_1 = C_1 b_0 + C_2 b_1 + a_1,$$

$$\pi_2 = C_0 b_0 + C_1 b_1 + C_2 b_2 + a_2,$$

$$\pi_3 = C_0 b_1 + C_1 b_2 + a_3, \quad \pi_4 = C_0 b_2.$$

Applying the Viète theorem to polynomial (34), a system of equations is built that relates the roots of polynomial (34) to its coefficients

$$p_1 + p_2 + p_3 + p_4 = -\frac{\pi_1}{\pi_0},$$

$$p_1 p_2 + p_1 p_3 + p_1 p_4 +$$

$$+ p_2 p_3 + p_2 p_4 + p_3 p_4 = \frac{\pi_2}{\pi_0}, \quad (35)$$

$$p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4 = -\frac{\pi_3}{\pi_0},$$

$$p_1 p_2 p_3 p_4 = \frac{\pi_4}{\pi_0}.$$

The roots of characteristic polynomial (34), which are the functions of the controller’s setting parameters (Fig. 1), can be arranged in different ways in the lines of the p -plane. As an option, we can assume that there are two real and two complex-coupled left roots. Their arrangement is shown in Fig. 6.

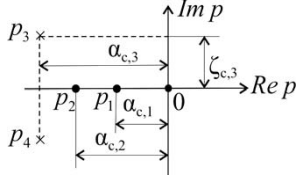


Fig. 6. Arranging the roots of polynomial (34) on the p -plane

The following ratios are established between the roots of characteristic polynomial (34):

$$\alpha_{c,2} = k_{\alpha} \alpha_{c,1},$$

$$\alpha_{c,3} = k_{\alpha,1} \alpha_{c,1}.$$

The degree of fluctuation of the system is calculated as the ratio of the imaginary part of the complex-coupled root to its real part $\mu = \frac{\zeta_{c,3}}{\alpha_{c,3}}$.

$$\text{Considering the value for } \alpha_{c,3}, \zeta_{c,3} = \mu k_{\alpha,1} \alpha_{c,1}$$

Choosing certain values of k_{α} and $k_{\alpha,1}$ allows us to obtain the desired arrangement for the roots of polynomial (33) on a complex p -plane.

To arrange the roots as shown in Fig. 6, one calculates the following values:

$$p_1 + p_2 + p_3 + p_4 = -\alpha_{c,1} (k_{\alpha} + 2k_{\alpha,1} + 1),$$

$$p_1 p_2 + p_1 p_3 + p_1 p_4 + p_2 p_3 + p_2 p_4 + p_3 p_4 = \alpha_{c,1}^2 (k_{\alpha} + 2k_{\alpha,1} (k_{\alpha} + 1) + k_{\alpha,1}^2 (1 + \mu^2)),$$

$$p_1 p_2 p_3 + p_1 p_2 p_4 + p_1 p_3 p_4 + p_2 p_3 p_4 = -\alpha_{c,1}^3 k_{\alpha,1} (2k_{\alpha} + k_{\alpha,1} (k_{\alpha} + 1) (\mu^2 + 1)),$$

$$p_1 p_2 p_3 p_4 = \alpha_{c,1}^4 k_{\alpha} k_{\alpha,1}^2 (\mu^2 + 1).$$

Taking into consideration the values of the left-hand sides in the system of equations (34), the following ratios are obtained:

$$\pi_0 \alpha_{c,1} (k_{\alpha} + 2k_{\alpha,1} + 1) = \pi_1,$$

$$\pi_0 \alpha_{c,1}^2 (k_{\alpha} + 2k_{\alpha,1} (k_{\alpha} + 1) + k_{\alpha,1}^2 (1 + \mu^2)) = \pi_2, \quad (36)$$

$$\pi_0 \alpha_{c,1}^3 k_{\alpha,1} (2k_{\alpha} + k_{\alpha,1} (k_{\alpha} + 1) (\mu^2 + 1)) = \pi_3,$$

$$\pi_0 \alpha_{c,1}^4 k_{\alpha} k_{\alpha,1}^2 (\mu^2 + 1) = \pi_4.$$

Taking into consideration the π_i , $i = \overline{0,3}$, values, the system of equations (36) takes the following form:

$$\alpha_{c,1} (C_2 b_0 + a_0) (k_{\alpha} + 2k_{\alpha,1} + 1) = C_1 b_0 + C_2 b_1 + a_1,$$

$$a_{c,1}^2 (C_2 b_0 + a_0) (k_{\alpha} + 2k_{\alpha,1} (k_{\alpha} + 1) + k_{\alpha,1}^2 (\mu^2 + 1)) = C_0 b_0 + C_1 b_1 + C_2 b_2 + a_2,$$

$$\alpha_{c,1}^3 k_{\alpha,1} (C_2 b_0 + a_0) (2k_{\alpha} + k_{\alpha,1} (k_{\alpha} + 1) (\mu^2 + 1)) = C_0 b_1 + C_1 b_2 + a_3,$$

$$\alpha_{c,1}^4 k_{\alpha} k_{\alpha,1}^2 (C_2 b_0 + a_0) (\mu^2 + 1) = C_0 b_2.$$

Upon obvious transformations, we obtain:

$$A_{11} C_0 + A_{12} C_1 + A_{13} C_2 = B_1,$$

$$A_{21} C_0 + A_{22} C_1 + A_{23} C_2 = B_2,$$

$$A_{31} C_0 + A_{32} C_1 + A_{33} C_2 = B_3,$$

$$A_{41} C_0 + A_{42} C_1 + A_{43} C_2 = B_4, \quad (37)$$

where $A_{11} = 0$; $A_{12} = b_0$;

$$A_{13} = b_1 - \alpha_{c,1} b_0 (k_{\alpha} + 2k_{\alpha,1} + 1);$$

$$A_{21} = b_0; A_{22} = b_1;$$

$$A_{23} = b_2 - \alpha_{c,1}^2 b_0 (k_{\alpha} + 2k_{\alpha,1} (k_{\alpha} + 1) + k_{\alpha,1}^2 (\mu^2 + 1));$$

$$A_{31} = b_1; A_{32} = b_2;$$

$$A_{33} = -b_0 \alpha_{c,1}^3 k_{\alpha,1} (2k_{\alpha} + k_{\alpha,1} (k_{\alpha} + 1) (\mu^2 + 1));$$

$$A_{41} = b_2; A_{42} = 0;$$

$$A_{43} = -\alpha_{c,1}^4 k_{\alpha} k_{\alpha,1} b_0 (\mu^2 + 1);$$

$$B_1 = a_1 - a_0 \alpha_{c,1} (k_{\alpha} + 2k_{\alpha,1} + 1);$$

$$B_2 = a_2 - a_0 \alpha_{c,1}^2 (k_{\alpha} + 2k_{\alpha,1} (k_{\alpha} + 1) + k_{\alpha,1}^2 (\mu^2 + 1));$$

$$B_3 = a_3 - \alpha_{c,1}^3 k_{\alpha,1} a_0 (2k_{\alpha} + k_{\alpha,1} (k_{\alpha} + 1) (\mu^2 + 1));$$

$$B_4 = -a_0 \alpha_{c,1}^4 k_{\alpha} k_{\alpha,1} (\mu^2 + 1).$$

Analysis of the system of equations (37) showed that condition (13) is not met and it is incompatible. One of the possible solutions to the redefined equation (35) is to minimize the residual function [15]

$$J(\bar{C}) = |\bar{A}\bar{C} - \bar{B}|^2, \quad (38)$$

relative to vector \bar{C} , which yields the following:

$$\bar{C}^* = (A^T A)^{-1} A^T \bar{B}, \quad (39)$$

where

$$\bar{C}^T = (C_0, C_1, C_2).$$

Note that the MATLAB software contains a built-in *lsqr* function that uses iterative procedures based on the bidiagonalization of Golub and Kohan [17], which is equivalent to the method of conjugate gradients but has better convergence. The method generates a sequence of approximations of the desired variables $\{\bar{C}_k\}$, such that the residual is $\|r_k\|^2$, decreased monotonously. In fact, the built-in *lsqr* function generates a pseudo-solution to equation (37) in a form similar to formula (39), the only difference is that such a built-in function is oriented to solving systems of linear equations of great dimensionality.

Another approach to solving the system of equations (37) is to search for a corrective matrix ΔA to the right-hand side of the matrix equation

$$A\bar{C} = \bar{B}, \tag{40}$$

such that the following system of equations

$$(A + \Delta A)\bar{C} = \bar{B},$$

is compatible, and the ΔA matrix is to meet the criterion of “smallness” [17, 18].

Under such a statement of the problem, it is unclear what values the elements of the ΔA matrix should accept in order to satisfy the criterion of “smallness”. In addition, if $\text{rang}(a) < n$, then the correction problem has no solution.

Alternatively, in [19], it is proposed to minimize the sum of squares of residuals

$$J_2(\bar{C}_b) = \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^m \left(\sum_{j=0}^n (a_{kj} - a_{lj}) C_{bj} \right)^2, \tag{41}$$

where $a_{k0} = B_k, a_{l0} = B_l, C_{b0} = -1, C_{bj} = C_{j-1}, j=1,2,3, a_{kl}$ are the elements of matrix $A, k, l = 1, m$.

Introduce designations: $\alpha_{kl}^{(0)} = a_{k0} - a_{l0}, \alpha_{kl}^{(1)} = a_{k1} - a_{l1}, \alpha_{kl}^{(2)} = a_{k2} - a_{l2}, \dots, \alpha_{kl}^{(j)} = a_{kj} - a_{lj}, \dots, \alpha_{kl}^{(n)} = a_{kn} - a_{ln}$. Since $n=3$, then

$$J_2(\bar{C}_b) = \frac{1}{2} \sum_{k=1}^m \sum_{l=1}^m (\bar{\alpha}_{kl}^T \bar{C}_b)^2, \tag{42}$$

where

$$\bar{\alpha}_{kl} = (\alpha_{kl}^{(0)}, \alpha_{kl}^{(1)}, \alpha_{kl}^{(2)}, \alpha_{kl}^{(3)})^T;$$

$$\bar{C}_b = (C_{b0}, C_{b1}, C_{b2}, C_{b3})^T.$$

The essence of expression (41) is that the following residuals are formed

$$z_k = a_{k1}C_0 + a_{k2}C_1 + a_{k3}C_2 - B_k,$$

$$z_l = a_{l1}C_0 + a_{l2}C_1 + a_{l3}C_2 - B_l.$$

Then

$$z_k - z_l = \sum_{j=0}^n (a_{kj} - a_{lj}) C_{bj}, \quad n=3$$

The sum of the squares of the difference in residuals $z_k - z_l$ leads to expression (41).

We minimize $J_2(\bar{C})$ for the desired values C_{b1}, C_{b2}, C_{b3} by using the condition of the minimum function of many variables

$$\frac{\partial J_2(\bar{C})}{\partial C_{bj}} = 0, \quad j=1, 2, 3.$$

As a result, we obtain the following system of linear equations:

$$\sum_{k=1}^m \sum_{l=1}^m (\bar{\alpha}_{kl}^T \bar{C}_b) \alpha_{kl}^{(j)} = 0, \quad j = \overline{1, n}, \quad n=3. \tag{43}$$

To compare the efficiency of calculating the parameters for the PID algorithm from formula (39) and by solving the system of linear algebraic equations (43), the MATLAB

environment was employed to construct the software for problems (39) and (43).

For values $a_0=6, a_1=4, a_2=7, a_3=1; b_0=1, b_1=3, b_2=5$. The parameters of the characteristic polynomial of transfer function (34) of the object are selected so that the conditions of stability are met. The degree of stability $\eta = \alpha_{c,1}$ is taken from the condition that the value $\alpha_{c,1}$ is to the left of the pole of transfer function (33), which is closer to the imaginary axis of the p -plane. When one takes into consideration the selected values of the parameters of the characteristic polynomial of transfer function (33), the nearest pole to the imaginary axis would accept the following value: $p_1 = -0.1532$. Therefore, we chose $\alpha_{c,1} = 0.9$. Other values for the algorithm were: $k_\alpha = 1.25, k_{\alpha,1} = 1.4$. The values for the degree of fluctuation μ are taken so that the closed system (Fig. 1) is robust – $\mu \in \{0.2; 0.4; 0.6; 0.8\}$.

First, the parameters for the PID algorithm were calculated from formula (39), which determines the pseudo-solution to equation (40). The calculation results are shown in Fig. 7 and are given in Table 3.

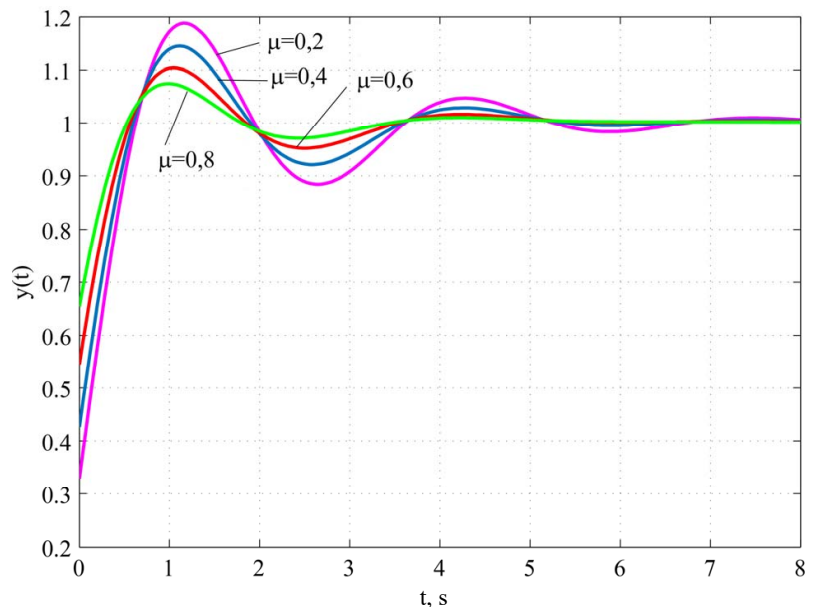


Fig. 7. Transition characteristics of the system built by using formula (39)

Table 3

Values of the PID-control algorithm configuration parameters and the control process quality indicators (pseudo-solution)

The degree of fluctuation, μ	Configuration parameter			Control quality indicator	
	C_0, s^{-1}	C_1	C_2, s	Overshoot, $\sigma, \%$	Control time, t_p, s
0.2	1.8502	10.0020	2.9422	18.0940	6.56
0.4	2.6534	11.7046	4.4507	14.0835	4.88
0.6	4.3080	15.0451	7.1863	10.1275	4.68
0.8	7.2934	20.7133	11.3213	7.2728	3.16

Now, the parameters for the PID algorithm can be found as a solution to the system of equations (43).

Plots of the transitional characteristics of an automatic control system (Fig. 1) are shown in Fig. 8.

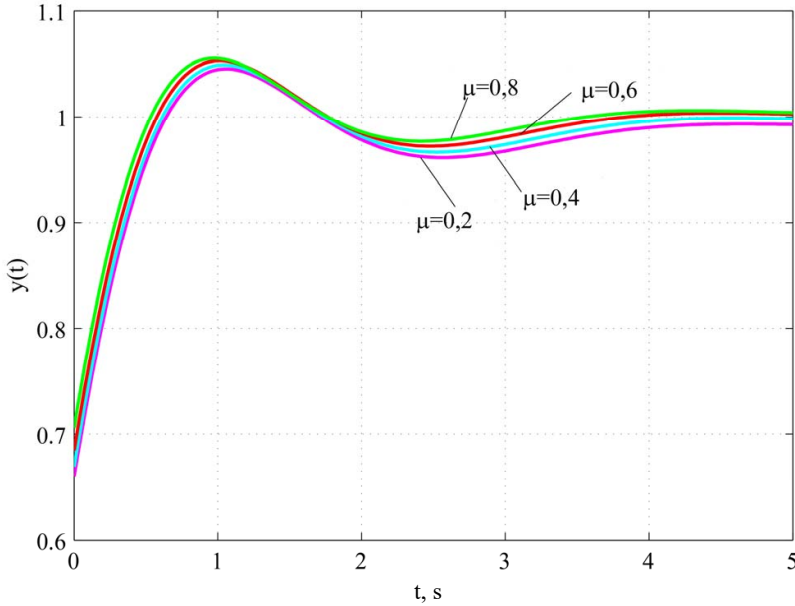


Fig. 8. Transition characteristics of the system shown in Fig. 1

The result of our calculations has established parameters for setting up the control PID algorithm, as well as the control quality indicators (Table 4).

Table 4

Values of the control PID-algorithm configuration parameters and the control process quality indicators at $k_{\alpha}=1.25$ and $k_{\alpha,1}=1.4$

The degree of fluctuation, μ	Configuration parameter			Control quality indicator	
	C_0, s^{-1}	C_1	C_2, s	Over-shoot, $\sigma, \%$	Control time, t_p, s
0.2	0.6384	14.9563	11.6415	5.3042	3.57
0.4	1.6535	16.4731	12.1286	5.0910	3.55
0.6	3.4759	19.1962	13.0030	5.0417	3.42
0.8	6.3440	23.4819	14.3792	5.1255	3.27

The comparative analysis of the two methods for calculating the configuration parameters of the PID algorithm reveals that the transformation of the initial problem to the system of equations (43) provides better quality indicators than the pseudo-solution (39) to equation (40). Thus, in the first case, the overshoot and control time do not exceed 18 % and 6.6 s; in the second case, 5.3 % and 5.6 s, which is much better than in the first case.

Now we can consider the possibility of achieving the compatibility of the equation system (40) by determining such a value for $\alpha_{c,1}$ that the Rouch–Capelli theorem conditions are met.

Since the rank of matrix A is equal to three, and the rank of the extended matrix $[A \mid B]$ – to four, we shall find such a value for $\alpha_{c,1}$ that the rank of the extended matrix has a dimensionality of 3. The extended matrix is square, the size of 4×4 . Taking into consideration that $\text{rang}(A)=3$, and when adding column B to matrix A , we obtain an extended square matrix, the rank of which is equal to four, the determinant of the matrix $[A|B]$ will be different from zero. The determinant of the extended matrix will be the function of the $\alpha_{c,1}$ variable. The opening of the determinant of the extended matrix generates a polynomial of the fourth power of the $\alpha_{c,1}$

variable, equating it to zero produces the following algebraic equation:

$$\eta_0 \alpha_{c,1}^4 + \eta_1 \alpha_{c,1}^3 + \eta_2 \alpha_{c,1}^2 + \eta_3 \alpha_{c,1} + \eta_4 = 0, \quad (44)$$

where $\eta_i, i=\overline{0,4}$ are the coefficients determined by the values of the parameters of transfer function (33) and the values of roots p_2, p_3, p_4 of the characteristic equation (34) of the closed system.

Assume the roots of equation (44) include several positive ones. The computational experiments have shown that the choice of the $\alpha_{c,1}$ value, which ensures meeting the Rouch–Capelli condition, is carried out under the following condition:

$$\alpha_{c,1}^{(0)} = \min_{\alpha_{c,1}^{(i)} > 0} (\alpha_{c,1}^{(i)}). \quad (45)$$

Condition (45) is selected for reasons of ensuring the stability of the automatic control system whose structural diagram is shown in Fig. 1.

The result of calculating the configuration parameters for the PID algorithm according to our method is shown in Fig. 9 and given in Table 5.

Table 5

Values of the control PID-algorithm configuration parameters and the control process quality indicators at $k_{\alpha}=1,25$ and $k_{\alpha,1}=1,4$

The degree of fluctuation, μ	Configuration parameter			Control quality indicator	
	C_0, s^{-1}	C_1	C_2, s	Over-shoot, $\sigma, \%$	Control time, t_p, s
0.2	20.0825	38.7476	21.9305	4.3549	2.87
0.4	24.6923	45.2681	24.2164	4.1443	2.75
0.6	34.9573	60.1487	29.5614	3.6241	2.53
0.8	60.0640	97.8943	43.6146	2.6198	1.28

Now, assume $b_0=0$. The other parameters of transfer function (33) were: $b_1=3, b_2=1; a_0=6, a_1=4, a_2=7, a_3=1$. The parameters for transfer function (31) are selected so that the control object is robust, that is, the following ratio holds: $a_1 a_2 - a_0 a_3 > 0$. The parameters at which the values of the roots can be changed on the p -plane were: $k_{\alpha}=1.25$ and $k_{\alpha,1}=1.4$.

The computational experiments have shown that when determining the parameters of the PID algorithm, it is necessary to take into consideration not only the compatibility of equation system (40) but also the requirements for the stability of the automatic control system. Under such an integrated approach to solving the problem of calculating the configuration parameters for the PID algorithm, it is necessary to choose the method that primarily ensures the stability of the automatic control system.

Applying the method for transforming the system of equations (40) to a compatible form has shown that the left root, $\alpha_{c,1}$, is close to the imaginary axis, which, first, does not allow for the necessary margin of system robustness, and, second, a value for the setting parameter C_0 is close to zero. This low C_0 value causes an unacceptably large static error of control.

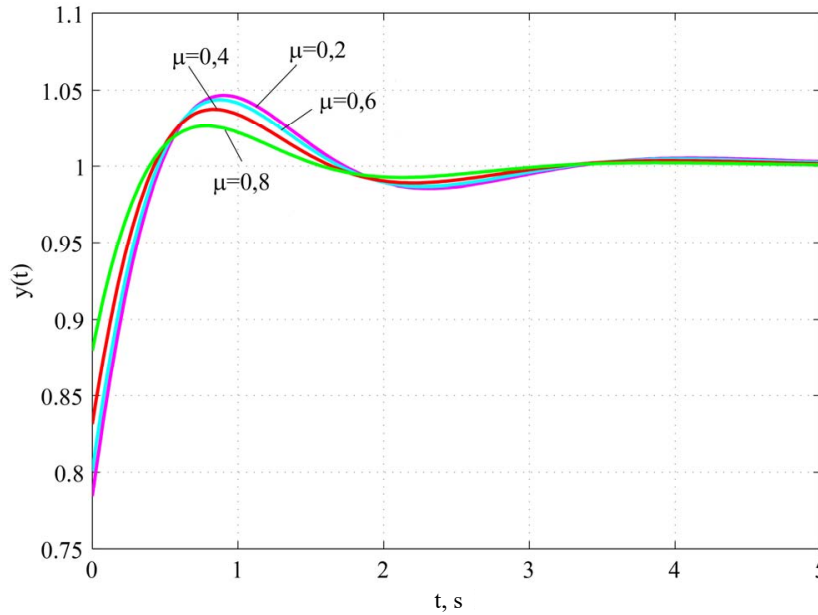


Fig. 9. Transition characteristics of the system that were obtained using condition (43)

Therefore, to calculate the configuration parameters for the PID algorithm, a method was chosen that is based on solving the system of equations (43).

The closest to the imaginary axis was the real left root $|\alpha_{c,1}|=0,9$.

For the set of values of the degree of fluctuation $\mu \in \{0,2;0,4;0,6;0,8\}$, the values of the parameters of the control algorithm were calculated and the quality indicators of the control process were determined (Table 6).

The transitional characteristics of a closed automatic control system (Fig. 1) with a single jump-like input value $u(t)$ are shown in Fig. 10.

The analysis of our results shows that the transition process in the automatic control system is oscillatory,

which is explained by the presence of a pair of left complex-coupled roots.

Table 6

Values of the PID-controller configuration parameters and the control process quality indicators at $k_{\alpha}=0.6$ and $k_{\alpha,1}=1.4$

The degree of fluctuation, μ	Configuration parameter			Control quality indicator	
	C_0, s^{-1}	C_1	C_2, s	Over-shoot, $\sigma, \%$	Control time, t_p, s
0.2	7.1794	9.9807	6.8063	8.7975	4.15
0.4	7.6444	10.2079	6.5755	10.2172	4.07
0.6	8.4196	10.5866	6.1909	12.8001	3.90
0.8	9.5048	11.1167	5.6525	17.0439	3.62

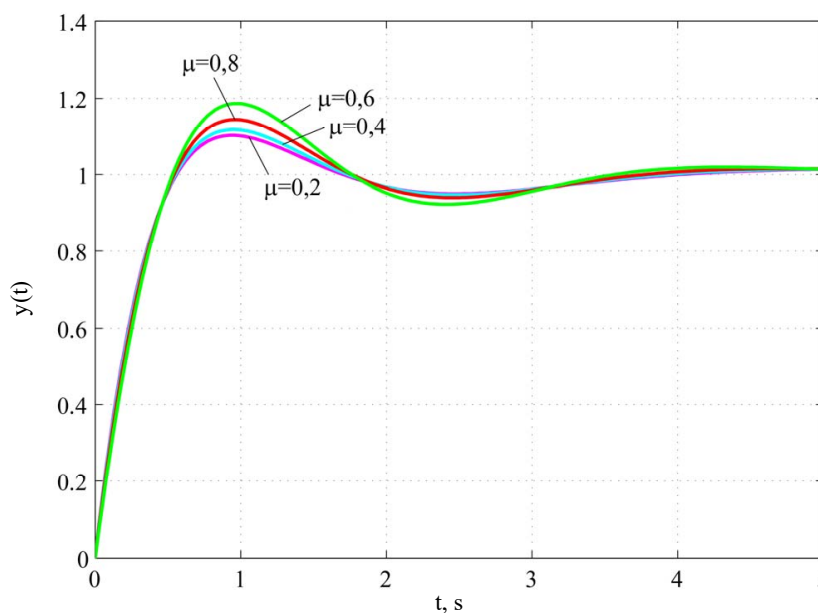


Fig. 10. Transitional characteristics of the system at $k_{\alpha}=1.25$ and $k_{\alpha,1}=1.4$

Consider the case where $b_0=b_1=0$, and $b_2=k$, where k is the transfer factor of the object. For this case, equation system (37) takes the following form:

$$\begin{aligned} \alpha_{c,1}a_0(k_\alpha + 2k_{\alpha,1}) - a_1 &= 0, \\ C_2b_2 &= \alpha_{c,1}^2a_0(k_\alpha + 2k_{\alpha,1}(k_\alpha + 1) + k_{\alpha,1}^2(\mu^2 + 1)) - a_2, \\ C_1b_2 &= \alpha_{c,1}^3k_{\alpha,1}a_0(2k_\alpha + k_{\alpha,1}(k_\alpha + 1)(\mu^2 + 1)) - a_3, \\ C_0b_2 &= a_0\alpha_{c,1}^4k_{\alpha,1}k_{\alpha,1}^2(\mu^2 + 1). \end{aligned} \quad (46)$$

The first equation of system (46) does not contain the desired values C_0 , C_1 , and C_2 . This means that the values of quantities k_α and $k_{\alpha,1}$, which determine the arrangement of roots p_2 , p_3 and p_4 on the p -plane (Fig. 6), cannot be selected arbitrarily; one of them, k_α or $k_{\alpha,1}$, depends on the other. Assume that the selected value is $k_{\alpha,1}$. Then, from the first equation of system (46), we determine

$$k_\alpha = \frac{a_1}{\alpha_{c,1}a_0} - 2k_{\alpha,1} - 1.$$

The k_α value determines the position of the root p_2 on the p -plane. To ensure the stability of the automatic control system (Fig. 1), the root p_2 must be left. This means that $k_\alpha > 0$, or $\frac{a_1}{\alpha_{c,1}a_0} > 2k_{\alpha,1} + 1$. The latter condition can be satisfied with a certain choice of the $\alpha_{c,1}$ value.

From other equations of system (46), we find

$$\begin{aligned} C_0 &= \frac{a_0}{b_2}\alpha_{c,1}^4k_{\alpha,1}k_{\alpha,1}^2(\mu^2 + 1), \\ C_1 &= \\ &= \frac{1}{b_2}\left(\alpha_{c,1}^3k_{\alpha,1}a_0(2k_\alpha + k_{\alpha,1}(k_\alpha + 1)(\mu^2 + 1)) - a_3\right), \\ C_2 &= \\ &= \frac{1}{b_2}\left(\alpha_{c,1}^2a_0(k_\alpha + 2k_{\alpha,1}(k_\alpha + 1) + k_{\alpha,1}^2(\mu^2 + 1)) - a_2\right). \end{aligned}$$

For the selected values $a_0=3$, $a_1=8$, $a_2=2$, $a_3=1$; $k=5$, $\alpha_{c,1}=0,4$, $k_{\alpha,1}=1,2$, using the software developed in the MATLAB environment, at values $\mu \in \{0.2; 0.4; 0.6; 0.8\}$, we built the plots of transition processes (Fig. 11) and determined quality indicators of the control process (Table 7).

In conclusion, we consider the case where the roots of characteristic polynomial (34) are real numbers, located in the left semi-plane of the p -plane, and the transfer function of an object is described by formula (33). As before, we believe that there are the following ratios between the roots of the characteristic equation: $p_1=-\alpha_{c,1}$, $p_2=-k_\alpha\alpha_{c,1}$, $p_3=-k_{\alpha,1}\alpha_{c,1}$, $p_4=-k_{\alpha,2}\alpha_{c,1}$, where $\alpha_{c,1}>0$, $k_\alpha>0$, $k_{\alpha,1}>0$, $k_{\alpha,2}>0$.

The system of equations (36) takes the following form:

$$\begin{aligned} C_1b_0 + C_2(b_1 - b_0\alpha_{c,1}q_1) &= a_0\alpha_{c,1}q_1 - a_1, \\ C_0b_0 + C_1b_1 + C_2(b_2 - b_0\alpha_{c,1}^2q_2) &= a_0\alpha_{c,1}^2q_2 - a_2, \\ C_0b_1 + C_1b_2 - C_2b_0\alpha_{c,1}^3q_3 &= a_0\alpha_{c,1}^3q_3 - a_3, \\ C_0b_2 - C_2b_0\alpha_{c,1}^4q_4 &= a_0\alpha_{c,1}^4q_4, \end{aligned} \quad (47)$$

where

$$\begin{aligned} q_1 &= 1 + k_\alpha + k_{\alpha,1} + k_{\alpha,2}, \\ q_2 &= k_\alpha + k_{\alpha,1} + k_{\alpha,2} + k_\alpha k_{\alpha,1} + k_\alpha k_{\alpha,2} + k_{\alpha,1}k_{\alpha,2}, \\ q_3 &= k_\alpha k_{\alpha,1} + k_\alpha k_{\alpha,2} + k_{\alpha,1}k_{\alpha,2} + k_\alpha k_{\alpha,1}k_{\alpha,2}, \\ q_4 &= k_\alpha k_{\alpha,1}k_{\alpha,2}. \end{aligned}$$

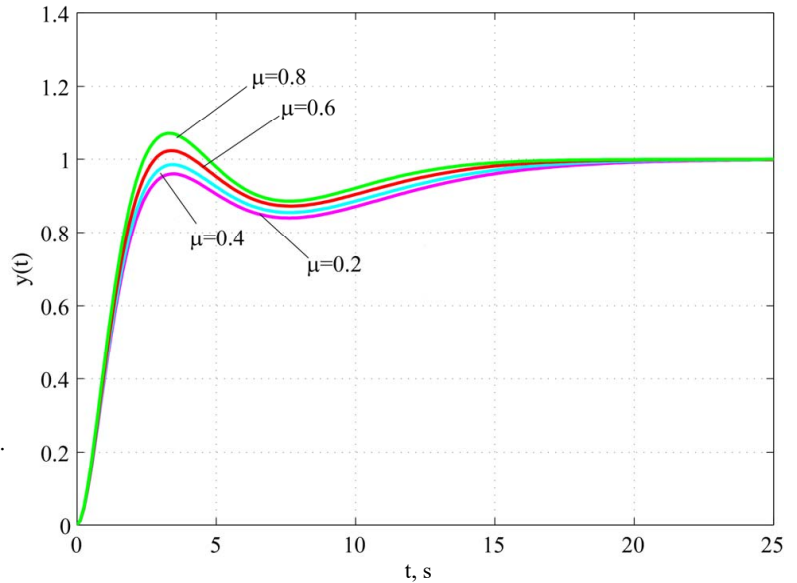


Fig. 11. Transition characteristics of the automatic control system (Fig. 1)

The system of algebraic equations (47) is represented in a matrix-vector form:

$$AC=B.$$

The elements of matrix A and vector B are as follows:

$$\begin{aligned} A_{11} &= 0, \quad A_{12} = b_0, \quad A_{13} = b_1 - b_0\alpha_{c,1}q_1; \\ A_{21} &= b_0, \quad A_{22} = b_1, \quad A_{23} = b_2 - b_0\alpha_{c,1}^2q_2; \\ A_{31} &= b_1, \quad A_{32} = b_2, \quad A_{33} = -b_0\alpha_{c,1}^3q_3; \\ A_{41} &= b_2, \quad A_{42} = 0, \quad A_{43} = -b_0\alpha_{c,1}^4q_4. \end{aligned}$$

Table 7

Values of the control PID-algorithm configuration parameters and the control process quality indicators at $k_{\alpha,1}=1.2$

The degree of fluctuation, μ	Configuration parameter			Control quality indicator	
	C_0, s^{-1}	C_1	C_2, s	Overshoot, $\sigma, \%$	Control time, t_p, s
0.2	0.0751	0.3464	1.0404	0	18.87
0.4	0.0838	0.3747	1.0570	0	17.87
0.6	0.0983	0.4219	1.0846	2.38	16.38
0.8	0.1185	0.4880	1.1234	7.07	14.75

$$B_1 = a_0 \alpha_{c,1} q_1 - a_1,$$

$$B_2 = a_0 \alpha_{c,1}^2 q_2 - a_2,$$

$$B_3 = a_0 \alpha_{c,1}^3 q_3 - a_3,$$

$$B_4 = a_0 \alpha_{c,1}^4 q_4.$$

So, we have a rectangular matrix of size $n \times m$, where $n=4$ and $m=3$. Since the condition $n > m$ holds, the system of equations (47) is redefined [20]. Fig. 10 shows the transition characteristics of a closed system (Fig. 1) for the following data: $a_0=6, a_1=4, a_2=7, a_3=7; b_0=1, b_1=3, b_2=5$. The parameters of the algorithm $k_\alpha, k_{\alpha 1}, k_{\alpha 2}$, which determine the position of the roots on the p -plane were as follows: $k_\alpha=1.1, k_{\alpha 1}=1.2, k_{\alpha 2}=1.3$. The value $\alpha_{c,1}$ was selected from the set $\alpha_{c,1} \in \{0,9; 0,8; 0,7\}$.

The configuration parameters of the control PID algorithm are calculated by solving the system of algebraic linear equations (47) using the built-in *lsqr* function, which is part of the MATLAB software [21]. Transitional characteristics of the system are shown in Fig. 12.

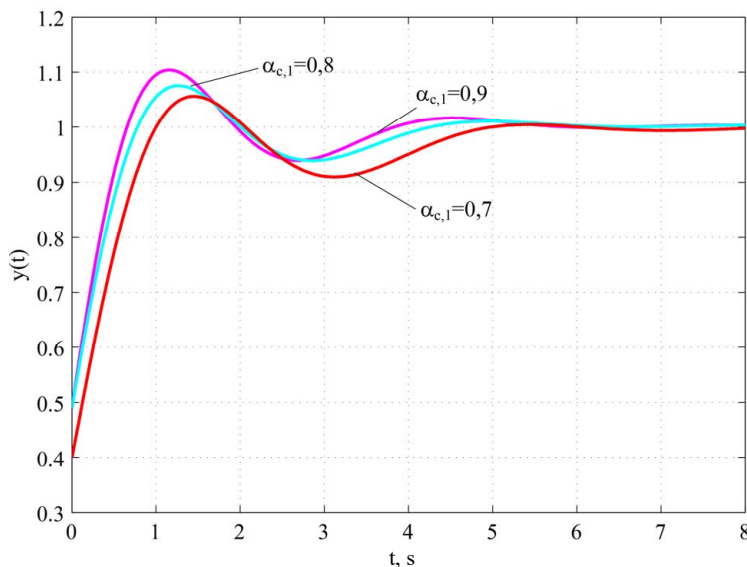


Fig. 12. Transition characteristics of the system (Fig. 1)

Thus, our results confirm the possibility of determining the configuration parameters of controllers by arranging the roots of a characteristic polynomial on a complex p -plane.

6. Discussion of the study results on calculating the configuration parameters of PID controllers by arranging poles on the p -plane

One of the possible directions of searching for a solution to the problem of determining the configuration parameters of PID controllers is a method based on the generalized Viète theorem, which makes it possible to reduce the formed problem to solving the system of linear algebraic equations, which is a significant advantage over those methods where the solution is based on the methods of nonlinear programming [6–9].

In the case where the transfer function of the control object is of the first and second orders, the problem of deter-

mining the configuration parameters of a PID controller is solved unequivocally as the solution to the systems of linear algebraic equations (17) and (27).

If the transfer function of the automatic control system is of the third and higher orders, then we obtain the redefined system of linear algebraic equations, which, in a general case, does not have an unambiguous solution. Only when the conditions of the Rouché–Capelli theorem are met, one can derive a solution to the algebraic equation system relative to the configuration parameters of the PID controller. For an object’s third-order transfer function, the condition of the Rouché–Capelli theorem can be satisfied by selecting the degree of stability of the system. This method of calculating the configuration parameters of the PID controller does not always guarantee the achievement of the necessary quality indicators of the control process.

Other alternatives may be to search for a pseudo-solution to the redefined system of linear algebraic equations, or a method that provides for a minimum of the criterion (42).

Our conclusions are based on the numerical experiments, the results of which are represented in the form of Tables 1–6. The quality of the control process was determined by constructing the plots of transitional characteristics (Fig. 2–12).

The analysis of our results shows that the overshoot and control time are within the permissible limits (overshoot does not exceed 30 %).

Further research into related topics implies determining the parameters for setting up industrial controllers for multiconnected systems.

7. Conclusions-

1. A concept of the method for calculating the configuration parameters of PID controllers has been proposed. It is shown that for the control object’s transfer functions of the first and second orders, the problem has an unambiguous solution. In the case where the order of the transfer function of the control object is of the third and higher orders, we obtain the redefined system of linear algebraic equations relative to the parameters of the controller configuration. For such a case, three methods for calculating the configuration parameters of PID controllers have been built. An appropriate method is selected by considering the requirements for the stability of the automatic control system and the quality indicators of the control process.

2. A procedure has been devised that makes it possible to select the necessary method of solving the problem in iterative mode, based on the order of the transfer function, requirements for the stability of the system, and indicators of the control process. The effectiveness of the method for calculating the configuration parameters of PID controllers for the predefined set of transfer functions of an object has been evaluated. Based on the transition characteristics of the automatic control system, the indicators of the control process have been determined: overshoot and time of control. According to the results of simulation modeling for the predefined class of transfer functions of objects, it was established that the overshoot is in the range of 0 to 27.9 %, which corresponds to the current norms.

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