

A non-axisymmetric problem of the theory of elasticity for a radial inhomogeneous cylinder of small thickness is studied. It is assumed that the elastic moduli are arbitrary positive piecewise continuous functions of a variable along the radius.

Using the method of asymptotic integration of the equations of the theory of elasticity, based on three iterative processes, a qualitative analysis of the stress-strain state of a radial inhomogeneous cylinder is carried out. On the basis of the first iterative process of the method of asymptotic integration of the equations of the theory of elasticity, particular solutions of the equilibrium equations are constructed in the case when a smooth load is specified on the lateral surface of the cylinder. An algorithm for constructing partial solutions of the equilibrium equations for special types of loads, the lateral surface of which is loaded by forces polynomially dependent on the axial coordinate, is carried out.

Homogeneous solutions are constructed, i.e., any solutions of the equilibrium equations that satisfy the condition of the absence of stresses on the lateral surfaces. It is shown that homogeneous solutions are composed of three types: penetrating solutions, solutions of the simple edge effect type, and boundary layer solutions. The nature of the stress-strain state is established. It is found that the penetrating solution and solutions having the character of the edge effect determine the internal stress-strain state of a radial inhomogeneous cylinder. Solutions that have the character of a boundary layer are localized at the ends of the cylinder and exponentially decrease with distance from the ends. These solutions are absent in applied shell theories.

Based on the obtained asymptotic expansions of homogeneous solutions, it is possible to carry out estimates to determine the range of applicability of existing applied theories for cylindrical shells. Based on the constructed solutions, it is possible to propose a new refined applied theory

Keywords: non-axisymmetric problem, radial inhomogeneous cylinder, asymptotic integration method, homogeneous solutions, boundary layer

BEHAVIOR OF SOLUTION OF THE ELASTICITY PROBLEM FOR A RADIAL INHOMOGENEOUS CYLINDER WITH SMALL THICKNESS

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1. Introduction

One of the properties of materials that affect the stress-strain state of elastic bodies is their inhomogeneity. In the study of inhomogeneous bodies, the real properties of materials are taken into account. Heterogeneous materials are widely used in various fields of technology. Various materials are being developed and created, the characteristics of which, in particular, the modulus of elasticity can change continuously along certain directions.

The complexity of the phenomena arising from the deformation of inhomogeneous shells has given rise to a number of applied theories based on various hypotheses. To assess the area of applicability of existing applied theories and in order to create new refined applied theories, it is relevant to analyze the stress-strain state of inhomogeneous shells from the standpoint of three-dimensional equations of the theory of elasticity. The study of inhomogeneous shells from the standpoint of three-dimensional equations of the theory of elasticity, more adequately take into account their mechanical and geometric structure. The study of the stress-strain state of inhomogeneous shells on the basis of three-dimensional equations of the theory of elasticity is associated with significant mathematical difficulties. Along with this, new qualitative and quantitative effects appear.

2. Literature review and problem statement

A number of studies are devoted to the study of problems in the theory of elasticity for a radial inhomogeneous

cylinder. In [1], the main achievements in the field of inhomogeneous elastic bodies are presented. In [2], an overview of the history of the development of research in the field of inhomogeneous elastic bodies is given and frequently used analytical, numerical-analytical methods are characterized. In [3], the problem of torsion of a radial layered cylinder is studied and a possible violation of Saint-Venant's principle in its classical formulation was shown.

In [4], the axisymmetric problem of the theory of elasticity is studied for a radial layered cylinder with alternating hard and soft layers. The existence of weakly damped boundary layer solutions is shown. A mechanical interpretation of penetrating solutions with a weak boundary layer is given.

In [5], on the basis of the method of asymptotic integration of the equations of the theory of elasticity, the axisymmetric problem of the theory of elasticity was studied for a radial inhomogeneous cylinder of small thickness, in the case when the elastic moduli vary along the radius according to a linear law. Three groups of solutions are obtained and the nature of the constructed solutions is explained. In [6], the problem of the theory of elasticity is studied for a radial inhomogeneous cylinder of small thickness with a fixed lateral surface. In [7], the problem of torsion of a radial inhomogeneous cylinder is studied in the case when the lateral surfaces of the cylinder are stress-free. In [8], the problem of torsion of a radial inhomogeneous cylinder with a fixed lateral surface was studied. In [9], an asymptotic method is used to study the behavior of solutions of an axisymmetric problem of elasticity theories for a radial inhomogeneous transversely isotropic cylinder of

small thickness. The analysis of the stress-strain state determined by homogeneous solutions is carried out.

In [10], the flexural deformation of a multilayer cylinder with cylindrical anisotropy is studied. In [11], the Almansi-Michell problem for an inhomogeneous anisotropic cylinder is investigated by a numerical-analytical method. An analogue of the classical Lamé problem for an isotropic hollow cylinder with Young's modulus depending on the radial coordinate and with a constant Poisson's ratio is considered in [12]. In [13], the stress-strain state of an inhomogeneous orthotropic cylinder with a given inhomogeneity is studied. In [14], the method of discrete orthogonalization is used to numerically solve a three-dimensional problem of the theory of elasticity for a radial inhomogeneous cylinder. In [15], on the basis of the spline collocation method and the finite element method, the three-dimensional stress-strain state of a radial inhomogeneous cylinder was studied and the numerical results obtained were compared. In [16], the analysis of the stress-strain state of a radial inhomogeneous cylinder subjected to uniform internal pressure is considered. In [17], an analysis was made of the problem of the theory of elasticity for a radial inhomogeneous hollow cylinder with a constant Poisson's ratio and Young's modulus, which is an exponential function of the radius. In [18], an analytical solution is obtained for an axisymmetric problem of the theory of elasticity for a radial inhomogeneous cylinder. It is assumed that the elastic properties of the cylinder material are arbitrary functions of the radial coordinate. The solutions to the problem are reduced to integral equations.

An analysis of the thermomechanical behavior of radial inhomogeneous cylinders is presented in [19]. Several numerical cases are investigated and conclusions are drawn about the general properties of thermal stresses in a cylinder. In [20], the axisymmetric problem of thermoelasticity for a radial inhomogeneous cylinder is studied. It is believed that the elastic properties of the material change in the radial direction by a power law. In [21], the thermoelasticity problem for an inhomogeneous cylinder was studied and the effect of the inhomogeneity of the stress-strain state of the cylinder was investigated. An analytical solution for an inhomogeneous cylinder subject to thermal loads, internal pressure and axial loads is presented. In [22], assuming that the thermoelastic parameters are power functions of the radial coordinate, the thermomechanical state of a radial inhomogeneous anisotropic cylinder is studied. In [23], the thermoelasticity problem is studied for a radial inhomogeneous cylinder, when the elastic moduli are linear functions of the radial coordinate. In [24], exact solutions were obtained for generalized plane problems for a piezoelectric cylinder with a power-law radial inhomogeneity.

In the above works, axisymmetric problems of the theory of elasticity are studied, in particular, when the elastic moduli are exponential or power-law functions of the radius. The nonaxisymmetric problem of the theory of elasticity in the case when the elastic moduli are arbitrary positive continuous functions of the radius is more complicated. Analysis of the stress-strain state of a radial inhomogeneous cylinder on the basis of three-dimensional equations of the theory of elasticity is reduced to the study of boundary value problems for systems of linear second-order partial differential equations with variable coefficients. Moreover, these coefficients include the elastic moduli, which are arbitrary positive con-

tinuous functions of the radius, which significantly complicates the construction of solutions to problems.

3. The aim and objectives of research

The aim of this research is to study the behavior of the solution to the problem of the theory of elasticity and to reveal the features of the stress-strain state for a radial inhomogeneous cylinder of small thickness. This will make it possible to estimate the range of applicability of various applied theories for radial inhomogeneous cylindrical shells. Based on the analysis performed, a new, more refined applied theory can be constructed. To achieve this goal, it is necessary to solve the following tasks:

- to build inhomogeneous solutions for a radial inhomogeneous cylinder;
- to construct homogeneous solutions for a radial inhomogeneous cylinder and obtain asymptotic formulas for displacements, stresses;
- to study the nature of stress-strain states corresponding to various types of homogeneous solutions.

4. Materials and methods of research

The three-dimensional stress-strain state of a radial inhomogeneous cylinder of small thickness is studied on the basis of the equations of the theory of elasticity. A complete system of equations of the theory of elasticity for a radial inhomogeneous cylinder in a cylindrical coordinate system is presented and a boundary value problem is formulated. Taking into account that the formulated boundary value problems include a small parameter characterizing the thickness of the cylinder, the method of asymptotic integration of the equations of the theory of elasticity is used to construct the solution. This method is one of the most effective methods for studying the three-dimensional stress state of inhomogeneous bodies of finite dimensions.

5. Research results on the behavior of the solution to the problem of the theory of elasticity for a radial inhomogeneous cylinder of small thickness

5.1. Construction of an inhomogeneous solution for a radial inhomogeneous cylinder

Let's consider a nonaxisymmetric problem of the theory of elasticity for a radial inhomogeneous cylinder of small thickness (the problem of bending deformation of a radial inhomogeneous cylinder) [25]. Let's refer the cylinder to the cylindrical coordinate system r, φ, z

$$r_1 \leq r \leq r_2; \quad 0 \leq \varphi \leq 2\pi; \quad -L \leq z \leq L.$$

The system of equilibrium equations in the absence of mass forces in the cylindrical coordinate system r, φ, z has the form [25]:

$$\begin{cases} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\varphi\varphi}}{r} = 0, \\ \frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \sigma_{\varphi z}}{\partial z} + \frac{2}{r} \sigma_{r\varphi} = 0, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\varphi z}}{\partial \varphi} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0. \end{cases} \quad (1)$$

Here $\sigma_{rr}, \sigma_{zz}, \sigma_{\varphi\varphi}, \sigma_{r\varphi}, \sigma_{\varphi z}, \sigma_{rz}$ – the stress tensor components, which are expressed in terms of the displacement vectors $v_r=v_r(r, \varphi, z), v_\varphi=v_\varphi(r, \varphi, z), v_z=v_z(r, \varphi, z)$ as follows [25]:

$$\begin{cases} \sigma_{rr} = (2G_* + \lambda_*) \frac{\partial v_r}{\partial r} + \lambda_* \left(\frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right), \\ \sigma_{zz} = (2G_* + \lambda_*) \frac{\partial v_z}{\partial z} + \lambda_* \left(\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} \right), \\ \sigma_{\varphi\varphi} = (2G_* + \lambda_*) \left(\frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_r}{r} \right) + \lambda_* \left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} \right), \\ \sigma_{r\varphi} = G_* \left(\frac{1}{r} \frac{\partial v_r}{\partial \varphi} + \frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r} \right), \\ \sigma_{rz} = G_* \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right), \sigma_{\varphi z} = G_* \left(\frac{\partial v_\varphi}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \varphi} \right). \end{cases} \quad (2)$$

Substituting (2) into (1), let's obtain the equilibrium equations in displacements

$$\begin{cases} \frac{\partial}{\partial \rho} \left[He^{-\varepsilon \rho} \frac{\partial u_\rho}{\partial \rho} + \lambda \varepsilon \left(e^{-\varepsilon \rho} \frac{\partial u_\varphi}{\partial \varphi} + e^{-\varepsilon \rho} u_\rho + \frac{\partial u_\xi}{\partial \xi} \right) \right] + \varepsilon^2 G \times \\ \times \left[e^{-\varepsilon \rho} \left(\frac{\partial^2 u_\rho}{\partial \varphi^2} + \frac{1}{\varepsilon} \frac{\partial^2 u_\varphi}{\partial \rho \partial \varphi} - 3 \frac{\partial u_\varphi}{\partial \varphi} + \frac{2}{\varepsilon} \frac{\partial u_\rho}{\partial \rho} - 2u_\rho \right) + \frac{1}{\varepsilon} \frac{\partial^2 u_\xi}{\partial \rho \partial \xi} + e^{\varepsilon \rho} \frac{\partial^2 u_\rho}{\partial \xi^2} \right] = 0, \\ \frac{\partial}{\partial \rho} \left[Ge^{-\varepsilon \rho} \left(\varepsilon \frac{\partial u_\rho}{\partial \varphi} + \frac{\partial u_\varphi}{\partial \rho} - \varepsilon u_\varphi \right) \right] + \varepsilon^2 He^{-\varepsilon \rho} \frac{\partial^2 u_\varphi}{\partial \varphi^2} + \\ + \varepsilon^2 (H + 2G) e^{-\varepsilon \rho} \frac{\partial u_\rho}{\partial \varphi} + \lambda \varepsilon e^{-\varepsilon \rho} \frac{\partial^2 u_\rho}{\partial \rho \partial \varphi} + \\ + \varepsilon^2 (G + \lambda) \frac{\partial^2 u_\xi}{\partial \xi \partial \varphi} + 2G \varepsilon e^{-\varepsilon \rho} \frac{\partial u_\varphi}{\partial \rho} - \\ - 2\varepsilon^2 Ge^{-\varepsilon \rho} u_\varphi + \varepsilon^2 Ge^{\varepsilon \rho} \frac{\partial^2 u_\varphi}{\partial \xi^2} = 0, \\ \frac{\partial}{\partial \rho} \left[G \left(e^{-\varepsilon \rho} \frac{\partial u_\xi}{\partial \rho} + \varepsilon \frac{\partial u_\rho}{\partial \xi} \right) \right] + \varepsilon^2 (G + \lambda) \frac{\partial^2 u_\varphi}{\partial \xi \partial \varphi} + \\ + \varepsilon^2 (G + \lambda) \frac{\partial u_\rho}{\partial \xi} + G \varepsilon e^{-\varepsilon \rho} \frac{\partial u_\xi}{\partial \rho} + \varepsilon^2 Ge^{-\varepsilon \rho} \frac{\partial^2 u_\xi}{\partial \varphi^2} + \\ + \varepsilon^2 He^{\varepsilon \rho} \frac{\partial^2 u_\xi}{\partial \xi^2} + \lambda \varepsilon \frac{\partial^2 u_\rho}{\partial \rho \partial \xi} = 0. \end{cases} \quad (3)$$

Here $\rho = \frac{1}{\varepsilon} \ln \left(\frac{r}{r_0} \right), \xi = \frac{z}{r_0}$ – new dimensionless variables, $\varepsilon = \frac{1}{2} \ln \left(\frac{r_2}{r_1} \right)$ – small parameter characterizing the thickness of the cylinder, $r_0 = \sqrt{r_1 r_2}, \rho \in [-1; 1], \xi \in [-l; l] \left(l = \frac{L}{r_0} \right); u_\rho = \frac{v_r}{r_0}, u_\varphi = \frac{v_\varphi}{r_0}, u_\xi = \frac{v_z}{r_0}; G = \frac{G_*}{G_0}, \lambda = \frac{\lambda_*}{G_0}, H = 2G + \lambda$ – di-

mensionless quantities; G_0 – some characteristic parameter that has the dimension of the shear modulus.

Let's assume that the Lamé elastic parameters $G=G(\rho), \lambda=\lambda(\rho)$ are arbitrary positive continuous functions of the variable ρ , the values of which can vary within the same order.

For bending deformation, the components u_ρ, u_ξ of the displacement vector are taken proportional to the cosine, and u_φ – the sine of the azimuthal angle φ [25]:

$$\begin{aligned} u_\rho &= u(\rho, \xi) \cos \varphi, \quad u_\varphi = v(\rho, \xi) \sin \varphi, \\ u_\xi &= w(\rho, \xi) \cos \varphi. \end{aligned} \quad (4)$$

Let's suppose that a load acts on the lateral part of the boundary

$$\begin{cases} \sigma_{\rho\rho} \Big|_{\rho=\pm 1} = f^\pm(\varphi, \xi), \\ \sigma_{\rho\varphi} \Big|_{\rho=\pm 1} = t^\pm(\varphi, \xi), \\ \sigma_{\rho\xi} \Big|_{\rho=\pm 1} = g^\pm(\varphi, \xi), \end{cases} \quad (5)$$

where $\sigma_{\rho\rho} = \frac{\sigma_{rr}}{G_0}, \sigma_{\rho\varphi} = \frac{\sigma_{r\varphi}}{G_0}, \sigma_{\rho\xi} = \frac{\sigma_{rz}}{G_0}$ – dimensionless quantities; $f^\pm(\varphi, \xi), t^\pm(\varphi, \xi), g^\pm(\varphi, \xi)$ – sufficiently smooth functions and are of order $O(1)$ with respect to ε .

Let's assume that arbitrary boundary conditions are set at the ends of the cylinder, leaving the cylinder in equilibrium.

Consider the construction of particular solutions of equations (3) satisfying boundary conditions (5), i.e., inhomogeneous solutions.

When constructing particular solutions (3), various techniques can be used.

Let's construct inhomogeneous solutions for problems (3), (5) by the method described in [26]. Let's assume that the following boundary conditions are set on the lateral surfaces of the cylinder:

$$\begin{cases} \sigma_{\rho\rho} \Big|_{\rho=\pm 1} = \tau_\pm \frac{\xi^m}{m!} \cos \varphi, \\ \sigma_{\rho\varphi} \Big|_{\rho=\pm 1} = q_\pm \frac{\xi^m}{m!} \sin \varphi, \\ \sigma_{\rho\xi} \Big|_{\rho=\pm 1} = h_\pm \frac{\xi^m}{m!} \cos \varphi. \end{cases} \quad (6)$$

Substituting (4) into (3) and (6), let's obtain:

$$\left(A_0 + \frac{\partial}{\partial \xi} A_1 + \frac{\partial^2}{\partial \xi^2} A_2 \right) \bar{a} = \bar{0}, \quad (7)$$

$$\left(B_0 + \frac{\partial}{\partial \xi} B_1 \right) \bar{a} = \bar{\sigma}_\pm \frac{\xi^m}{m!}. \quad (8)$$

Here $\bar{a}(\rho, \xi) = (u(\rho, \xi), v(\rho, \xi), w(\rho, \xi))^T$,

$$A_0 = \begin{vmatrix} \partial((H\partial + \lambda\varepsilon)e^{-\varepsilon\rho}) + & \partial(\lambda\varepsilon e^{-\varepsilon\rho}) + & 0 \\ +\varepsilon Ge^{-\varepsilon\rho}(2\partial - 3\varepsilon) & +\varepsilon Ge^{-\varepsilon\rho}(\partial - 3\varepsilon) & 0 \\ -\partial(\varepsilon Ge^{-\varepsilon\rho}) - & \partial(Ge^{-\varepsilon\rho}(\partial - \varepsilon)) + & 0 \\ -e^{-\varepsilon\rho}(\varepsilon^2(H + 2G) + \lambda\varepsilon\partial) & + (2\varepsilon G\partial - \varepsilon^2(H + 2G))e^{-\varepsilon\rho} & \partial(Ge^{-\varepsilon\rho}\partial) + \\ 0 & 0 & +\varepsilon Ge^{-\varepsilon\rho}(\partial - \varepsilon) \end{vmatrix},$$

$$A_1 = \begin{vmatrix} 0 & 0 & \varepsilon(\partial(\lambda) + G\partial) \\ 0 & 0 & -\varepsilon^2(G + \lambda) \\ \varepsilon^2(G + \lambda) + \lambda\varepsilon\partial + \varepsilon\partial(G) & \varepsilon^2(G + \lambda) & 0 \end{vmatrix},$$

$$A_2 = \begin{vmatrix} \varepsilon^2 G e^{\varepsilon\rho} & 0 & 0 \\ 0 & \varepsilon^2 G e^{\varepsilon\rho} & 0 \\ 0 & 0 & \varepsilon^2 H e^{\varepsilon\rho} \end{vmatrix}, \quad B_1 = \begin{vmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ G & 0 & 0 \end{vmatrix},$$

$$B_0 = \begin{vmatrix} \left(\frac{H}{\varepsilon}\partial + \lambda\right)e^{-\varepsilon\rho} & \lambda e^{-\varepsilon\rho} & 0 \\ -G e^{-\varepsilon\rho} & \left(\frac{G}{\varepsilon}\partial - G\right)e^{-\varepsilon\rho} & 0 \\ 0 & 0 & \frac{G e^{-\varepsilon\rho}}{\varepsilon}\partial \end{vmatrix},$$

$$\partial = \frac{\partial}{\partial\rho},$$

$$\bar{\sigma}_\pm = (\tau_\pm, q_\pm, h_\pm)^T.$$

Let's note that, having a set of solutions for different integers "m", it is possible to construct solutions for arbitrary boundary conditions, given by smooth functions, by approximating them in advance by polynomials.

Let's first consider an auxiliary problem. Let's suppose that on the lateral surfaces, instead of a power-law load, a load is given:

$$\begin{aligned} \sigma_{\rho\rho}|_{\rho=\pm 1} &= \tau_\pm e^{\alpha\xi} \cos\varphi, \quad \sigma_{\rho\varphi}|_{\rho=\pm 1} = \\ &= q_\pm e^{\alpha\xi} \sin\varphi, \quad \sigma_{\rho\xi}|_{\rho=\pm 1} = h_\pm e^{\alpha\xi} \cos\varphi. \end{aligned} \tag{9}$$

Taking into account (4), from (9) let's obtain:

$$\left(B_0 + \frac{\partial}{\partial\xi} B_1\right)\bar{a} = \bar{\sigma}_\pm e^{\alpha\xi} \quad \text{at } \rho = \pm 1. \tag{10}$$

Let's seek solutions to problems (7), (10) in the form:

$$\bar{a}(\rho, \xi) = \bar{b}(\rho) e^{\alpha\xi}, \tag{11}$$

where $\bar{b}(\rho) = (\tilde{u}(\rho), \tilde{v}(\rho), \tilde{w}(\rho))^T$.

After substituting (11) into (7), (10):

$$\begin{cases} (A_0 + \alpha A_1 + \alpha^2 A_2)\bar{b} = \bar{0}, \\ (B_0 + \alpha B_1)\bar{b}|_{\rho=\pm 1} = \bar{\sigma}_\pm. \end{cases} \tag{12}$$

Solutions (12) are a meromorphic function of the spectral parameter α . Its poles coincide with the spectrum of the homogeneous problem:

$$\begin{cases} (A_0 + \alpha A_1 + \alpha^2 A_2)\bar{b} = \bar{0}, \\ (B_0 + \alpha B_1)\bar{b}|_{\rho=\pm 1} = \bar{0}. \end{cases} \tag{13}$$

$\alpha=0$ is a fourfold point of the spectrum of the homogeneous problem (13). In a neighborhood of zero, solutions (12) have the form:

$$\bar{b}(\rho) = \alpha^{-4} \sum_{k=0}^{\infty} \alpha^k \bar{b}_k(\rho). \tag{14}$$

Let's introduce the operator

$$P(\cdot) = \frac{1}{(m+4)!} \lim_{\alpha \rightarrow 0} \frac{d^{(m+4)} \alpha^4(\cdot)}{d\alpha^{m+4}}. \tag{15}$$

If the operator $P(\cdot)$ is applied to the right-hand side of (10), then let's obtain the expression on the right-hand side of (8).

Let's substitute (14) into (11), and then into (7), (10). Next, act on (7), (10) by operator (15) and, equating the coefficients at the same powers of ξ , let's obtain a recurrent system of boundary value problems to determine $\bar{b}_k(\rho)$:

$$\begin{cases} A_0 \bar{b}_0 = \bar{0}, \\ B_0 \bar{b}_0|_{\rho=\pm 1} = \bar{0}. \end{cases} \tag{16}$$

$$\begin{cases} A_0 \bar{b}_1 + A_1 \bar{b}_0 = \bar{0}, \\ (B_0 \bar{b}_1 + B_1 \bar{b}_0)|_{\rho=\pm 1} = \bar{0}. \end{cases} \tag{17}$$

$$\begin{cases} A_0 \bar{b}_2 + A_1 \bar{b}_1 + A_2 \bar{b}_0 = \bar{0}, \\ (B_0 \bar{b}_2 + B_1 \bar{b}_1)|_{\rho=\pm 1} = \bar{0}. \end{cases} \tag{18}$$

$$\begin{cases} A_0 \bar{b}_3 + A_1 \bar{b}_2 + A_2 \bar{b}_1 = \bar{0}, \\ (B_0 \bar{b}_3 + B_1 \bar{b}_2)|_{\rho=\pm 1} = \bar{0}. \end{cases} \tag{19}$$

$$\begin{cases} A_0 \bar{b}_{4+k} + A_1 \bar{b}_{3+k} + A_2 \bar{b}_{2+k} = \bar{0}, \\ (B_0 \bar{b}_{4+k} + B_1 \bar{b}_{3+k})|_{\rho=\pm 1} = \bar{\sigma}_\pm \delta_{0k}. \end{cases} \tag{20}$$

Here δ_{0k} – the Kronecker symbol, $k = \overline{0; m}$.

The solution satisfying the boundary condition (8) takes the form:

$$\bar{a}(\rho; \xi) = \sum_{k=0}^{m+4} \frac{\xi^{m+4-k}}{(m+4-k)!} \bar{b}_k(\rho).$$

The system of boundary value problems (16)–(20) is solved by the small parameter method.

If the thickness of the cylinder is sufficiently small, and the load given on the lateral surfaces is sufficiently smooth, then to construct inhomogeneous solutions it is advisable to use the first iterative process of the asymptotic method [27, 28]. Using the first iterative process, let's construct particular solutions (3) satisfying the boundary conditions (5).

Let's seek solutions (3), (5) in the form:

$$\begin{aligned} u_\rho(\rho, \xi, \varphi) &= \varepsilon^{-1} (u_{\rho_0} + \varepsilon u_{\rho_1} + \dots), \\ u_\varphi(\rho, \xi, \varphi) &= \varepsilon^{-1} (u_{\varphi_0} + \varepsilon u_{\varphi_1} + \dots), \\ u_\xi(\rho, \xi, \varphi) &= \varepsilon^{-1} (u_{\xi_0} + \varepsilon u_{\xi_1} + \dots). \end{aligned} \tag{21}$$

Substitution of (21) into (3), (5) leads to a system whose successive integration over ρ gives the relations for the expansion coefficients u_ρ, u_φ, u_ξ :

$$\begin{aligned} u_{\rho_0} &= C_1^*(\xi, \varphi), \quad u_{\varphi_0} = C_2^*(\xi, \varphi), \quad u_{\xi_0} = C_3^*(\xi, \varphi), \\ u_{\rho_1} &= -\left(\frac{\partial C_2^*}{\partial \varphi} + \frac{\partial C_3^*}{\partial \xi} + C_1^*\right) \int_0^\rho \frac{\lambda}{H} dt + C_4^*(\xi, \varphi), \end{aligned}$$

$$u_{\varphi_1} = \rho \left(C_2^* - \frac{\partial C_1^*}{\partial \varphi} \right) + C_3^*(\xi, \varphi), \quad (22)$$

$$u_{\xi_1} = -\rho \frac{\partial C_1^*}{\partial \xi} + C_6^*(\xi, \varphi),$$

where

$$C_1^*(\xi, \varphi) = \frac{f(\xi, \varphi)}{g_0} - \frac{t_0}{g_0} \frac{\partial C_3^*}{\partial \xi} - \frac{\partial C_2^*}{\partial \varphi},$$

$$\begin{cases} G_0 \frac{\partial^2 C_2^*}{\partial \xi \partial \varphi} + G_0 \frac{\partial^2 C_3^*}{\partial \varphi^2} + \left(g_0 - \frac{t_0^2}{g_0} \right) \frac{\partial^2 C_3^*}{\partial \xi^2} = \\ = - \left(g(\xi, \varphi) + \frac{t_0}{g_0} \frac{\partial f}{\partial \xi} \right), \\ \frac{\partial^2 C_3^*}{\partial \xi \partial \varphi} + \frac{\partial^2 C_2^*}{\partial \xi^2} = - \frac{1}{G_0} \left(t(\xi, \varphi) + \frac{\partial f}{\partial \varphi} \right), \end{cases}$$

$$\begin{aligned} C_4^*(\xi) &= \frac{-\partial C_2^*}{\partial \varphi} - \frac{t_0}{g_0} \frac{\partial C_6^*}{\partial \xi} + \frac{Q_0^*}{g_0} \frac{\partial C_2^*}{\partial \varphi} + \\ &+ \frac{(g_1 - g_0)}{g_0} \left(\frac{\partial^3 C_2^*}{\partial \xi^2 \partial \varphi} + \frac{\partial^3 C_2^*}{\partial \varphi^3} \right) + \frac{(Q_0^* + g_1)}{g_0} C_1^* + \\ &+ \frac{(2g_1 - g_0)}{g_0} \frac{\partial^2 C_1^*}{\partial \varphi^2} + \frac{(2t_1 - t_0)}{g_0} \frac{\partial^2 C_1^*}{\partial \xi^2} + \frac{Q_1^*}{g_0} \frac{\partial C_3^*}{\partial \xi} + \\ &+ \frac{(g_1 - g_0)}{g_0} \left(\frac{\partial^3 C_3^*}{\partial \xi^3} + \frac{\partial^3 C_3^*}{\partial \xi \partial \varphi^2} \right) + \frac{2}{g_0} \frac{\partial t^-}{\partial \varphi} + \\ &+ \frac{2}{g_0} \frac{\partial g^-}{\partial \xi} + \frac{1}{g_0} f^- \int_{-1}^1 \frac{2G}{H} d\rho, \end{aligned}$$

$$\begin{aligned} G_0 \frac{\partial^2 C_5^*}{\partial \xi \partial \varphi} + G_0 \frac{\partial^2 C_6^*}{\partial \varphi^2} + \left(g_0 - \frac{t_0^2}{g_0} \right) \frac{\partial^2 C_6^*}{\partial \xi^2} &= \\ = \left(Q_4^* - \frac{t_0(Q_0^* + g_1)}{g_0} \right) \frac{\partial C_1^*}{\partial \xi} + \\ + \left(g_1 + \frac{t_0(t_0 - 2t_1)}{g_0} \right) \frac{\partial^3 C_1^*}{\partial \xi^3} + \\ + \left(g_1 + \frac{t_0(g_0 - 2g_1)}{g_0} \right) \frac{\partial^3 C_1^*}{\partial \xi \partial \varphi^2} + \\ + \frac{t_0(g_0 - g_1)}{g_0} \left(\frac{\partial^4 C_2^*}{\partial \xi^3 \partial \varphi} + \frac{\partial^4 C_2^*}{\partial \varphi^3 \partial \xi} \right) + \\ + \left(G_0 + t_0 - 2G_1 - \right. \\ \left. - 2t_1 + Q_4^* - \frac{t_0}{g_0} Q_0^* \right) \frac{\partial^2 C_2^*}{\partial \varphi \partial \xi} + \\ + \left(Q_5^* - \frac{t_0}{g_0} Q_1^* \right) \frac{\partial^2 C_3^*}{\partial \xi^2} + \\ + \frac{t_0(g_0 - g_1)}{g_0} \left(\frac{\partial^4 C_3^*}{\partial \xi^4} + \frac{\partial^4 C_3^*}{\partial \xi^2 \partial \varphi^2} \right) + \\ + G_0 \frac{\partial^2 C_3^*}{\partial \varphi^2} - 2g^- - \frac{2t_0}{g_0} \left(\frac{\partial^2 g^-}{\partial \xi^2} + \frac{\partial^2 t^-}{\partial \varphi \partial \xi} \right) - \\ - \frac{\partial f^-}{\partial \xi} \int_{-1}^1 \frac{\lambda}{H} d\rho - \frac{t_0}{g_0} \frac{\partial f^-}{\partial \xi} \int_{-1}^1 \frac{2G}{H} d\rho, \end{aligned}$$

$$\begin{aligned} G_0 \frac{\partial^2 C_5^*}{\partial \xi^2} + G_0 \frac{\partial^2 C_6^*}{\partial \xi \partial \varphi} &= (Q_2^* - Q_0^*) \frac{\partial^2 C_6^*}{\partial \varphi^2} + \\ + (g_1 + 2t_1 + t_0) \frac{\partial^3 C_1^*}{\partial \xi^2 \partial \varphi} + \\ + (Q_3^* - Q_1^*) \frac{\partial^2 C_3^*}{\partial \xi \partial \varphi} + (Q_2^* - Q_0^*) \frac{\partial C_1^*}{\partial \varphi} + \\ + (g_0 - g_1) \left[\begin{aligned} &\left(\frac{\partial^3 C_1^*}{\partial \varphi^3} + \frac{\partial^4 C_3^*}{\partial \xi^3 \partial \varphi} + \right. \\ &\left. + \frac{\partial^4 C_3^*}{\partial \xi \partial \varphi^3} + \frac{\partial^4 C_2^*}{\partial \xi^2 \partial \varphi^2} + \frac{\partial^4 C_2^*}{\partial \varphi^4} \right) - \\ &- 2G_1 \frac{\partial^2 C_2^*}{\partial \xi^2} - 2 \frac{\partial f^-}{\partial \varphi} - 2 \frac{\partial^2 t^-}{\partial \varphi^2} - 2 \frac{\partial^2 g^-}{\partial \xi \partial \varphi} - 4t^-, \end{aligned} \right] \\ t(\xi, \varphi) &= t^+(\xi, \varphi) - t^-(\xi, \varphi), \quad g(\xi, \varphi) = g^+(\xi, \varphi) - g^-(\xi, \varphi), \\ f(\xi, \varphi) &= f^+(\xi, \varphi) - f^-(\xi, \varphi), \end{aligned}$$

$$\begin{aligned} Q_0^* &= \int_{-1}^1 \frac{2G}{H} \left(\int_{-1}^{\rho} \frac{4G(G+\lambda)}{H} dt \right) d\rho + \\ &+ \int_{-1}^1 \frac{4G(G+\lambda)}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho, \\ Q_1^* &= \int_{-1}^1 \frac{4G(G+\lambda)}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho + \\ &+ 2 \int_{-1}^1 G \left(\int_{-1}^{\rho} \frac{2G\lambda}{H} dt \right) d\rho \\ Q_2^* &= 2(g_0 - g_1) - \int_{-1}^1 \frac{\lambda}{H} \left(\int_{-1}^{\rho} \frac{4G(G+\lambda)}{H} dt \right) d\rho + \\ &+ \int_{-1}^1 \frac{4G(G+\lambda)}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho, \\ Q_3^* &= \int_{-1}^1 \frac{4G(G+\lambda)}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho - \\ &- \int_{-1}^1 \lambda \left(\int_{-1}^{\rho} \frac{2G\lambda}{H} dt \right) d\rho + 2(G_0 + t_0 - G_1 - t_1), \\ Q_4^* &= t_0 - t_1 + \int_{-1}^1 \frac{2G\lambda}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho - \\ &- \int_{-1}^1 \frac{\lambda}{H} \left(\int_{-1}^{\rho} \frac{4G(G+\lambda)}{H} dt \right) d\rho, \\ Q_5^* &= G_0 + t_0 - 2(G_1 + t_1) + \int_{-1}^1 \frac{2G\lambda}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho - \\ &- \int_{-1}^1 \frac{\lambda}{H} \left(\int_{-1}^{\rho} \frac{4G(G+\lambda)}{H} dt \right) d\rho, \\ g_k &= \int_{-1}^1 \frac{4G(G+\lambda)}{H} \rho^k d\rho, \quad t_k = \int_{-1}^1 \frac{2G\lambda}{H} \rho^k d\rho, \quad G_k = \int_{-1}^1 G \rho^k d\rho. \end{aligned}$$

(21), (22) allow one to obtain asymptotic formulas for the components of the stress tensor.

5. 2. Construction of a homogeneous solution for a radial inhomogeneous cylinder

Any solution of the equilibrium equations (3) that satisfies the condition of the absence of stresses on the lateral surfaces is called a homogeneous solution.

Let's start building homogeneous solutions. In (5), let's substitute $f^{\pm}(\varphi, \xi) = t^{\pm}(\varphi, \xi) = g^{\pm}(\varphi, \xi) = 0$:

$$\begin{cases} \sigma_{\rho\rho}|_{\rho=\pm 1} = 0, \\ \sigma_{\rho\varphi}|_{\rho=\pm 1} = 0, \\ \sigma_{\rho\xi}|_{\rho=\pm 1} = 0. \end{cases} \quad (23)$$

Let's assume that the stresses are set at the ends of the cylinder

$$\begin{aligned} \sigma_{\rho\xi}|_{\xi=\pm l} &= f_{1s}(\rho) \cos \varphi, \sigma_{\varphi\xi}|_{\xi=\pm l} = \\ &= f_{2s}(\rho) \sin \varphi, \sigma_{\xi\xi}|_{\xi=\pm l} = f_{3s}(\rho) \cos \varphi. \end{aligned} \quad (24)$$

Here, $f_{1s}(\rho), f_{2s}(\rho), f_{3s}(\rho)$ ($s=1,2$) – sufficiently smooth functions having order $O(1)$ with respect to ε .

Let's seek solutions (3), (23) in the form:

$$\begin{aligned} u_{\rho}(\rho, \xi, \varphi) &= \tilde{u}(\rho) e^{\alpha\xi} \cos \varphi, \\ u_{\varphi}(\rho, \xi, \varphi) &= \tilde{v}(\rho) e^{\alpha\xi} \sin \varphi, \\ u_{\xi}(\rho, \xi, \varphi) &= \tilde{w}(\rho) e^{\alpha\xi} \cos \varphi. \end{aligned} \quad (25)$$

Substituting (25) into (3), (23):

$$\begin{cases} \left[He^{-\varepsilon\rho} \tilde{u}' + \lambda \varepsilon (e^{-\varepsilon\rho} (\tilde{u} + \tilde{v}) + \alpha \tilde{w}) \right]' + \\ + \varepsilon G \left[e^{-\varepsilon\rho} (\tilde{v}' - 3\varepsilon(\tilde{u} + \tilde{v}) + 2\tilde{u}') + \right. \\ \left. + \alpha \tilde{w}' + \alpha^2 \varepsilon e^{\varepsilon\rho} \tilde{u} \right] = 0, \\ \left[Ge^{-\varepsilon\rho} (\tilde{v}' - \varepsilon(\tilde{u} + \tilde{v})) \right]' - \\ - e^{-\varepsilon\rho} \left[\varepsilon^2 (H + 2G) \tilde{u} + \lambda \varepsilon \tilde{u}' - \right. \\ \left. - 2\varepsilon G \tilde{v}' + \varepsilon^2 (H + 2G) \tilde{v} \right] + \\ + \varepsilon^2 Ge^{\varepsilon\rho} \alpha^2 \tilde{v} - \varepsilon^2 (G + \lambda) \alpha \tilde{w} = 0, \\ \left[G(e^{-\varepsilon\rho} \tilde{w}' + \varepsilon \alpha \tilde{u}) \right]' + \\ + \varepsilon^2 (G + \lambda) \alpha (\tilde{u} + \tilde{v}) + \\ + \varepsilon Ge^{-\varepsilon\rho} (\tilde{w}' - \varepsilon \tilde{w}) + \\ + \varepsilon^2 He^{\varepsilon\rho} \alpha^2 \tilde{w} + \lambda \varepsilon \alpha \tilde{u}' = 0. \end{cases} \quad (26)$$

$$\begin{cases} \left(He^{-\varepsilon\rho} \tilde{u}' + \lambda \varepsilon (e^{-\varepsilon\rho} (\tilde{u} + \tilde{v}) + \alpha \tilde{w}) \right) \Big|_{\rho=\pm 1} = 0, \\ Ge^{-\varepsilon\rho} (\tilde{v}' - \varepsilon(\tilde{u} + \tilde{v})) \Big|_{\rho=\pm 1} = 0, \\ G(e^{-\varepsilon\rho} \tilde{w}' + \varepsilon \alpha \tilde{u}) \Big|_{\rho=\pm 1} = 0. \end{cases} \quad (27)$$

To solve (26), (27), let's use the asymptotic method [27, 28] based on three iterative processes.

Homogeneous solutions corresponding to the first iteration process can be obtained from (22) if to put $f^{\pm}(\varphi, \xi) = t^{\pm}(\varphi, \xi) = g^{\pm}(\varphi, \xi) = 0$ in them:

$$\begin{aligned} u_{\rho}^{(1)}(\rho, \xi, \varphi) &= \\ &= \left[\left(\frac{\xi^3}{6} + \frac{t_0}{2g_0} \xi \right) C + \left(\frac{\xi^2}{2} + \frac{t_0}{2g_0} \right) D + \right. \\ &\quad \left. + \varepsilon (D + \xi C) \left(Q_0 + \frac{2G_0}{g_0} \int_0^{\rho} \frac{\lambda}{H} dt \right) + O(\varepsilon^2) \right] \cos \varphi, \\ u_{\varphi}^{(1)}(\rho, \xi, \varphi) &= \\ &= \left[\left(-\frac{\xi^3}{6} + \frac{t_0}{2g_0} \xi \right) C + \left(-\frac{\xi^2}{2} + \frac{t_0}{2g_0} \right) D + \right. \\ &\quad \left. + \varepsilon (D + \xi C) \left(Q_0 + \rho \frac{t_0}{g_0} \right) + O(\varepsilon^2) \right] \sin \varphi, \end{aligned} \quad (28)$$

$$\begin{aligned} u_{\xi}^{(1)}(\rho, \xi, \varphi) &= \\ &= \left[-\left(2 + \frac{3t_0}{2g_0} + \frac{\xi^2}{2} \right) C - \xi D + \right. \\ &\quad \left. + \varepsilon \left(\left(Q_1 - \frac{\xi^2}{2} \rho - \rho \frac{t_0}{2g_0} \right) C - \rho \xi D \right) + O(\varepsilon^2) \right] \cos \varphi, \end{aligned}$$

where C, D are arbitrary constants,

$$\begin{aligned} Q_0 &= \frac{t_1}{g_0} - \frac{t_0 g_1}{2g_0^2} + \frac{t_0}{g_0} \int_{-1}^1 \frac{G}{H} \left(\int_{-1}^{\rho} \frac{4G(G+\lambda)}{H} dt \right) d\rho - \\ &- \frac{1}{g_0} \int_{-1}^1 \frac{G}{H} \left(\int_{-1}^{\rho} \frac{2G\lambda}{H} dt \right) d\rho - \frac{G_0}{g_0^2} \int_{-1}^1 \frac{4G(G+\lambda)}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho, \end{aligned}$$

$$\begin{aligned} Q_1 &= -\frac{3g_1}{G_0} + \frac{t_0}{g_0 G_0} \int_{-1}^1 \frac{\lambda}{H} \left(\int_{-1}^{\rho} \frac{4G(G+\lambda)}{H} dt \right) d\rho - \\ &- \frac{1}{G_0} \int_{-1}^1 \frac{\lambda}{H} \left(\int_{-1}^{\rho} \frac{2G\lambda}{H} dt \right) d\rho + \\ &+ \frac{2}{g_0} \int_{-1}^1 \frac{2G\lambda}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho + \frac{2t_0(G_1 + t_1)}{G_0 g_0} + \\ &+ \frac{(G_0 + 2t_0)}{G_0} \left[\frac{t_1}{g_0} - \frac{t_0 g_1}{2g_0^2} + \right. \\ &\quad \left. + \frac{t_0}{g_0^2} \int_{-1}^1 \frac{G}{H} \left(\int_{-1}^{\rho} \frac{4G(G+\lambda)}{H} dt \right) d\rho - \right. \\ &\quad \left. - \frac{1}{g_0} \int_{-1}^1 \frac{G}{H} \left(\int_{-1}^{\rho} \frac{2G\lambda}{H} dt \right) d\rho - \right. \\ &\quad \left. - \frac{G_0}{g_0^2} \int_{-1}^1 \frac{4G(G+\lambda)}{H} \left(\int_0^{\rho} \frac{\lambda}{H} dt \right) d\rho \right]. \end{aligned}$$

These solutions correspond to the fourfold eigenvalue $\alpha_0 = 0$. The stress corresponding to solution (28) is as follows:

$$\begin{aligned} \sigma_{\rho\rho}^{(1)} &= \sigma_{\rho\varphi}^{(1)} = \sigma_{\rho\xi}^{(1)} = 0, \\ \sigma_{\xi\xi}^{(1)} &= \left[(D + \xi C) \left(\frac{2G\lambda}{H} \frac{t_0}{g_0} - \frac{4G(G+\lambda)}{H} \right) + O(\varepsilon) \right] \cos \varphi, \\ \sigma_{\varphi\xi}^{(1)} &= \left[\frac{2G(g_0 + t_0)}{g_0} C + O(\varepsilon) \right] \sin \varphi, \end{aligned}$$

$$\sigma_{\varphi\varphi}^{(1)} = \left[(D + \xi C) \left(\frac{4G(G + \lambda)}{H} \frac{t_0}{g_0} - \frac{2G\lambda}{H} \right) + O(\varepsilon) \right] \cos \varphi. \quad (29)$$

The solution corresponding to the second iterative process will be sought in the following form:

$$\begin{aligned} \tilde{u}^{(2)}(\rho) &= \tilde{u}_{20}(\rho) + \varepsilon \tilde{u}_{21}(\rho) + \dots, \\ \tilde{v}^{(2)}(\rho) &= \varepsilon (\tilde{v}_{20}(\rho) + \varepsilon \tilde{v}_{21}(\rho) + \dots), \\ \tilde{w}^{(2)}(\rho) &= \varepsilon^{\frac{1}{2}} (\tilde{w}_{20}(\rho) + \varepsilon \tilde{w}_{21}(\rho) + \dots), \\ \alpha &= \varepsilon^{\frac{1}{2}} (\alpha_0 + \varepsilon \alpha_1 + \dots). \end{aligned} \quad (30)$$

Substituting (30) into (26), (27) after some transformations let's obtain:

$$\begin{aligned} u_{\rho}^{(2)}(\rho, \xi, \varphi) &= \sum_{j=1}^4 T_j U_{\rho j}^{(2)}, \\ u_{\varphi}^{(2)}(\rho, \xi, \varphi) &= \varepsilon \sum_{j=1}^4 T_j U_{\varphi j}^{(2)}, \\ u_{\xi}^{(2)}(\rho, \xi, \varphi) &= \varepsilon^{\frac{1}{2}} \sum_{j=1}^4 T_j U_{\xi j}^{(2)}, \end{aligned} \quad (31)$$

where

$$U_{\rho j}^{(2)} = \left[1 + \varepsilon \left(\alpha_{0j}^2 \int_0^{\rho} \frac{\lambda}{H} x dx - \int_0^{\rho} \frac{\lambda}{H} dx + \frac{(t_0 - \alpha_{0j}^2 g_1)}{g_0} \int_0^{\rho} \frac{\lambda}{H} dx \right) + O(\varepsilon^2) \right] \times \exp \left(\varepsilon^{\frac{1}{2}} (\alpha_{0j} + \varepsilon \alpha_{1j} + \dots) \xi \right) \cos \varphi,$$

$$\begin{aligned} U_{\varphi}^{(2)} &= \left[\rho + \frac{g_0^2 - \alpha_{0j}^2 g_0 (2G_1 + t_1) + (G_0 + t_0) (\alpha_{0j}^2 g_1 - t_0)}{\alpha_{0j}^2 G_0 g_0} + O(\varepsilon) \right] \times \\ &\times \exp \left(\varepsilon^{\frac{1}{2}} (\alpha_{0j} + \varepsilon \alpha_{1j} + \dots) \xi \right) \sin \varphi, \end{aligned}$$

$$\begin{aligned} U_{\xi j}^{(2)} &= \left[-\alpha_{0j} \rho + \frac{(\alpha_{0j}^2 g_1 - t_0)}{\alpha_{0j} g_0} + O(\varepsilon) \right] \times \\ &\times \exp \left(\varepsilon^{\frac{1}{2}} (\alpha_{0j} + \varepsilon \alpha_{1j} + \dots) \xi \right) \cos \varphi. \end{aligned}$$

To determine α_{0j} :

$$\begin{aligned} (g_1^2 - g_0 g_2) \alpha_0^4 + 2(g_0 t_1 - g_1 t_0) \alpha_0^2 + \\ + (t_0^2 - 2g_0 G_0 - t_0 g_0) = 0. \end{aligned}$$

The stress corresponding to the second iterative process is as follows:

$$\begin{aligned} \sigma_{\rho\rho}^{(2)} &= \left(-\alpha_{0j}^4 \int_{-1}^{\rho} \left(\int_{-1}^x \frac{4G(G + \lambda)}{H} y dy \right) dx + \right. \\ &\left. + \alpha_{0j}^2 \left[\int_{-1}^{\rho} \left(\int_{-1}^x \frac{2G\lambda}{H} dy \right) dx - \int_{-1}^{\rho} \frac{2G\lambda}{H} x dx \right] + \right. \\ &\left. + \left[\alpha_{0j}^2 \times \int_{-1}^{\rho} \left(\int_{-1}^x \frac{4G(G + \lambda)}{H} dy \right) dx + \int_{-1}^{\rho} \frac{2G\lambda}{H} x dx \right] \right) \times \\ &\times \frac{(\alpha_{0j}^2 g_1 - t_0)}{g_0} + O(\varepsilon) \\ &\times \exp \left(\varepsilon^{\frac{1}{2}} (\alpha_{0j} + \varepsilon \alpha_{1j} + \dots) \xi \right) \cos \varphi, \end{aligned}$$

$$\begin{aligned} \sigma_{\rho\varphi}^{(2)} &= \varepsilon \sum_{j=1}^4 T_j \left(\int_{-1}^{\rho} \frac{4G(G + \lambda)}{H} dx - \right. \\ &\left. - \alpha_{0j}^2 \left[2 \int_{-1}^{\rho} x G dx + \int_{-1}^{\rho} \frac{2G\lambda}{H} x dx \right] + \right. \\ &\left. + \frac{(\alpha_{0j}^2 g_1 - t_0)}{g_0} \left[\int_{-1}^{\rho} G dx + \int_{-1}^{\rho} \frac{2G\lambda}{H} dx \right] - \right. \\ &\left. - \frac{1}{g_0 G_0} \left[\begin{aligned} &g_0^2 - \alpha_{0j}^2 g_0 \times \\ &\times (2G_1 + t_1) + \\ &+ (G_0 + t_0) \times \\ &\times (\alpha_{0j}^2 g_1 - t_0) \end{aligned} \right] \right) \times \\ &\times \int_{-1}^{\rho} G dx + O(\varepsilon) \\ &\times \exp \left(\varepsilon^{\frac{1}{2}} (\alpha_{0j} + \varepsilon \alpha_{1j} + \dots) \xi \right) \sin \varphi, \end{aligned}$$

$$\begin{aligned} \sigma_{\rho\xi}^{(2)} &= \varepsilon^{\frac{1}{2}} \sum_{j=1}^4 T_j \alpha_{0j} \left(\alpha_{0j}^2 \int_{-1}^{\rho} \frac{4G(G + \lambda)}{H} x dx - \right. \\ &\left. - \int_{-1}^{\rho} \frac{2G\lambda}{H} dx - \frac{(\alpha_{0j}^2 g_1 - t_0)}{g_0} \right) \times \\ &\times \left[\int_{-1}^{\rho} \frac{4G(G + \lambda)}{H} dx \right] + O(\varepsilon) \\ &\times \exp \left(\varepsilon^{\frac{1}{2}} (\alpha_{0j} + \varepsilon \alpha_{1j} + \dots) \xi \right) \cos \varphi, \end{aligned} \quad (32)$$

$$\sigma_{\varphi\varphi}^{(2)} = \sum_{j=1}^4 T_j \left\langle \frac{4G(G+\lambda)}{H} + \left[\frac{(\alpha_{0j}^2 g_1 - t_0)}{g_0} - \alpha_{0j}^2 \rho \right] \frac{2G\lambda}{H} + O(\varepsilon) \right\rangle \times \exp\left(\varepsilon^{\frac{1}{2}}(\alpha_{0j} + \varepsilon\alpha_{1j} + \dots)\xi\right) \cos\varphi,$$

$$\sigma_{\varphi\xi}^{(2)} = \varepsilon^{\frac{1}{2}} \sum_{j=1}^4 T_j G \left\langle \frac{2\alpha_{0j}\rho + \frac{1}{\alpha_{0j}G_0g_0} \begin{bmatrix} g_0^2 - \alpha_{0j}^2 g_0 \times \\ \times(2G_1 + t_1) + \\ + (G_0 + t_0) \times \\ \times(\alpha_{0j}^2 g_1 - t_0) \end{bmatrix}}{\alpha_{0j}g_0} + O(\varepsilon) \right\rangle \times \exp\left(\varepsilon^{\frac{1}{2}}(\alpha_{0j} + \varepsilon\alpha_{1j} + \dots)\xi\right) \sin\varphi,$$

$$\sigma_{\xi\xi}^{(2)} = \sum_{j=1}^4 T_j \left\langle \frac{2G\lambda}{H} + \frac{4G(G+\lambda)}{H} \times \left[\frac{\alpha_{0j}^2 g_1 - t_0}{g_0} - \alpha_{0j}^2 \rho \right] + O(\varepsilon) \right\rangle \times \exp\left(\varepsilon^{\frac{1}{2}}(\alpha_{0j} + \varepsilon\alpha_{1j} + \dots)\xi\right) \cos\varphi.$$

The solution corresponding to the third iterative process is sought in the form:

$$\begin{aligned} \tilde{u}^{(3)}(\rho) &= \varepsilon(\tilde{u}_{30}(\rho) + \varepsilon\tilde{u}_{31}(\rho) \dots), \\ \tilde{v}^{(3)}(\rho) &= \varepsilon(\tilde{v}_{30}(\rho) + \varepsilon\tilde{v}_{31}(\rho) \dots), \\ \tilde{w}^{(3)}(\rho) &= \varepsilon(\tilde{w}_{30}(\rho) + \varepsilon\tilde{w}_{31}(\rho) \dots), \\ \alpha &= \varepsilon^{-1}(\beta_0 + \varepsilon\beta_1 + \dots). \end{aligned} \tag{33}$$

After substituting (33) into (26), (27) for the first terms of the expansion, let's obtain:

$$N(\beta_0)\bar{f}_0 = \bar{0}, \tag{34}$$

where

$$N(\beta_0)\bar{f}_0 = \{\tau(\beta_0)\bar{f}_0, L_*(\beta_0)\bar{f}_0 = \bar{0} \text{ at } \rho = \pm 1\},$$

$$\begin{aligned} \tau(\beta_0)\bar{f}_0 &= (N_0 + \beta_0 N_1 + \beta_0^2 N_2)\bar{f}_0, L_*(\beta_0)\bar{f}_0 = \\ &= (L_0 + \beta_0 L_1)\bar{f}_0, \end{aligned}$$

$$N_0 = \begin{bmatrix} \partial_1(H\partial_1) & 0 & 0 \\ 0 & \partial_1(G\partial_1) & 0 \\ 0 & 0 & \partial_1(G\partial_1) \end{bmatrix},$$

$$N_1 = \begin{bmatrix} 0 & 0 & \partial_1(\lambda) + G\partial_1 \\ 0 & 0 & 0 \\ \partial_1(G) + \lambda\partial_1 & 0 & 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} G & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & H \end{bmatrix},$$

$$L_0 = \begin{bmatrix} H\partial_1 & 0 & 0 \\ 0 & G\partial_1 & 0 \\ 0 & 0 & G\partial_1 \end{bmatrix}, L_1 = \begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ G & 0 & 0 \end{bmatrix},$$

$$\bar{f}_0 = (\tilde{u}_{30}, \tilde{v}_{30}, \tilde{w}_{30})^T.$$

Spectral problem (34) is divided into two independent problems with respect to $(\tilde{u}_{30}, \tilde{w}_{30})^T$ and \tilde{v}_{30} . These problems, respectively, coincide with the problem describing the potential solution and the vortex solution of a plate inhomogeneous over the thickness [26].

At the next stage, let's obtain a boundary value problem for determining $\bar{f}_1 = (\tilde{u}_{31}, \tilde{v}_{31}, \tilde{w}_{31})^T$ and β_1

$$\begin{cases} \tau(\beta_0)\bar{f}_1 = \tau(\beta_0, \beta_1)\bar{f}_0, \\ M(\beta_0)\bar{f}_0|_{\rho=\pm 1} = (\rho M_0 + \beta_1 E_6 + E_7)\bar{f}_0|_{\rho=\pm 1}, \end{cases} \tag{35}$$

where

$$\begin{aligned} \tau(\beta_0, \beta_1) &= \rho E_0 + E_1 + \beta_0 E_2 + \\ &+ \beta_0^2 \rho E_3 + 2\beta_0 \beta_1 E_4 + \beta_1 E_5, \end{aligned}$$

$$E_0 = \begin{bmatrix} \partial_1(H\partial_1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_1(G\partial_1) \end{bmatrix},$$

$$E_1 = \begin{bmatrix} \lambda\partial_1 - \partial_1(\lambda) & -G\partial_1 - \partial_1(\lambda) & 0 \\ \lambda\partial_1 + \partial_1(G) & \partial_1(G) - G\partial_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (G+\lambda) \\ -(G+\lambda) & -(G+\lambda) & 0 \end{bmatrix},$$

$$E_3 = \begin{bmatrix} -G & 0 & 0 \\ 0 & -2G & 0 \\ 0 & 0 & -H \end{bmatrix},$$

$$E_4 = \begin{bmatrix} -G & 0 & 0 \\ 0 & -G & 0 \\ 0 & 0 & -H \end{bmatrix},$$

$$E_5 = \begin{bmatrix} 0 & 0 & -\partial_1(\lambda) - G\partial_1 \\ 0 & 0 & 0 \\ -\partial_1(G) - \lambda\partial_1 & 0 & 0 \end{bmatrix},$$

$$E_6 = \begin{bmatrix} 0 & 0 & -\lambda \\ 0 & 0 & 0 \\ -G & 0 & 0 \end{bmatrix},$$

$$E_7 = \begin{bmatrix} -\lambda & -\lambda & 0 \\ G & G & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solvability condition (35) is the orthogonality of the right-hand side of the solution to the adjoint problem:

$$N^*(\beta_0) \bar{f}_0^* = N(-\bar{\beta}_0) \bar{f}_0^* = \bar{0},$$

where $\bar{f}_0^* = (\bar{u}_0^*, \bar{v}_0^*, \bar{w}_0^*)^T$.

Satisfying this condition, for β_1 let's obtain:

$$\beta_1 = \frac{K_2}{K_1},$$

where

$$K_1 = \int_{-1}^1 \left\{ \begin{aligned} & \left[(\lambda \bar{w}'_{30})' + G \bar{w}'_{30} \right] \bar{u}_0^* + \\ & \left[(G \bar{u}'_{30})' + \lambda \bar{u}'_{30} \right] \bar{w}_0^* + \\ & + 2\beta_0 \left[G(\bar{u}_{30} \bar{u}_0^* + \bar{v}_{30} \bar{v}_0^*) + \right. \\ & \left. + H \bar{w}_{30} \bar{w}_0^* \right] \end{aligned} \right\} d\rho,$$

$$K_2 = \int_{-1}^1 \left\{ \begin{aligned} & \beta_0 \left\{ \begin{aligned} & (G + \lambda) \left(\bar{w}_{30} \bar{v}_0^* - \right. \\ & \left. - \bar{v}_{30} \bar{w}_0^* - \bar{u}_{30} \bar{w}_0^* \right) - \\ & - \rho \left[\begin{aligned} & (\lambda \bar{w}'_{30})' \bar{u}_0^* + G \bar{w}'_{30} \bar{u}_0^* + \\ & + (G \bar{u}'_{30})' \bar{w}_0^* + \lambda \bar{u}'_{30} \bar{w}_0^* \end{aligned} \right] \end{aligned} \right\} - \\ & - 2\beta_0^2 \rho \left[\begin{aligned} & G \bar{u}_{30} \bar{u}_0^* + \\ & + H \bar{w}_{30} \bar{w}_0^* + G \bar{v}_{30} \bar{v}_0^* \end{aligned} \right] + \\ & + \left[\begin{aligned} & \lambda \bar{u}'_{30} - (\lambda \bar{u}_{30})' - \\ & - (\lambda \bar{v}_{30})' - G \bar{v}'_{30} \end{aligned} \right] \bar{u}_0^* + \\ & + \left[\begin{aligned} & (G \bar{v}'_{30})' + \lambda \bar{v}'_{30} + \\ & + (G \bar{u}'_{30})' - G \bar{v}'_{30} \end{aligned} \right] \bar{v}_0^* \end{aligned} \right\} d\rho.$$

The solutions corresponding to the third iteration process are:

$$u_\rho^{(3)} = \varepsilon \sum_{k=1}^{\infty} F_K \left[\begin{aligned} & -\beta_{0k}^{-3} (p_0 \Psi_k'')' - 2\beta_{0k}^{-1} p_1 \Psi_k' + \\ & + \beta_{0k}^{-1} (p_2 \Psi_k)' + O(\varepsilon) \end{aligned} \right] \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \cos \varphi, \tag{36}$$

$$u_\varphi^{(3)} = \varepsilon \sum_{k=1}^{\infty} F_K \left[\bar{v}_{30}^{(k)} + O(\varepsilon) \right] \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \sin \varphi,$$

$$u_\xi^{(3)} = \varepsilon \sum_{k=1}^{\infty} F_K \left[p_0 \beta_{0k}^{-2} \Psi_k'' - p_2 \Psi_k + O(\varepsilon) \right] \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \cos \varphi.$$

Here $\psi_k(\rho)$ are solutions to the generalized spectral Papkovich problem [26]

$$\begin{aligned} & (p_0 \Psi_k''(\rho))'' + \\ & + \beta_{0k}^2 \left(2(p_1 \Psi_k'(\rho))' - (p_2 \Psi_k(\rho))'' - p_2 \Psi_k''(\rho) \right) + \\ & + \beta_{0k}^4 p_0 \Psi_k(\rho) = 0, \end{aligned}$$

$$p_0 = \frac{2G + \lambda}{4G(G + \lambda)}; \quad p_1 = \frac{1}{2G}; \quad p_2 = \frac{\lambda}{4G(G + \lambda)},$$

$\tilde{v}_{30}^{(k)}$ are the solutions to the boundary value problem

$$\begin{cases} \left(G(\tilde{v}_{30}^{(k)}(\rho))' \right)' + \beta_{0k}^2 G \tilde{v}_{30}^{(k)}(\rho) = 0, \\ \left. G(\tilde{v}_{30}^{(k)}(\rho))' \right|_{\rho=\pm 1} = 0. \end{cases}$$

The stresses corresponding to the third iteration process are:

$$\begin{aligned} \sigma_{\rho\rho}^{(3)} &= \sum_{k=1}^{\infty} F_k \left[\beta_{0k} \Psi_k + O(\varepsilon) \right] \times \\ & \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \cos \varphi, \\ \sigma_{\rho\varphi}^{(3)} &= \sum_{k=1}^{\infty} F_k G \left[(\tilde{v}_{30}^{(k)})' + O(\varepsilon) \right] \times \\ & \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \sin \varphi, \\ \sigma_{\varphi\xi}^{(3)} &= \sum_{k=1}^{\infty} F_k G \left[\beta_{0k} \tilde{v}_{30}^{(k)} + O(\varepsilon) \right] \times \\ & \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \sin \varphi, \\ \sigma_{\rho\xi}^{(3)} &= \sum_{k=1}^{\infty} F_k G \left[-2p_1 \Psi_k' + O(\varepsilon) \right] \times \\ & \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \cos \varphi, \\ \sigma_{\varphi\varphi}^{(3)} &= \lambda (p_0 - p_2) \sum_{k=1}^{\infty} F_K \left[\beta_{0k}^{-1} \Psi_k'' + \beta_{0k} \Psi_k + O(\varepsilon) \right] \times \\ & \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \cos \varphi, \\ \sigma_{\xi\xi}^{(3)} &= \sum_{k=1}^{\infty} F_K \left[\beta_{0k}^{-1} \Psi_k'' + O(\varepsilon) \right] \times \\ & \times \exp(\varepsilon^{-1} (\beta_{0k} + \varepsilon \beta_{1k} + \dots) \xi) \cos \varphi, \end{aligned} \tag{37}$$

The general solution (26), (27) will be the sum of solutions (28), (31), (36) corresponding to the above three iterative processes:

$$u_\rho(\rho, \xi, \varphi) = u_\rho^{(1)} + u_\rho^{(2)} + u_\rho^{(3)},$$

$$u_\xi(\rho, \xi, \varphi) = u_\xi^{(1)} + u_\xi^{(2)} + u_\xi^{(3)},$$

$$u_\varphi(\rho, \xi, \varphi) = u_\varphi^{(1)} + u_\varphi^{(2)} + u_\varphi^{(3)}.$$

5.3. Qualitative analysis of stress-strain states corresponding to different types of homogeneous solutions

Let's give a characteristic of the stress-strain states determined by homogeneous solutions. During bending deformation, the stresses $\sigma_{\xi\xi}$, $\sigma_{\rho\xi}$, $\sigma_{\varphi\xi}$ in the cross section are statically equivalent to the transverse force \tilde{X} and bending moment \tilde{M}_b relative to the OY axis in the $z=0$ plane. For \tilde{X} and \tilde{M}_b there are [25]:

$$\tilde{X} = \int_0^{2\pi} \int_{r_1}^{r_2} (\sigma_{rz} - \sigma_{\varphi z}) r dr d\varphi, \tag{38}$$

$$\tilde{M}_b = z\tilde{X} - \int_0^{2\pi} \int_{r_1}^{r_2} r^2 \sigma_{zz} dr d\varphi. \tag{39}$$

In dimensionless coordinates (38) and (39) take the form:

$$X = \pi \varepsilon \int_{-1}^1 (\sigma_{\rho\xi}^* - \sigma_{\varphi\xi}^*) e^{2\varepsilon\rho} d\rho, \tag{40}$$

$$M_b = \pi \varepsilon \xi \int_{-1}^1 (\sigma_{\rho\xi}^* - \sigma_{\varphi\xi}^*) e^{2\varepsilon\rho} d\rho - \pi \varepsilon \int_{-1}^1 \sigma_{\xi\xi}^* e^{3\varepsilon\rho} d\rho. \tag{41}$$

Here, asterisks denote the factors at $\cos\varphi, \sin\varphi$ in the corresponding stress expressions and $X = \frac{\tilde{X}}{G_0 r_0^2}, M_b = \frac{\tilde{M}_b}{G_0 r_0^3}$ – dimensionless quantities.

Let’s investigate the connection of homogeneous solutions with the transverse force X and the bending moment M_b in the section $\xi = \text{const}$.

Let’s represent the displacements in the form:

$$\begin{aligned} u_\rho &= u_\rho^{(1)} + \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} \tilde{u}_k(\rho) \cos \varphi, \\ u_\varphi &= u_\varphi^{(1)} + \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} \tilde{v}_k(\rho) \sin \varphi, \\ u_\xi &= u_\xi^{(1)} + \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} \tilde{w}_k(\rho) \cos \varphi. \end{aligned} \tag{42}$$

The second term includes displacements determined by the second and third groups of solutions.

For stresses:

$$\begin{aligned} \sigma_{\rho\xi} &= \sigma_{\rho\xi}^{(1)} + \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} E_{k1}(\rho) \cos \varphi, \\ \sigma_{\varphi\xi} &= \sigma_{\varphi\xi}^{(1)} + \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} E_{k2}(\rho) \sin \varphi, \\ \sigma_{\xi\xi} &= \sigma_{\xi\xi}^{(1)} + \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} E_{k3}(\rho) \cos \varphi. \end{aligned} \tag{43}$$

Here

$$\begin{aligned} E_{k1}(\rho) &= G \left[\frac{e^{-\varepsilon\rho}}{\varepsilon} \tilde{w}'_k(\rho) + \alpha_k \tilde{u}_k(\rho) \right], \\ E_{k2}(\rho) &= G \left[\alpha_k \tilde{v}_k(\rho) - e^{-\varepsilon\rho} \tilde{w}_k(\rho) \right], \\ E_{k3}(\rho) &= H \alpha_k \tilde{w}_k(\rho) + \lambda e^{-\varepsilon\rho} \left[\frac{\tilde{u}'_k(\rho)}{\varepsilon} + \tilde{u}_k(\rho) + \tilde{v}_k(\rho) \right]. \end{aligned}$$

After substituting (43) into (40), (41):

$$\begin{aligned} X &= -\pi \varepsilon \left[\frac{2G_0(g_0 + t_0)}{g_0} C + O(\varepsilon) \right] + \\ &+ \pi \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} \int_{-1}^1 G \left[e^{-\varepsilon\rho} (\tilde{w}'_k(\rho) + \varepsilon \tilde{w}_k(\rho)) + \right. \\ &\left. + \varepsilon \alpha_k (\tilde{u}_k(\rho) - \tilde{v}_k(\rho)) \right] e^{2\varepsilon\rho} d\rho, \\ M_b &= \pi \varepsilon \left[\frac{2G_0(g_0 + t_0)}{g_0} D + O(\varepsilon) \right] + \\ &+ \pi \sum_{k=1}^{\infty} C_k^* \xi e^{\alpha_k \xi} \int_{-1}^1 G \left[e^{-\varepsilon\rho} (\tilde{w}'_k(\rho) + \varepsilon \tilde{w}_k(\rho)) + \right. \\ &\left. + \varepsilon \alpha_k (\tilde{u}_k(\rho) - \tilde{v}_k(\rho)) \right] e^{2\varepsilon\rho} d\rho - \\ &- \pi \sum_{k=1}^{\infty} C_k^* e^{\alpha_k \xi} \int_{-1}^1 \left[H \alpha_k \varepsilon \tilde{w}_k(\rho) + \right. \\ &\left. + \lambda e^{-\varepsilon\rho} (\tilde{u}'_k(\rho) + \varepsilon (\tilde{u}_k(\rho) + \tilde{v}_k(\rho))) \right] e^{3\varepsilon\rho} d\rho. \end{aligned} \tag{44}$$

Let’s subtract from the first equation of system (26) the second equation of this system

$$\begin{aligned} &\left[H e^{-\varepsilon\rho} \tilde{u}'_k(\rho) + \lambda \varepsilon (e^{-\varepsilon\rho} (\tilde{v}_k(\rho) + \tilde{u}_k(\rho)) + \alpha_k \tilde{w}_k(\rho)) \right]' + \\ &+ \varepsilon G \left[e^{-\varepsilon\rho} (\tilde{v}'_k(\rho) - 3\varepsilon (\tilde{u}_k(\rho) + \tilde{v}_k(\rho)) + 2\tilde{u}'_k(\rho)) + \right. \\ &\left. + \alpha_k \tilde{w}'_k(\rho) + \varepsilon \alpha_k^2 e^{\varepsilon\rho} \tilde{u}_k(\rho) \right] - \\ &- \left[G e^{-\varepsilon\rho} (\tilde{v}'_k(\rho) - \varepsilon (\tilde{u}_k(\rho) + \tilde{v}_k(\rho))) \right]' + \\ &+ e^{-\varepsilon\rho} \left[\varepsilon^2 (H + 2G) \tilde{u}_k(\rho) + \lambda \varepsilon \tilde{u}'_k(\rho) - \right. \\ &\left. - 2\varepsilon G \tilde{v}'_k(\rho) + \varepsilon^2 (H + 2G) \tilde{v}_k(\rho) \right] - \\ &- \varepsilon^2 \alpha_k^2 G e^{\varepsilon\rho} \tilde{v}_k(\rho) + \varepsilon^2 (G + \lambda) \alpha_k \tilde{w}_k(\rho) = 0. \end{aligned} \tag{46}$$

Multiplying both sides of (46) by $e^{\varepsilon\rho}$ and integrating within the limits $[-1; 1]$, let’s obtain:

$$\begin{aligned} &\int_{-1}^1 \left[H e^{-\varepsilon\rho} \tilde{u}'_k(\rho) + \right. \\ &\left. + \lambda \varepsilon (e^{-\varepsilon\rho} (\tilde{v}_k(\rho) + \tilde{u}_k(\rho)) + \alpha_k \tilde{w}_k(\rho)) \right] e^{\varepsilon\rho} d\rho - \\ &- \int_{-1}^1 \left[G e^{-\varepsilon\rho} (\tilde{v}'_k(\rho) - \varepsilon (\tilde{u}_k(\rho) + \tilde{v}_k(\rho))) \right] e^{\varepsilon\rho} d\rho + \\ &+ \int_{-1}^1 \left[G \varepsilon \tilde{v}'_k(\rho) - 3\varepsilon^2 G (\tilde{u}_k(\rho) + \tilde{v}_k(\rho)) + \right. \\ &\left. + \alpha_k \varepsilon G e^{\varepsilon\rho} \tilde{w}'_k(\rho) + \right. \\ &\left. + \alpha_k^2 \varepsilon^2 e^{2\varepsilon\rho} G (\tilde{u}_k(\rho) - \tilde{v}_k(\rho)) + \right. \\ &\left. + \varepsilon^2 (H + 2G) (\tilde{u}_k(\rho) + \tilde{v}_k(\rho)) + \varepsilon H \tilde{u}'_k(\rho) - \right. \\ &\left. - 2\varepsilon G \tilde{v}'_k(\rho) + \varepsilon^2 (G + \lambda) \alpha_k \tilde{w}_k(\rho) e^{\varepsilon\rho} \right] d\rho = 0. \end{aligned} \tag{47}$$

By integrating by parts using boundary conditions (27), it follows from (47)

$$\int_{-1}^1 G \left[e^{-\varepsilon\rho} (\tilde{w}'_k(\rho) + \varepsilon \tilde{w}_k(\rho)) + \varepsilon \alpha_k (\tilde{u}_k(\rho) - \tilde{v}_k(\rho)) \right] e^{2\varepsilon\rho} d\rho = 0. \tag{48}$$

Let’s multiply both sides of the third equation of system (26) by $e^{2\varepsilon\rho}$ and, integrating within the limits $[-1; 1]$, let’s obtain:

$$\begin{aligned} &\int_{-1}^1 G \left[(e^{-\varepsilon\rho} \tilde{w}'_k(\rho) + \varepsilon \alpha_k \tilde{u}_k(\rho)) \right]' e^{2\varepsilon\rho} d\rho + \\ &+ \int_{-1}^1 \left[\varepsilon^2 (G + \lambda) \alpha_k (\tilde{u}_k(\rho) + \tilde{v}_k(\rho)) + \right. \\ &\left. + \varepsilon G e^{-\varepsilon\rho} (\tilde{w}'_k(\rho) - \varepsilon \tilde{w}_k(\rho)) + \right. \\ &\left. + \varepsilon^2 H e^{\varepsilon\rho} \alpha_k^2 \tilde{w}_k(\rho) + \lambda \varepsilon \alpha_k \tilde{u}'_k(\rho) \right] e^{2\varepsilon\rho} d\rho = 0. \end{aligned} \tag{49}$$

By integrating by parts using boundary condition (27) from (49):

$$\begin{aligned} &\int_{-1}^1 \left[H \varepsilon \alpha_k \tilde{w}_k(\rho) + \right. \\ &\left. + \lambda e^{-\varepsilon\rho} (\tilde{u}'_k(\rho) + \varepsilon (\tilde{u}_k(\rho) + \tilde{v}_k(\rho))) \right] e^{3\varepsilon\rho} d\rho = \\ &= \int_{-1}^1 G \left[e^{-\varepsilon\rho} (\tilde{w}'_k(\rho) + \varepsilon \tilde{w}_k(\rho)) + \right. \\ &\left. + \varepsilon \alpha_k (\tilde{u}_k(\rho) - \tilde{v}_k(\rho)) \right] e^{2\varepsilon\rho} d\rho. \end{aligned} \tag{50}$$

Taking into account (48) from (50) let’s obtain:

$$\int_{-1}^1 \left[H \alpha_k \varepsilon \tilde{w}_k(\rho) + \lambda e^{-\varepsilon\rho} (\tilde{u}'_k(\rho) + \varepsilon (\tilde{u}_k(\rho) + \tilde{v}_k(\rho))) \right] e^{3\varepsilon\rho} d\rho = 0. \tag{51}$$

(44), (45) taking into account (48) and (51) take the form:

$$X = -\pi\varepsilon \left(\frac{2G_0(t_0 + g_0)}{g_0} C + O(\varepsilon) \right), \quad (52)$$

$$M_b = \pi\varepsilon \left(\frac{2G_0(t_0 + g_0)}{g_0} D + O(\varepsilon) \right). \quad (53)$$

Let's expand the shear force X and the bending moment M_b in series in ε

$$X = \varepsilon X_0 + \varepsilon^2 X_1 + \dots \quad M_b = \varepsilon M_0 + \varepsilon^2 M_1 + \dots \quad (54)$$

Substituting (54) into (52), (53) for C and D :

$$C = -\frac{g_0}{2\pi G_0(t_0 + g_0)} X_0, \quad (55)$$

$$D = \frac{g_0}{2\pi G_0(t_0 + g_0)} M_0. \quad (56)$$

Thus, the constants C and D are determined through the principal parts of the shear force X and the bending moment M_b .

Solution (28) corresponding to the first asymptotic process together with (31) determines the internal stress-strain state of the cylinder. Solution (31) defines a simple edge effect in a radial inhomogeneous cylinder. In the first terms of the expansion in the parameter ε , solution (28) together with (31) can be considered as a solution according to the applied Kirchhoff-Love theory. The stress state corresponding to solution (36) has the character of a boundary layer and is localized at the ends of the cylinder. The first terms of its asymptotic expansion are equivalent to the Saint-Venant's edge effect of an inhomogeneous plate [26].

Thus, the analysis of the solutions shows that the stress state of a radial inhomogeneous cylinder consists of three types: an internal stress state, a simple edge effect, and a boundary layer.

Let's consider the question of the fulfillment of boundary conditions (24) at the ends of the cylinder. Let's seek the solution in the form (42). To determine the coefficients C_k^* ($k=1,2,\dots$), let's use the Lagrange variational principle. The variational principle takes the following form [29, 30]:

$$\sum_{s=1}^2 \int_{-1}^1 \left[\begin{aligned} & (\sigma_{\rho\xi}^* - f_{1s}) \delta u_\rho^* + \\ & (\sigma_{\varphi\xi}^* - f_{2s}) \delta u_\varphi^* + \\ & (\sigma_{\xi\xi}^* - f_{3s}) \delta u_\xi^* \end{aligned} \right] e^{2\varepsilon\rho} d\rho = 0. \quad (57)$$

Here, asterisks denote the factors at $\cos\varphi, \sin\varphi$ in the corresponding expressions for stresses and displacements. After substituting (42) into (57), assuming δC_k^* independent variations, let's obtain an infinite system of linear algebraic equations with respect to C_k^* :

$$\sum_{k=1}^{\infty} F_{jk} C_k^* = d_{0j}; \quad (j=1,2,\dots). \quad (58)$$

Here

$$F_{jk} = \int_{-1}^1 \begin{bmatrix} E_{k1}(\rho) \tilde{u}_j(\rho) + \\ + E_{k2}(\rho) \tilde{v}_j(\rho) + \\ -1 \\ + E_{k3}(\rho) \tilde{w}_j(\rho) \end{bmatrix} e^{2\varepsilon\rho} d\rho \times \\ \times \left(\exp((\alpha_k + \alpha_j)l) + \exp(-(\alpha_k + \alpha_j)l) \right),$$

$$d_{0j} = \int_{-1}^1 \begin{bmatrix} f_{11}(\rho) \tilde{u}_j(\rho) + \\ + f_{21}^*(\rho) \tilde{v}_j(\rho) + \\ -1 \\ + f_{31}^*(\rho) \tilde{w}_j(\rho) \end{bmatrix} e^{2\varepsilon\rho} d\rho \exp(\alpha_j l) +$$

$$+ \int_{-1}^1 \begin{bmatrix} f_{12}(\rho) \tilde{u}_j(\rho) + \\ + f_{22}^*(\rho) \tilde{v}_j(\rho) + \\ -1 \\ + f_{32}^*(\rho) \tilde{w}_j(\rho) \end{bmatrix} e^{2\varepsilon\rho} d\rho \exp(-\alpha_j l),$$

$$f_{2s}^*(\rho) = f_{2s}(\rho) - \frac{2G_0(g_0 + t_0)}{g_0} C + O(\varepsilon),$$

$$f_{3s}^*(\rho) = f_{3s}(\rho) - (D + (1)^{s+1} C) \times \\ \times \left(\frac{2G\lambda}{H} \frac{t_0}{g_0} - \frac{4G(G+\lambda)}{H} \right) + O(\varepsilon),$$

Using the smallness of the parameter ε , let's construct an asymptotic solution to system (58). Let's clarify the assumptions regarding the external load, taking into account that

$$\sigma_{\xi\xi}^{(2)} = O(1), \quad \sigma_{\rho\xi}^{(2)} = O\left(\varepsilon^{\frac{1}{2}}\right), \quad \sigma_{\varphi\xi}^{(2)} = O\left(\varepsilon^{\frac{1}{2}}\right).$$

The tangential stresses given at the ends of the cylinder are decomposed as follows:

$$f_{1s} = f_{1s}^{(1)} + f_{1s}^{(2)}, \quad f_{2s} = f_{2s}^{(1)} + f_{2s}^{(2)},$$

where

$$f_{1s}^{(1)} = \frac{1}{2} \int_{-1}^1 f_{1s} d\rho, \quad f_{1s}^{(2)} = f_{1s} - f_{1s}^{(1)},$$

$$f_{2s}^{(1)} = \frac{1}{2} \int_{-1}^1 f_{2s} d\rho, \quad f_{2s}^{(2)} = f_{2s} - f_{2s}^{(1)}, \quad (s=1,2).$$

Let's note that

$$\int_{-1}^1 \sigma_{\rho\xi}^* d\rho = \\ = \varepsilon^{\frac{1}{2}} \sum_{j=1}^4 T_j \alpha_{0j} \left\{ \frac{(g_1^2 - g_0 g_2) \alpha_{0j}^2 + g_0 t_1 - g t_0}{g_0} + \right\} \times \\ \times \exp\left(\frac{\alpha_{0j} \xi}{\sqrt{\varepsilon}}\right), \\ \int_{-1}^1 \sigma_{\varphi\xi}^* d\rho = \\ = \varepsilon^{\frac{1}{2}} \sum_{j=1}^4 T_j \left\{ \begin{aligned} & -2g_1 G \alpha_{0j}^2 + 2G_1 \alpha_{0j} + \\ & + 2G_0(g_0 + t_0) + O\left(\varepsilon^{\frac{1}{2}}\right) \end{aligned} \right\} \exp\left(\frac{\alpha_{0j} \xi}{\sqrt{\varepsilon}}\right). \quad (59)$$

Based on (59), it is assumed that $f_{1s}^{(1)} = O\left(\varepsilon^{\frac{1}{2}}\right)$, $f_{2s}^{(1)} = O\left(\varepsilon^{\frac{1}{2}}\right)$.

Then the quantities $f_{1s}^{(2)}$, $f_{2s}^{(2)}$ can be of the same order as $f_{3s}(\rho)$:

$$f_{1s}^{(2)} = O(1), \quad f_{2s}^{(2)} = O(1), \quad (s=1,2). \quad (60)$$

Arbitrary constants $T_j, F_k (j=1,4, k=1,2,...)$ will be sought in the form:

$$T_j = T_{j0} + \epsilon T_{j1} + \dots, F_k = F_{k0} + \epsilon F_{k1} + \dots \quad (61)$$

After substituting (61) into (58), taking into account (60), let's obtain the following systems of linear algebraic equations:

$$\sum_{j=1}^4 q_{kj} T_{j0} = \gamma_k \quad (k = \overline{1;4}), \quad (62)$$

$$\sum_{k=1}^{\infty} t_{nk} F_{k0} = d_n \quad (n = 1,2,...). \quad (63)$$

Here

$$q_{kj} = \left[\begin{array}{c} \alpha_{0j}^3 (g_1 - g_2) + \alpha_{0j} (t_1 - t_0) + \\ \frac{\alpha_{0j} (\alpha_{0j}^2 g_1 - t_0) (g_1 - g_0)}{g_0} + \\ \frac{\alpha_{0k} ((g_2 g_0 - g_1^2) \alpha_{0j}^2 + g_1 t_0 - t_1 g_0)}{g_0} \end{array} \right] \times \\ \times \left(\exp \left(\frac{(\alpha_{0k} + \alpha_{0j}) l}{\sqrt{\epsilon}} \right) + \exp \left(\frac{-(\alpha_{0k} + \alpha_{0j}) l}{\sqrt{\epsilon}} \right) \right),$$

$$\gamma_k = \int_{-1}^1 \left\{ \begin{array}{l} \left[\tau_1(\rho) + \left(\frac{(\alpha_{0k}^2 g_1 - t_0)}{\alpha_{0k} g_0} - \alpha_{0k} \rho \right) f_{31}(\rho) \right] \exp \left(\frac{\alpha_{0k} l}{\sqrt{\epsilon}} \right) + \\ \left[\tau_2(\rho) + \left(\frac{(\alpha_{0k}^2 g_1 - t_0)}{\alpha_{0k} g_0} - \alpha_{0k} \rho \right) f_{32}(\rho) \right] \exp \left(\frac{-\alpha_{0k} l}{\sqrt{\epsilon}} \right) \end{array} \right\} d\rho,$$

$$\tau_s = \lim_{\epsilon \rightarrow 0} \frac{f_{1s}^{(1)}}{\sqrt{\epsilon}},$$

$$t_{nk} = \int_{-1}^1 \left\{ \begin{array}{l} \left[\Psi'_k \left[\begin{array}{c} \alpha_{0n}^{-3} (p_0 \Psi''_n)' + \\ + 2 p_1 \alpha_{0n}^{-1} \Psi'_n - \alpha_{0n}^{-1} (p_2 \Psi_n)' \end{array} \right] + \right. \\ \left. + \alpha_{0k} G \tilde{v}_{30}^{(k)} \tilde{v}_{30}^{(n)} + \alpha_{0k}^{-1} \Psi''_k (p_0 \alpha_{0n}^{-2} \Psi''_n - 2 p_2 \Psi_n) \right] \end{array} \right\} d\rho \times \\ \times \left[\exp \left(\frac{(\alpha_{0k} + \alpha_{0n}) l}{\epsilon} \right) + \exp \left(\frac{-(\alpha_{0k} + \alpha_{0n}) l}{\epsilon} \right) \right],$$

$$d_n = \int_{-1}^1 \left\langle \left[\begin{array}{c} f_{11}(\rho) \times \\ \left[\begin{array}{c} -\alpha_{0n}^{-3} (p_0 \Psi''_n)' - \\ - 2 p_1 \alpha_{0n}^{-1} \Psi'_n + \\ + \alpha_{0n}^{-1} (p_2 \Psi_n)' \end{array} \right] \times \\ + f_{21}^{(0)} \tilde{v}_{30}^{(n)} + f_{31}^{(0)}(\rho) \times \\ \times (\alpha_{0n}^{-2} p_0 \Psi''_n - p_2 \Psi_n) \end{array} \right] \exp \left(\frac{\alpha_{0n} l}{\epsilon} \right) + \left[\begin{array}{c} f_{12}(\rho) \times \\ \left[\begin{array}{c} -\alpha_{0n}^{-3} (p_0 \Psi''_n)' - \\ - 2 p_1 \alpha_{0n}^{-1} \Psi'_n + \\ + \alpha_{0n}^{-1} (p_2 \Psi_n)' \end{array} \right] \times \\ + f_{22}^{(0)}(\rho) \tilde{v}_{30}^{(n)} + f_{32}^{(0)}(\rho) \times \\ \times (\alpha_{0n}^{-2} p_0 \Psi''_n - p_2 \Psi_n) \end{array} \right] \exp \left(\frac{-\alpha_{0n} l}{\epsilon} \right) \right] \right\rangle d\rho,$$

$$f_{2s}^{(0)} = \lim_{\epsilon \rightarrow 0} f_{2s}^*(\rho), \quad f_{3s}^{(0)} = \lim_{\epsilon \rightarrow 0} f_{3s}^*(\rho).$$

The definition of the constants $T_{jt}, F_{kt} (t=1, 2, 3...)$ invariably reduces to systems whose matrices coincide with the matrices of systems (62), (63).

The system of infinite linear algebraic equations (63) is always solvable under physically meaningful conditions imposed on the right-hand side [26, 29, 30]. The decidability and convergence of the reduction method for (63) is proved in [26].

6. Discussion of the results of studying the behavior of the solution to the problem of elasticity theory for a radial inhomogeneous cylinder of small thickness

An asymptotic theory of a radial inhomogeneous cylinder of small thickness is constructed. Assuming that the elastic moduli are arbitrary positive continuous functions along the radius, the values of which vary within the same order, boundary value problems (3), (5) are formulated.

An algorithm for constructing particular solutions of equilibrium equations (3) for special types of loads (6) is presented. Having a set of such solutions, it is possible to construct solutions of the equilibrium equations for arbitrary boundary conditions specified on the lateral surface by smooth functions, having previously approximated them by polynomials. In the case when a smooth load is specified on the lateral surface of the cylinder, using the first iterative process of the asymptotic integration method, particular solutions (21), (22) of equilibrium equation (3), i.e., inhomogeneous solutions, are constructed.

Using the method of asymptotic integration of the equations of the theory of elasticity, homogeneous solutions are constructed, i.e. any solutions of equilibrium equations (3) satisfying homogeneous boundary conditions (23). It is shown that the first iterative process corresponds to a certain penetrating solution (28). The stress state determined by the penetrating solution (28) is equivalent to the shear force X and the bending moment M_b relative to the OY axis in the $z=0$ plane. The second iterative process determines the solution (31) of the edge effect type, which is similar to the edge effect in the applied theory of shells. Solutions (36) corresponding to the third iterative process have the character of a boundary layer. These solutions are localized at the ends of the cylinder, and they decrease exponentially with distance from the ends. Boundary layer solutions are absent in applied shell theories. The first terms of its asymptotic expansion are equivalent to the Saint-Venant's edge effect in the theory of inhomogeneous plates. Penetrating solution (28) and solutions (31), which have the character of the edge effect, determine the internal stress-strain state of a radial inhomogeneous cylinder. It is

found that homogeneous solutions are composed of three types: penetrating solutions, solutions such as a simple edge effect, and boundary layer solutions.

It should be noted that the method of asymptotic integration of the equations of the theory of elasticity does not allow to correctly solve the problem of the theory of elasticity for thick radial inhomogeneous cylinders. The division of the stress-strain state into internal and boundary layer solutions

is valid only for a cylinder of small thickness. The obtained asymptotic formulas (28), (31), (36), which are typical for

thin cylinders, are not suitable for radial inhomogeneous thick cylinders.

When the values of the elastic modulus for a radial inhomogeneous cylinder do not change within the same order of magnitude, but strongly differ from each other, then a weak boundary layer appears. Then the processes of determining the penetrating solution and the weak boundary layer are not separated. In such cases, the asymptotic integration method is not effective for solving the problem of elasticity theory.

7. Conclusions

1. In the case when a smooth load is specified on the lateral surface of a radial inhomogeneous cylinder having order $O(1)$ with respect to ε , the inhomogeneous solutions are constructed by an asymptotic method. In particular, when the lateral surface of the cylinder is loaded by forces polynomially depending on the axial coordinate, the construction of inhomogeneous solutions is reduced to solving a recurrent system of boundary value problems.

2. Using the method of separation of variables, the formulated boundary value problem is reduced to a spectral problem. As $\varepsilon \rightarrow 0$, three groups of asymptotic solutions of the spectral problem are obtained by the method of asymptotic integration based on three iterative processes. The overall solution will be the sum of the solutions of the respective three iterative processes.

3. On the basis of asymptotic analysis, the features of the stress-strain state in a radial inhomogeneous cylinder are revealed. The solutions corresponding to the first iteration process are penetrating solutions. Penetrating solutions are determined through the shear force X and the bending moment M_b relative to the OY axis in the $z=0$ plane. The stress state corresponding to the second iteration process represents edge effects in the applied theory of shells. The third iterative process determines solutions that have the character of a boundary layer and are localized at the ends of the cylinder. It is found that the penetrating solution and solutions having the character of the edge effect determine the internal stress-strain state of a radial inhomogeneous cylinder.

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