The problem of optimal control over a finite time interval for a mathematical model of a three-sector economic cluster is posed. The economic system is reduced by means of transformations to the optimal control problem for one class of nonlinear systems with coefficients depending on the state of the control object. Two optimal control problems for one class of nonlinear systems with and without control constraints are considered. The nonlinear objective functional in these problems depends on the control and state of the object. Then, using the results of solving optimal control problems on a finite interval, an algorithm for solving the problem for a nonlinear system of a three-sector economic cluster is developed. A nonlinear control based on the feedback principle using Lagrange multipliers of a special kind is found. The results obtained for nonlinear systems are used to construct the control parameters of a mathematical model of a three-sector economic cluster at a finite time interval with a given functional and various initial conditions. The results of the system state calculation are shown in the figures, the optimal controls satisfy the given constraints. The optimal distribution of labor and investment resources for a three-sector economic cluster is determined. They ensure that the system is brought into an equilibrium state and satisfy balance ratios. These results are useful for practice and are important because there are a number of optimal control problems when it is necessary to transfer a system from an initial state to a desired final state for a given time interval. Such problems often arise for an economic system when a certain level of development is required

Keywords: optimal control problem, three-sector economic cluster, Lagrange multiplier method, nonlinear system, quadratic functional

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DEVELOPMENT OF AN ALGORITHM FOR SOLVING THE PROBLEM OF OPTIMAL CONTROL ON A FINITE INTERVAL FOR A NONLINEAR SYSTEM OF A THREE-SECTOR ECONOMIC CLUSTER

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1. Introduction

The researches proposed in this paper belong to one of the promising and rapidly developing areas of mathematical control theory in recent years. The relevance of such research is reasoned by the fact that control tasks are encountered in almost all areas of human activity: these are complex technical systems and technological processes. In these systems, there are issues of achieving the goal by selecting the optimum control actions, taking into account various constraints (system trajectory requirements, control constraints).

The problem of optimal control for dynamic systems can be formulated as the problem of finding a program control or constructing a synthesizing control depending on the system state and the current time moment. In the first case, the problem can be solved using the Pontryagin maximum principle [1], which reduces the initial problem of optimal control to a two-point boundary value problem of ordinary differential equations. In the general case, the Pontryagin maximum principle gives necessary optimality conditions and allows us to obtain the program control depending on the current moment of time. In the second case, the Bellman dynamic programming method [2] or Krotov sufficient optimality conditions [3] can be used. It should be noted that it is difficult to apply these methods directly to obtain the optimal control law.

In practice, we meet a large number of optimal control problems for economic systems, which are non-linear systems with coefficients that depend on the state of the control object. Economic systems are required to achieve a certain level of economic development over a given planning horizon.

A cluster (in economics) is a group of interconnected organizations, infrastructures, producers, suppliers, research institutes, universities, etc., concentrated on a certain territory, complementing each other and reinforcing competitive advantages of individual companies and the cluster as a whole. Aspects of structure and functioning of clusters were first studied elaborately in [4]. The three-sector economic cluster model and the necessary conditions for optimal balanced economic growth were given by [5, 6]. The extensive work of [7] provided a basis of the mathematical theory of the infinite-horizon optimal control of dynamic systems using the Pontryagin maximum principle and an example of a two-sector model of optimal economic growth with an occasional jump in prices.

It should be noted that the controllability criteria for nonlinear systems were obtained in [8], and for discrete systems in [9]. The work [10] considered the optimization of discrete processes with bounded control.

One of the methods for designing non-linear controllers is the method based on using the Riccati equation, whose parameters depend on the state of the object. At the same time, an inefficient algorithm in terms of computational volume, which requires multiple integration of matrix differential Riccati equations with state-dependent coefficients, is proposed. It should be noticed that the ambiguity of representation of non-linear system like a linear system structure, but with state-dependent parameters, also the absence of sufficiently universal algorithms of solving the Riccati equation, whose parameters as well depend on state rise many possible sub-optimal solutions.

Therefore, researches devoted to developing an algorithm for solving the problem of optimal control of a non-linear system to determine a non-linear control based on the feedback principle, allowing to find synthesizing controls with control constraints, which depend on the system state and the current time are relevant.

2. Literature review and problem statement

In [11], the optimal sampling times of the compensator as a function of different sampling output frequencies are discussed and comparison results for different quality indicators are given. The proposed algorithm is general and will require further analysis and investigation to build control of different systems. And in [12], a class of nonlinear systems, for which there exists a coordinate representation (diffeomorphism) that translates the initial system into a system with a linear fundamental part and nonlinear feedback, is considered. The matrix of the controller gain is found by solving the matrix equations of Riccati type with parameters that depend on the state of the system. Note that it is not possible to solve the Riccati equation in the general case. It is necessary to approximate the solution and then it is possible to obtain a suboptimal control.

Such a system is also studied for one class of nonlinear uncertain systems and based on adaptive dynamic programming (ADP) methods based on neuro-observables have been developed for the optimal control problem with continuous-time [13] and neural network approximation design of optimal control with guaranteed costs [14]. In these works, the optimal control problem of a nonlinear system is transformed to the optimal control problem of a nominal system by changing the cost function. The main result is to develop an adaptive learning algorithm for solving the optimal control problem of uncertain nonlinear systems. In practical applications, there are a variety of uncertainties in nonlinear systems in the presence of constraints on the control actions. However, in this work, only a single network is used instead of the typical dual network, which provides a smaller computational burden, but initial stabilizing control requirements may arise.

In [15], an ADP-based control algorithm has been proposed to solve the problem of optimal control of nonlinear systems with saturating actuators and non-quadratic cost functionals. In these works, the solutions of the Hamilton-Jacobi-Bellman equation are reduced to the solution of matrix Riccati type equations with parameters depending on the system state. New algorithms for solving the problem of optimal control of nonlinear systems have been developed in the framework of ADP in [16]. The authors proposed an adaptive algorithm for nonlinear systems with bounded inputs and the influence of possible uncertainties has not been taken into account.

In [17], an adaptive neural decentralized control method is proposed for a class of uncertain stochastic nonlinear strongly coupled systems with multiple inputs and multiple outputs. The main feature of this work is that the proposed approach allows one to control stochastic systems with strongly interconnected nonlinearities in both drift and diffusion, which are functions of all states of the whole system. And in [18], the problem of adaptive neural control of triangular nonlinear systems with unmodelled dynamics and dynamic perturbations is considered. The main advantage of this study is that the neural network-based tracking method is developed for uncertain nonlinear systems. In [19], optimal control of power flows in a microgrid network (MGs) based on the Pontryagin Maximum Principle (PMP) is presented. In [20], the focus is on the design of near-optimal controllers based on adaptive criticism for nonlinear decentralized feedback problems with continuous time. Note that in these works, the requirement for an initial stabilizing control is a rather serious limiting condition, and especially for uncertain nonlinear systems. Stabilizing control is difficult to obtain in practice in the presence of constraints on the control actions. In [15–20], solving Hamilton-Jacobi-Bellman equation is reduced to solving matrix equation similar to Riccati's state-dependent parameters. At the same time, an inefficient algorithm in terms of computational volume, which requires multiple integration of matrix differential Riccati equations with state-dependent coefficients, is proposed. However, it should be noticed that the ambiguity of representation of non-linear system like a linear system structure, but with state-dependent parameters, also the absence of sufficiently universal algorithms of solving the Riccati equation, whose parameters as well depend on state rise many possible sub-optimal solutions.

We consider the optimal control problem for a three-sector economy model consisting of the material sector (i=0), the capital generating sector (i=1) and the consumer sector (i=2). It is assumed that each sector produces its aggregate product: the material sector produces objects of labor (fuel, electricity, raw materials and other materials); the capital-generating sector produces means of labor (machines, equipment, industrial buildings, etc.); the consumer sector produces consumer goods.

In accordance with [5], the mathematical model consists of:

1. Three Cobb-Douglas type functions of the product specific output:

$$x_i = \theta_i A_i k_i^{\alpha_i},\ \ A_i > 0,\ \ 0 < \alpha_i < 1,\ \ (i = 0,1,2). \tag{1}$$

2. Three differential equations describing the dynamics of the capital-labor ratio:

$$\dot{k}_i = -\lambda_i k_i + \frac{s_i}{\theta_i} x_1,\ \ k_i(0) = k_{i0},\ \ \lambda_i > 0,\ \ (i = 0,1,2). \tag{2}$$

3. Three balance equations:

$$s_0+s_1+s_2=1,\ \ s_i\geq 0,\ \ (i=0,1,2). \tag{3}$$

$$\theta_0+\theta_1+\theta_2=1,\ \theta_i\geq 0,\ \ (i=0,1,2). \tag{4}$$

$$(1+\beta_0)x_0=\beta_1 x_1+\beta_2 x_2,\ \ \beta_i\geq 0,\ \ (i=0,1,2). \tag{5}$$

Here the state of the system (capital-labor ratio) is described by a vector k_0, k_1, k_2; (s_0, s_1, s_2, θ_0, θ_1, θ_2) is a control vector, where $s0$, $s1$, $s2$ are shares of sectors in an investment resources distribution and θ_0, θ_1, θ_2 are shares of sectors in a workforce distribution; x_i is products specific outputs (the number of products in the i-th sector per worker); α_i is the coefficient of elasticity of funds; λ_i is the capital-labor ratio; β_i is direct material costs of manufacturing products in the i-th mentioned sector of a cluster (i=0,1,2).

We will consider the problem of transition of a nonlinear system from an initial state (k_{00}, k_{10}, k_{20}) of capital-labor ratio to the state (k_{0s}, k_{1s}, k_{2s}) over the time interval $[0, T]$. As a desired final state (k_{0s}, k_{1s}, k_{2s}) we choose a steady state of the system, which is determined by equating the right sides of differential equations (2) to zero:

$$k_{0s}=\frac{s_0\theta_1 A_1 k_{1s}^{\alpha_1}}{\lambda_0\theta_0},\ \ k_{1s}=\left(\frac{s_1 A_1}{\lambda_1}\right)^{\frac{1}{1-\alpha_1}},\ \ k_{2s}=\frac{s_2\theta_1 A_1 k_{1s}^{\alpha_1}}{\lambda_2\theta_2}. \tag{6}$$

The capital-labor ratio values of the capital-labor ratio k_{is} (i=0,1,2) in the equilibrium state (6) depend on the controls (s_0, s_1, s_2, θ_0, θ_1, θ_2), for which we can select the values (s_{0s}, s_{1s}, s_{2s}, θ_{0s}, θ_{1s}, θ_{2s}), the result of solving the nonlinear programming problem in order to maximize the specific consumption: $x_2\to$max. The system of differential equations (2) in vector form is:

$$\dot{y}(t)=Ay(t)+Bf(y(t))u(t),\ \ \ t\in[0,T], \tag{7}$$

where

$$y_1=k_1,\ \ y_2=k_2,\ \ y_3=k_0,$$

$$u_1=s_1,\ \ u_2=\frac{s_2\theta_1}{\theta_2},\ \ u_3=\frac{s_0\theta_1}{\theta_0},$$

$$f(y)=f_1(y_1)=k_1^{\alpha_1},\ \ f_2(y_2)=k_2^{\alpha_2},\ \ f_3(y_3)=k_0^{\alpha_0},$$

$$A=\begin{pmatrix}-\lambda_1 & 0 & 0\\ 0 & -\lambda_2 & 0\\ 0 & 0 & -\lambda_3\end{pmatrix},\ B=\begin{pmatrix}A_1 & 0 & 0\\ 0 & A_1 & 0\\ 0 & 0 & A_1\end{pmatrix}.$$

Here $y=(y_1, y_2, y_3)^*$ is a vector of the object state, $u=(u_1, u_2, u_3)^*$ is the control vector. The initial and final states of the system are given:

$$y(0)=y_0,\ \ y(T)=y_T. \tag{8}$$

Note that the desired final state of the system $y(T)=y_s$ is an equilibrium state in which per capita consumption is maximized and ensures a balanced growth of the economy.

To solve the initial state into an equilibrium state $y(T)=y_s$ in a time interval and minimize the cost functional:

$$J(u)=\frac{1}{2}\int_0^T\left\{\begin{array}{l}[y(t)-y_s]^* Q[y(t)-y_s]+\\ +[f(y)u(t)-f_s u_s]^*\times\\ \times R[f(y)u(t)-f_s u_s]\end{array}\right\}dt, \tag{9}$$

where Q and R are positive semidefinite and positive definite (3×3) matrices, respectively.

Studies of the stability of nonlinear systems in the form (7) were carried out by [21, 22] describing optimal stabilization of these systems over an infinite time interval.

It follows from the analysis of scientific researches that the proposed algorithms for solving the problem of optimal control of non-linear systems lead to the solution of Riccati equation, which requires multiple integration of matrix differential equations with coefficients that depend on the state of the system. An option to overcome this problem is to search for new approaches to develop algorithms to solve the problems of optimal control of non-linear systems and to provide the required constraint in the form of algebraic equation. All this suggests that research on optimal control problems for one class of nonlinear systems under external influences and with constraints on the controlling actions is promising.

3. The aim and objectives of the study

The aim of the study is to develop an algorithm for solving the problem of optimal control on a finite interval for a non-linear system of a three-sector economic cluster. This algorithm will make it possible to apply in practice the obtained scientific results in economic systems in order to achieve a certain level of economic development on a given planning horizon.

To achieve this aim, the following objectives are accomplished:

– to find a synthesizing control $u(t,y)$ such that the pair $\{y(t),u(t)\}$ corresponding to it delivers the minimum value to the functional (9), where $y(t)$ is the solution of the differential equation (7) under the control $u(t)=u(y(t),t)$;

– to find a synthesizing control $u(t,y)$ such that the pair $\{y(t),u(t)\}$ corresponding to it delivers the minimum value to the functional (9), where $y(t)$ is the solution of the differential equation (7) under the control $u(t)=u(y(t),t)$ in the presence of restrictions on controls of the form (3), (4);

– to develop an algorithm for solving the problem of constructing control parameters for a three-sector economic cluster.

4. Materials and methods

This method also allows taking into account the constraints on the values of control. It should be emphasized that there is a more general statement of the optimal control. The problem is considered for a three-sector economic cluster, where the share of labor and investment resources for all three sectors of the economy can be changed simultaneously.

In practice, there are a number of optimal control problems where it is necessary to move a system from an initial state to a desired final state over a specified time interval. Such problems often arise for an economic system, when it is required to achieve a certain level of economic development on a given planning horizon. To solve this problem, we construct an auxiliary functional with Lagrange multipliers of a special kind. For that, we add a system of differential equations (7) with multiplier $\lambda=K(t)(y(t)-y_s)+q(t)$ to the expression for functional (9). As a result, we get the following functional:

$$L(u)=\int_0^T\left\{\begin{matrix}\frac{1}{2}\left[y(t)-y_s\right]^*Q\left[y(t)-y_s\right]+\\+\frac{1}{2}\left[f(y)u(t)-f_su_s\right]^*\times\\\times R\left[f(y)u(t)-f_su_s\right]+\\+\left[K(t)\left(y(t)-y_s\right)+q(t)\right]^*\times\\\times\left[\begin{matrix}A\left(y(t)-y_s\right)+\\+B\left(f(y)u-f_su_s\right)-\dot{y}(t)\end{matrix}\right]\end{matrix}\right\}\mathrm{d}t, \quad (10)$$

where $q(t)$ is a vector of dimension (3×1); $K(t)$ is a symmetric positive definite (3×3) matrix.

Consider the following functions:

$$v(y,t)=\frac{1}{2}\left[y(t)-y_s\right]^*K(t)\left[y(t)-y_s\right]+$$
$$+\left[y(t)-y_s\right]^*q(t),$$

$$\frac{\partial v(y,t)}{\partial y}=K(t)\left[y(t)-y_s\right]+q(t). \quad (11)$$

$$M(y,u,t)=\frac{1}{2}\left[y(t)-y_s\right]^*\left[Q+\dot{K}(t)\right]\left[y(t)-y_s\right]+$$
$$+\frac{1}{2}\left[f(y)u(t)-f_su_s\right]^*R\left[f(y)u(t)-f_su_s\right]+$$
$$+\left[K(t)\left(y(t)-y_s\right)+q(t)\right]^*\times$$
$$\times\left[A\left(y(t)-y_s\right)+B\left(f(y)u-f_su_s\right)\right]+\left(y(t)-y_s\right)\dot{q}(t) \quad (12)$$

Using (11) and (12), we get the following representation for the Lagrange functional (10):

$$L(y,u)=v(y_0,t_0)+\int_0^T M(y,u,t)\mathrm{d}t. \quad (13)$$

For the given task, the principle of liberation from ties is as follows: the original task of optimal control with constraints is reduced to another task, without constraints. At the same time, a new task is formulated so that its solution would be the solution of the original problem [15, 16]. The multiplier $\left(K\left(y(t)-y_s\right)+q(t)\right)$ removes constraints on $\left\{x(t),u(t)\right\}$, in the form of a system of differential equations (7), and functions $\left\{\lambda_1(t),\ \lambda_2(t)\right\}$ – corresponding constraints, imposed on control (3), (4). Such representation of the functional (13) allows reducing the initial problem on a conditional extremum to the problem on an unconditional extremum.

5. Results of the study of optimal control problems and the development of an algorithm for solving the problem for a nonlinear system of a three-sector economic cluster

5. 1. Solution of the optimal control problem (7)–(9)

To find a solution of the problem, it is necessary to determine matrices $K(t)$, $W(t,T)$ and vector $q(t)$ in accordance with the proposed algorithm [23]. Applying common methods of differential calculus, from (12) we determine a control function that minimizes $M(y,u,t)$:

$$f(y)u(t)-f_su_s=-R^{-1}B^*\left[K(t)\left(y(t)-y_s\right)+q(t)\right]. \quad (14)$$

Matrices $K(t)$, $W(t,T)$ and vector $q(t)$ are defined as follows:

$$\dot{K}(t)+K(t)A+A^*K(t)-$$
$$-K(t)BR^{-1}B^*K(t)+Q=0,\quad K(0)=K_0. \quad (15)$$

$$\dot{W}(t,T)=W(t,T)A_1^*(t)+$$
$$+A_1(t)W(t,T)-B_1,\qquad W(T,T)=0. \quad (16)$$

$$\dot{q}(t)=-A_1^*(t)q(t),\quad q(0)=W^{-1}(0,T)\left(y(0)-y_s\right), \quad (17)$$

where

$$A_1(t)=A-BR^{-1}B^*K(t),\quad B_1=BR^{-1}B^*. \quad (18)$$

Assuming that the solutions of the equations (15)–(17) exist and the conditions (14) are satisfied, we can represent the differential equations that determine the law of the system motion, as follows:

$$\dot{y}(t)=A_1(t)\left(y(t)-y_s\right)-B_1q(t),\quad y(0)=y_0. \quad (19)$$

Using the solutions of differential equations (17) and (19), and following [23], we find that the state of system (19) corresponding to the control (14) will satisfy $y(T)=y_s$ at the final time T.

$$y(t)-y_s=W(t,T)q(t),\quad t\in[t_0,T]. \quad (20)$$

Now, the obtained results for the optimal control problem can be formulated as the following assertion:

Theorem 1. The pair of functions in problem (7)–(9) is optimal if and only if:

1. $y(t)$ satisfies the differential equation:

$$\dot{y}(t)=A_1(t)\left(y(t)-y_s\right)-B_1q(t) \quad (21)$$

with the conditions $y(t_0)=y_0$, $y(T)=y_s$.

2. Control $u(t)$ is defined as follows:

$$u(y,t)=f^{-1}(y)\left\{f_su_s-R^{-1}B^*\left[K(t)\left(y(t)-y_s\right)+q(t)\right]\right\}. \quad (22)$$

Matrices $K(t)$ and $W(t,T)$ are the solutions of equations (15) and (16), and vector $q(t)$ satisfies the differential equation (17).

5. 2. Solving the problem with limited control

We write the system of differential equations (7) in vector form using the following notation:

$$\dot{y}(t)=Ay(t)+BD\left(y(t)\right)u(t),\quad y(0)=y_0,\quad t\in[0,T],$$

$$D\left(y(t)\right)=\begin{pmatrix}y_1^{\alpha_1}&0&0\\0&y_1^{\alpha_1}&0\\0&0&y_1^{\alpha_1}\end{pmatrix}. \quad (23)$$

Here, $y(t)=(y_1,y_2,y_3)^*$ is a state vector of the object, $u(t)=(u_1,u_2,u_3)^*$ is a control vector. The components of the control vector $u(t)=(u_1,u_2,u_3)^*$ satisfy two-sided constraints of the following form:

$$\gamma_1 \le u \le \gamma_2,\quad 0<\gamma_{1i} \le u_i \le \gamma_{2i} <1,\quad (i=0,1,2). \tag{24}$$

which are obtained from the initial constraints (3), (4).

In [24], optimal control problems with fixed ends of trajectories for a linearized system of an economic cluster are considered.

It is required to find a synthesizing control $u(y,t)$ that takes the system (23) from a given initial state $y(0)=y_0$ to the desired equilibrium state $y(T)=y_s$ in a time interval $[0,T]$, while minimizing the functional:

$$J(u)=\frac{1}{2}\int_0^T \left\{\begin{array}{l}[y(t)-y_s]^* Q[y(t)-y_s]+\\ +[D(y)u(t)-D_s u_s]^* \times\\ \times R[D(y)u(t)-D_s u_s]\end{array}\right\}\mathrm{d}t+$$
$$+\frac{1}{2}[y(T)-y_s]^* F[y(T)-y_s], \tag{25}$$

where Q is a positively semidefinite (3×3) matrix, and R, F, $D(y(t))$ are positive definite matrices of dimension (3×3), $D_s=D(y_s)$. The symbol (*) means the operation of transposing a matrix or a vector.

To solve the problem, we add to the expression for the functional (25) the system of differential equations (23) with the factor $\lambda=K(t)(y(t)-y_s)+q(t)$, as well as the following expression:

$$\lambda_1^*(t)D^*(y)RD(y)[\gamma_1-u(t)]+\lambda_2^*(t)D^*(y)RD(y)\times$$
$$\times[u(t)-\gamma_2]+\lambda_3^*(t)[y(t)-y_s-W(t,T)q(t)], \tag{26}$$

where $\lambda_1\ge 0$, $\lambda_2\ge 0$. As a result, we obtain the following functional:

$$L(y,u)=\int_0^T \left\{\begin{array}{l}\frac{1}{2}[y(t)-y_s]^* Q[y(t)-y_s]+\\ +\frac{1}{2}[D(y)u(t)-D_s u_s]^* \times\\ \times R[D(y)u(t)-D_s u_s]+\\ +[K(t)(y(t)-y_s)+q(t)]^* \times\\ \times\left[\begin{array}{l}A(y(t)-y_s)+\\ +B(D(y)u(t)-D_s u_s)-\dot{y}(t)\end{array}\right]+\\ +\lambda_1^*(t)D^*(y)RD(y)[\gamma_1-u(t)]+\\ +\lambda_2^*(t)D^*(y)RD(y)[u(t)-\gamma_2]+\\ +\lambda_3^*(t)[y(t)-y_s-W(t,T)q(t)]\end{array}\right\}\mathrm{d}t+$$
$$+\frac{1}{2}[y(T)-y_s]^* F[y(T)-y_s], \tag{27}$$

where $q(t)$ is a vector of dimension $(n\times 1)$, and $K(t)$ is a symmetric positive definite matrix of dimension $(n\times n)$.

For the problem under consideration, the principle of releasing from bonds is as follows: the original problem of optimal control with constraints is reduced to another problem, but without any limitations. We introduce the following functions:

$$v(y,t)=\frac{1}{2}[y(t)-y_s]^* K(t)[y(t)-y_s]+$$
$$+[y(t)-y_s]^* q(t),$$
$$\frac{\partial v(y,t)}{\partial y}=K(t)[y(t)-y_s]+q(t). \tag{28}$$

$$M(y,u,t)=\frac{1}{2}[y(t)-y_s]^*[Q+\dot{K}(t)][y(t)-y_s]+$$
$$+\frac{1}{2}[D(y)u(t)-D_s u_s]^* R[D(y)u(t)-D_s u_s]+$$
$$+\left[\begin{array}{l}K(t)(y(t)-y_s)+\\ +q(t)\end{array}\right]^*\left[\begin{array}{l}A(y(t)-y_s)+\\ +B(D(y)u(t)-D_s u_s)\end{array}\right]+$$
$$+\lambda_1^*(t)D^*(y)RD(y)[\gamma_1-u(t)]+$$
$$+\lambda_2^*(t)D^*(y)RD(y)[u(t)-\gamma_2]+$$
$$+\lambda_3^*(t)[y(t)-y_s-W(t,T)q(t)]+(y(t)-y_s)\dot{q}(t). \tag{29}$$

Then the following representation of the functional holds [24]:

$$L(y,u)=v(y_0,t_0)+\int_0^T M(y,u,t)\mathrm{d}t-$$
$$-v(y(T),T)+\frac{1}{2}[y(T)-y_s]^* F[y(T)-y_s]. \tag{30}$$

The desired control is determined from the relation:

$$D(y)u(t)-D_s u_s=$$
$$=-R^{-1}B^*[K(t)(y(t)-y_s)+q(t)]-D(y)[\lambda_2-\lambda_1], \tag{31}$$

where the matrices $K(t)$, $W(t,T)$ and the vector $q(t)$ satisfy in the interval $tt_0,T]$ the differential equations:

$$\dot{K}(t)+K(t)A+A^*K(t)-$$
$$-K(t)BR^{-1}B^*K(t)+Q=0,\quad K(T)=K_T. \tag{32}$$

$$\dot{W}(t,T)=W(t,T)A_1^*(t)+A_1(t)W(t,T)-B_1,$$
$$W(T,T)=(F-K_T)^{-1}, \tag{33}$$

$$\dot{q}(t)=-A_1^*(t)q(t)+W^{-1}(t,T)BD(y)D(y)\phi(y,t),$$
$$q(0)=(F-K(T))(y(0)-y_s). \tag{34}$$

Here we used the following notation:

$$A_1(t)=A-BR^{-1}B^*K(t),\quad B_1=BR^{-1}B^*,$$
$$\phi(y,t)=\lambda_1(y,t)-\lambda_2(y,t),$$
$$\lambda_1(y,t)=\max\{0,\gamma_1-\omega(y,t)\}\ge 0,$$
$$\lambda_2(y,t)=\max\{0,\omega(y,t)-\gamma_2\}\ge 0, \tag{35}$$

$$\omega(y,t)=D^{-1}(y)D_s u_s-$$
$$-D^{-1}(y)R^{-1}B^*[K(t)(y(t)-y_s)+q(t)]. \tag{36}$$

Let there exist solutions of equations (32), (33), then the differential equations defining the law of motion of the system can be represented in the following form:

$$\dot{y}(t)=A_1(t)(y(t)-y_s)-$$
$$-B_1q(t)+BD(y)\phi(y,t),\quad y(t_0)=y_0. \tag{37}$$

We note that the initial condition for the differential equation (34) is determined from the following relations:

$$y(t)-y_s=W(t,T)q(t),\ \ t\in[t_0,T]. \tag{38}$$

The results established for the problem of optimal control are formulated in the form of the following assertion.

Theorem 2. Let Q be a positively semidefinite matrix, and R, F, $D(y)$ be positive definite matrices in the time interval $t_0 \le t \le T$; the matrix $W_0=W(t_0,T)$ is positive definite. Suppose that system (23) is completely controllable at the instant time t_0. Then for the optimality of the pair $(y(t),u(t))$ in problem (23)–(25), it is necessary and sufficient that:

1. $y(t)$ satisfies the differential equation:

$$\dot{y}(t)=A_1(t)(y(t)-y_s)-$$
$$-B_1q(t)+BD(y)\phi(y,t),\quad y(t_0)=y_0.$$

2. Control $u(y,t)$ is defined as follows:

$$u(y,t)=$$
$$=D^{-1}(y)\begin{pmatrix} D_s u_s - R^{-1}B^*\times \\ \times\left[K(t)(y(t)-y_s)+q(t)\right]\end{pmatrix}+\phi(y,t). \tag{39}$$

Matrices $K(t)$ and $W(t,T)$ are the solutions of equations (32) and (33), vector $q(t)$ satisfies the differential equation (34), and the vector-valued function $\varphi(y,t)$ is determined by formula (35).

5. 3. Algorithm for solving the optimal control problem and constructing control parameters for a three-sector economic cluster

We describe an algorithm for solving the optimal control problem (1)–(5), which can conveniently be implemented with a computer.

Step 1. Integrate the system of differential equations (15) and (16) determine the matrices $K(t)$ and $W(t,T)$ over the interval $[0,T]$ under conditions $K(0)=K_0$ and $W(T,T)=0$.

It should be noted that K_0 is an arbitrary symmetric and positive definite matrix. If we set different initial conditions for the matrix differential equation (15), then we obtain different $K(t)$ and $W(t,T)$ matrices. However, this gives the same vector-function $u(t)$ of the form (22), because the problem has a unique solution. When calculating vector $q(t)$ with formula (17), the influence of matrix $K(t)$ is compensated.

Step 2. Set the conditions $y(0)=y_0$ and calculate $q_0=$ $=W^{-1}(0,T)(y_0-y_s)$.

Step 3. Integrate the system of differential equations (17), (21) in interval $[0,T]$ with the initial conditions $y(0)=y_0$, $q(0)=q_0$. It is possible to output the results and a graph (if needed) of the optimal trajectory $y(t)$ and optimal control $u(t)$ in the process of integration of the system (17), (21).

Step 4. Using the results of step 3, the optimal trajectory $y(t)$ (capacity to labor ratio) and the optimal control $u(t)$, we find $v(t)$ from the following relationship:

$$f_i(y_i)=y_i^{\alpha_i},\ \ (i=0,1,2).$$

$$v=\frac{\beta_1A_1f_1(y_1)+\beta_2A_2f_2(y_2)\dfrac{1-u_1(t)}{u_2(t)}}{(1-\beta_0)A_0f_3(y_3)\dfrac{1-u_1(t)}{u_3(t)}+\beta_2A_2f_2(y_2)\dfrac{1-u_1(t)}{u_2(t)}} \tag{40}$$

ensure the fulfillment of conditions (5); then we determine the optimal distribution of investment resources $(s_0(t),s_1(t),s_2(t))$

$$s_0=v(1-u_1(t)),\ \ s_1=u_1(t),\ \ s_2=(1-v)(1-u_1(t)), \tag{41}$$

ensure the fulfillment of conditions (3); and optimal distribution of labor resources $(\theta_0(t),\theta_1(t),\theta_2(t))$

$$\theta_0=\frac{v(1-s_1)\theta_1}{u_3(t)},\ \ \theta_1=\frac{1}{1+\dfrac{s_0}{u_3(t)}+\dfrac{s_2}{u_2(t)}},$$
$$\theta_2=\frac{(1-v)(1-s_1)\theta_1}{u_2(t)} \tag{42}$$

ensure the fulfillment of conditions (4).

Numerical calculations were carried out on a computer using the Maple application package with the following parameter values indicated in Table 1.

Table 1

Parameter values for a three-sector economic model of a cluster

I	α_i	β_i	λ_i	A_i	s_{is}	θ_{is}	k_{is}
0	0.46	0.39	0.05	6.19	0.2763	0.3944	966.4430
1	0.68	0.29	0.05	1.35	0.4476	0.2562	2410.1455
2	0.49	0.52	0.05	2.71	0.2761	0.3494	1090.1238

Matrices Q and R are determined by formula (25) for two variants. Since matrix $Q(t)$ is stationary, it follows from the differential matrix Riccati equation (32) that matrix $K(t)$ will also be stationary in the time interval $[t_0,T]$ and $K_T=F$.

We use two variants of the initial conditions.

Here we solve the optimal control problem with the chosen initial condition in the form:

$$y(t_0)-y_s=(-200,\ \ -100,\ \ -100)^*. \tag{43}$$

Matrices Q and R were chosen as:

$$Q=\begin{pmatrix}0.25\cdot10^{-4} & 0 & 0\\ 0 & 10^{-4} & 0\\ 0 & 0 & 10^{-4}\end{pmatrix},\ R=\begin{pmatrix}\dfrac{1}{8100} & 0 & 0\\ 0 & \dfrac{1}{1600} & 0\\ 0 & 0 & \dfrac{1}{1250}\end{pmatrix}$$

accordingly, matrix K_T is equal to:

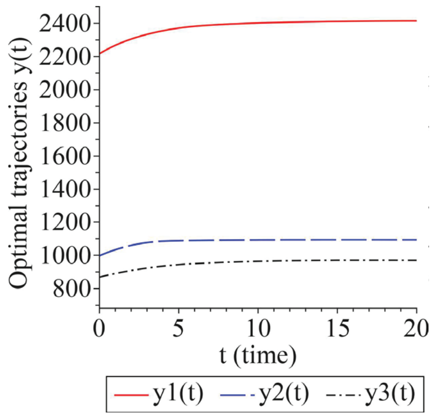
$$K_T=\begin{pmatrix}0.037\cdot10^{-3} & 0 & 0\\ 0 & 0.168\cdot10^{-3} & 0\\ 0 & 0 & 0.188\cdot10^{-3}\end{pmatrix}.$$

It is required to transfer the system (23) into an equilibrium state $y(T)=(k_{0s},k_{1s},k_{2s})$ over the time interval $[t_0,T]=[0,20]$, while minimizing functional (25).

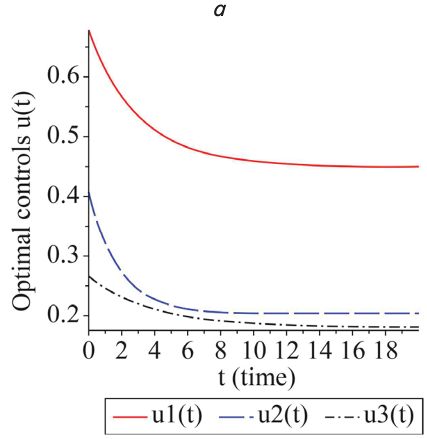
The obtained optimal trajectories and optimal controls are presented in Fig. 1, *a*, *b*. As can be seen from Fig. 1, *a*, the found optimal controls ensure that the trajectories of the system (fund-labor ratio) are brought to the state of equilibrium. The optimal controls found in Fig. 1, *b* do not exceed the limits of region U. Using the formulae (40)–(42), the optimal distribution of labor $(\theta_0(t),\theta_1(t),\theta_2(t))$ and investment resources$(s_0(t),s_1(t),s_2(t))$ is determined. Fig. 2, *a*, *b* show the changes in resources that bring system (2) to a state of equilibrium and satisfy balance ratios (3)–(5). It follows from Fig. 1, *a* that the obtained optimum trajectory values $y_1(T)\to k_{1s}$=2410.15, $y_2(T)\to k_{2s}$=1090.12, $y_3(T)\to k_{0s}$=966.45. Fig. 1, *b* shows that the values of the difference of the fundamentals $y_i(T)$ and the steady-state fundamentals y_{is}, $(y_1(T)-y_{1s})\to 0$; $(y_2(T)-y_{2s})\to 0$; $(y_3(T)-y_{3s})\to 0$ and the optimal control values at the finite time at T=20: $u_1(T)$, $u_2(T)$, $u_3(T)$ satisfy the given constraints and are within the domain. Fig. 2, *a*, *b* shows graphs of optimal allocation of investment and labor resources, with investment resources $s_1(T)\to s_{1s}$=0.4476, $s_2(T)\to s_{2s}$=0.2761, $s_0(T)\to s_{0s}$=0.2763, labor resources $\theta_1(T)\to\theta_{1s}$=0.2562, $\theta_2(T)\to\theta_{2s}$=0.3494, $\theta_0(T)\to\theta_{0s}$=0.3944.

In this variant, we also solve the problem of optimal control for the values of the initial state of the system $y(t_0)$, which were chosen in the following form:

$$y(t_0)-y_s=(-600,\quad -200,\quad 200)^*. \qquad (44)$$

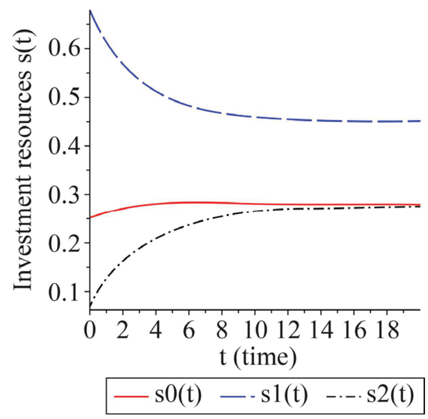


a

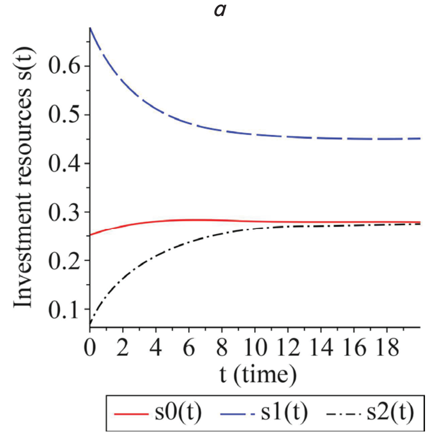


b

Fig. 1. Graphs for optimum trajectories and optimum controls: *a* — Graphs of the optimal trajectories (capacity to labor ratio) for the system (7) with the initial condition (43) under the control (22); *b* — Graphs of the optimal controls for the system (7) with the initial condition (43)



a



b

Fig. 2. Graphs for optimal allocation of investment and labor resources: *a* — Graphs of the optimal allocations of investment resources for the balance ratios (3)—(5) for the system (7) with the initial condition (43); *b* — Graphs of the optimal distribution of labor resources for the balance ratios (3)—(5) for the system (7) with the initial condition (43)

In this variant, matrices Q and R were chosen as:

$$Q=\begin{pmatrix} 0.0277\cdot 10^{-4} & 0 & 0 \\ 0 & 0.25\cdot 10^{-4} & 0 \\ 0 & 0 & 0.25\cdot 10^{-4} \end{pmatrix},$$

$$R=\begin{pmatrix} \frac{1}{8100} & 0 & 0 \\ 0 & \frac{1}{1600} & 0 \\ 0 & 0 & \frac{1}{1250} \end{pmatrix}.$$

Matrix K_T is equal to:

$$K_T=\begin{pmatrix} 0.107\cdot 10^{-4} & 0 & 0 \\ 0 & 0.77\cdot 10^{-4} & 0 \\ 0 & 0 & 0.85\cdot 10^{-4} \end{pmatrix}.$$

The results of the system state calculations are shown in Fig. 3, *a*, *b*. It can be seen from Fig. 4 that the optimal controls do not go beyond the range U determined by the constraints (9). For the example under consideration, these restrictions have the form:

$$0.1 \le u_i \le 0.9, \quad (i=0,1,2). \tag{45}$$

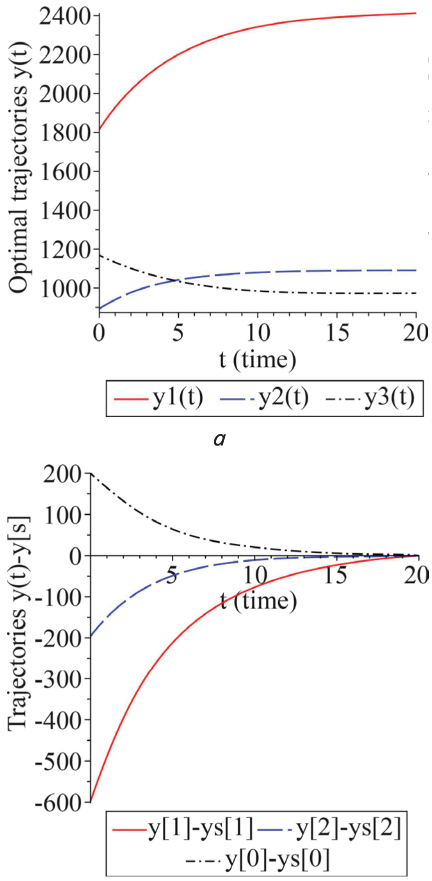


Fig. 3. Graphs for optimum trajectories and ($y(t)-y(t_s)$) trajectories: *a* – Graphs of optimal trajectories (capacity labor ratio) for the system (23) with the initial condition (44) under the control (39); *b* – Graphs of trajectories ($y(t)-y(t_s)$)

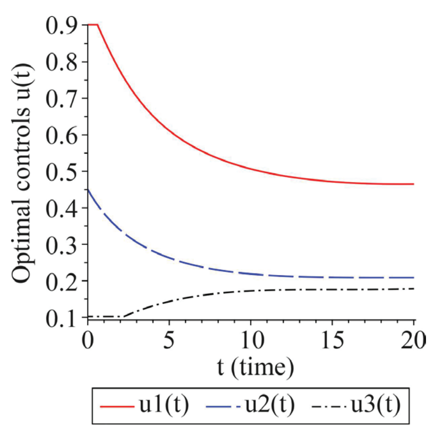


Fig. 4. Graphs of the optimal controls for the system (23) with the initial condition (44)

In this second variant, the control components $u_1(y)$ and $u_3(y)$ lie on the boundary of the region U in the time interval $[0,t_1]$, $[0,t_2]$, respectively, then for $t_1(t_1,T]$, $t_2(t_2,T]$, the lines enter the interior of the region U. Switching controls occurs at times t_1=0.637; t_2=2.234 for components $u1(t)$, $u3(t)$ respectively. It follows from Fig. 3, *a* that the obtained optimum trajectory values $y_1(T)\to k_{1s}$=2410.15, $y_2(T)\to k_{2s}$=1090.12, $y_3(T)\to k_{0s}$=966.45. It follows from Fig. 3, *b* that the values of the difference of the fundamentals $y_i(T)$ and steady-state fundamentals y_{is} $(y_1(T)-y_{1s})\to 0$; $(y_2(T)-y_{2s})\to 0$; $(y_3(T)-y_{3s})\to 0$ and optimal control values (Fig. 4) at the finite time at T=20: $u_1(T)$, $u_2(T)$, $u_3(T)$ satisfy the given constraints. Fig. 5, *a*, *b* shows graphs of optimal allocation of investment and labor resources, where investment resources $s_1(T)\to s_{1s}$=0.4476, $s_2(T)\to s_{2s}$=0.2761, $s_0(T)\to s_{0s}$=0.2763, labor resources $\theta 1(T)\to\theta 1s$=0.2562, $\theta_2(T)\to\theta_{2s}$=0.3494, $\theta_0(T)\to\theta_{0s}$=0.3944.

Using formulas (40)–(42), the optimal distribution of labor $(\theta_0(t),\theta_1(t),\theta_2(t))$ and investment resources $(s_0(t),s_1(t),s_2(t))$ is determined.

Fig. 5, *a*, *b* shows the changes in resources that ensure the system (23) is brought to an equilibrium state and satisfies the balance ratios (3)–(5).

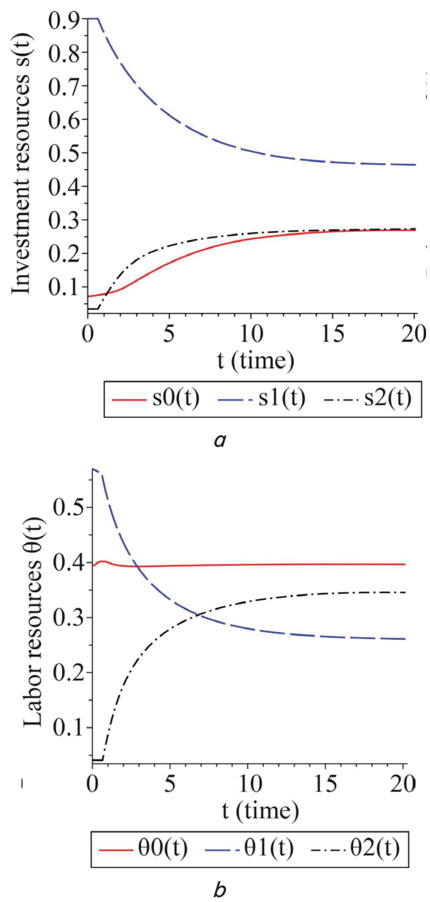


Fig. 5. Graphs for optimal allocation of investment and labor resources: *a* – Graphs of the optimal allocations of investment resources for the balance ratios (3)–(5) for the system (23) with the initial condition (44); *b* – Graphs of the optimal distribution of labor resources for the balance ratios (3)–(5) for the system (23) with the initial condition (44)

Numerical examples with two variants of the initial condition are considered. Fig. 1, 2 show trajectories at $y(t_0)-y_s$= =(–200, –100, –100)*, where control is performed by formula (22). The balance relations (3)–(5) are satisfied.

Fig. 3–5 show optimum trajectories at $y(t_0)-y_s$=(–600, –200, 200)*. In this case controls $u_1(t)$, $u_2(t)$, $u_3(t)$ take values satisfying the given constraints (45) and the balance relations (3)–(5) are fulfilled.

6. Discussion of the results of research and development of an algorithm for solving optimal control problems for the economic control object

To solve nonlinear optimal control problems in which the right parts of the ordinary differential equations are first reduced to a formally linear form by state and control, where the coefficients of all matrices may depend on the state of the system. The synthesizing feedback control is constructed by solving the corresponding linear-quadratic optimal control problems, where the coefficients of the weight matrices in the optimality criterion may also depend on the state variables. Then the matrix of the regulator gain is found by solving the matrix equations of Riccati type for the control problems on a finite time interval. As numerous experiments have shown, this heuristic approach in the numerical solution of the matrix Riccati equations, whose coefficients also depend on state, generates many possible sub-optimal solutions. But considering the complexity of construction and the importance for applications of controls in the form of feedback laws in nonlinear systems, this approach has been widespread in the literature in solving optimal control problems without control constraints. In this work, an algorithm for solving the problem of optimal control for one class of nonlinear controllable systems on a finite time interval is developed. It is shown that it is possible to construct a nonlinear synthesizing control (39) in problems with control constraints by means of an auxiliary functional with Lagrange multipliers of special form (30). At the same time, a more effective algorithm in terms of computation volume of Riccati equations (32), which does not require multiple integration of matrix differential equations with state-dependent coefficients, is proposed.

The results obtained for the nonlinear system, are used in the construction of the control parameters for the mathematical model of the economic control object. Numerical calculations for the mathematical model of the economic object have been performed using the proposed control method of the nonlinear system. The figures show the optimal trajectories, and the synthesizing controls take values that satisfy the given constraints (45) and balance relations (3)–(5) are fulfilled.

One of the drawbacks of the results of the presented research is the need to account for uncertain nonlinearities and external perturbations. The next stage of the research work is the need to consider the increasing complexity of mathematical models, taking into account nonlinearities, perturbations, increasing the dimensionality of the state and control vectors. At the same time, it is obvious that one may encounter such difficulties as taking into account various control constraints and non-linear characteristics of the system.

7. Conclusions

1. For solving the problem, auxiliary functional with Langrage multipliers of special form (13) is constructed. A non-linear synthesizing control (22) is constructed with the help of Langrage multiplier of special form. In doing so, we propose an efficient algorithm in terms of the computation volume of the Riccati equation (15) that does not require multiple integration of matrix differential equations with state-dependent coefficients. The state of the system $y(t)$ satisfies differential equation (21), and the synthesizing control $u(y,t)$ is defined by formula (22).

2. To solve the problem, a new approach to constructing synthesizing control based on the feedback principle and with control constraints is proposed. The possibility of constructing a non-linear synthesizing control (39) in problems with control constraints by means of an auxiliary functional with Lagrange multipliers of special form (30) is shown. In this case, we propose a more efficient algorithm in terms of the computational volume of the Riccati equations (32) that does not require multiple integration of matrix differential equations with state-dependent coefficients. The problem is solved using special Lagrange multipliers, taking into account the constraints on the controls. The synthesizing control is defined as follows:

$$u(y,t)=D^{-1}(y)\begin{pmatrix} D_s u_s - R^{-1}B^{*}\times \\ \times\left[K(t)\left(y(t)-y_s\right)+q(t)\right]\end{pmatrix}+\varphi(y,t).$$

The matrices $K(t)$ and $W(t,T)$ are solutions of equations (32) and (33), vector $q(t)$ satisfies differential equation (34), and vector function $\varphi(y,t)$ is defined by formula (35).

3. Algorithm of solving the problem of optimal control (1)–(5) is described in p. 5. 3. Numerical calculations for the mathematical model of the economic object have been performed using the proposed control method of the non-linear system. Numerical examples with two variants of the initial condition are considered. The figures show the optimal trajectories and the controls take values that satisfy the given constraints (45) and balance relations (3)–(5) are fulfilled.

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